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QUASICONFORMAL EXTENSIONS OF HARMONIC MAPPINGS WITH A COMPLEX PARAMETER

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Abstract

In this paper, we study quasiconformal extensions of harmonic mappings. Utilizing a complex parameter, we build a bridge between the quasiconformal extension theorem for locally analytic functions given by Ahlfors [‘Sufficient conditions for quasiconformal extension’, *Ann. of Math. Stud.* **79** (1974), 23–29] and the one for harmonic mappings recently given by Hernández and Martín [‘Quasiconformal extension of harmonic mappings in the plane’, *Ann. Acad. Sci. Fenn. Math.* **38** (2) (2013), 617–630]. We also give a quasiconformal extension of a harmonic Teichmüller mapping, whose maximal dilatation estimate is asymptotically sharp.

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1. Introduction

If a C^2-complex-valued function f of the unit disk $\mathbb{D}$ satisfies the Laplacian equation $\Delta f(z) = 4f_{z\bar{z}} = 0$, then it is said to be a *harmonic mapping*. It is known that f has a canonical representation $f = h + \bar{g}$, where h and g are analytic on $\mathbb{D}$. We also call $\nu = g'/h'$ the *second Beltrami coefficient* of f. Lewy's theorem [10] says that a harmonic mapping f is locally univalent if and only if its Jacobian $J_f = |h'|^2 - |g'|^2$ does not vanish. If a harmonic mapping is of the representation $f = h + \alpha\bar{h}$, where h is a conformal mapping and α is a constant such that $0 < |\alpha| < 1$, then it is called a *harmonic Teichmüller mapping* (see [4]).

The *pre-Schwarzian derivative* P_f of a locally univalent harmonic mapping f is defined on $\mathbb{D}$ by

$$P_f = \frac{\partial}{\partial z}\log|J_f| = \frac{h''}{h'} - \frac{\bar{\nu}\nu'}{1-|\nu|^2}.$$

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In particular, if f is a locally univalent analytic mapping, then $P_f = f''/f'$.

Becker [2] stated that if a locally univalent analytic mapping f in the unit disk $\mathbb{D}$ satisfies

$$\sup_{z\in\mathbb{D}}|P_f|(1-|z|^2) \le 1,$$

then f is univalent on $\mathbb{D}$. Later, Becker and Pommerenke [3] showed that the constant 1 is sharp. Moreover, if

$$\sup_{z\in D}|P_f|(1-|z|^2) \le k < 1, \tag{1.1}$$

then it also has a continuous extension $\widetilde{f}$ to $\overline{\mathbb{D}}$ and $\widetilde{f}(\partial\mathbb{D})$ is a quasicircle [2]. For every f satisfying (1.1), Ahlfors [1] gave an explicit quasiconformal extension of the complex plane $\mathbb{C}$ onto itself with the infinity fixed.

Theorem A. *If a locally univalent analytic mapping f in the unit disk $\mathbb{D}$ satisfies (1.1), then it admits a homeomorphic extension*

$$F(z) = \begin{cases} \tilde{f}(z) & \text{if } |z| \le 1, \\ f\Big(\dfrac{1}{\bar{z}}\Big) + u\Big(\dfrac{1}{\bar{z}}\Big) & \text{if } |z| > 1, \end{cases}$$

where $u(z) = f'(z)(1-|z|^2)/\bar{z}$, for $z \in \mathbb{D}\backslash\{0\}$. Moreover, the mapping F is a $(1+k)/(1-k)$-quasiconformal in the complex plane $\mathbb{C}$, coinciding with f in $\mathbb{D}$.

Hernández and Martín [8] proved that if a sense-preserving harmonic mapping $f = h + \overline{g}$ in the unit disk $\mathbb{D}$ with the second Beltrami coefficient $\nu(z)$ satisfies

$$|P_f|(1-|z|^2) + |\nu^*(z)| \le 1, \qquad \nu^*(z) = \frac{\nu'(z)(1-|z|^2)}{1-|\nu(z)|^2}, z \in \mathbb{D},$$

then f is univalent in $\mathbb{D}$. Moreover, the constant 1 is sharp. Recently, Hernández and Martín [7] showed the following theorem.

Theorem B. *If a sense-preserving harmonic mapping f in the unit disk with a second Beltrami coefficient $\nu(z)$ satisfies*

$$|P_f|(1-|z|^2) + |\nu^*(z)| \le k < 1, \quad z \in \mathbb{D}, \tag{1.2}$$

then f has a continuous and homeomorphic extension $\widetilde{f}$ to $\overline{\mathbb{D}}$. It also admits an explicit homeomorphic extension of the complex plane $\mathbb{C}$ with

$$F(z) = \begin{cases} \tilde{f}(z) & \text{if } |z| \le 1, \\ f\Big(\dfrac{1}{\bar{z}}\Big) + U\Big(\dfrac{1}{\bar{z}}\Big) & \text{if } |z| > 1, \end{cases} \tag{1.3}$$

where

$$U(z) = \frac{h'(z)(1-|z|^2)}{\bar{z}} + \frac{\overline{g'(z)}(1-|z|^2)}{z}, \quad z \in \mathbb{D}\backslash\{0\}.$$

Furthermore, Hernández and Martín [7] gave the maximal dilatation estimate of the homeomorphic extension.

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THEOREM C. *If a sense-preserving harmonic mapping f satisfies (1.2) and $\|\nu\|_\infty = \sup_{z\in\mathbb{D}}|\nu(z)| < 1$, then $\tilde{f}(\partial\mathbb{D})$ is a quasicircle and f can be extended to a quasiconformal mapping in the complex plane $\mathbb{C}$. Indeed, the mapping defined by (1.3) is an explicit K-quasiconformal extension of f whenever*

$$k < \frac{1 - \|\nu\|_\infty}{1 + \|\nu\|_\infty}.$$

The constant K is equal to

$$K = \frac{1 + k + (1 - k)\|\nu\|_\infty}{1 - k - (1 + k)\|\nu\|_\infty}.$$

As the first result of this paper, for a sense-preserving harmonic mapping f, we study the stability of its quasiconformal extension in a complex parameter $\lambda \in \overline{\mathbb{D}}$. We prove the following theorem.

THEOREM 1.1. *Let $f = h + \bar{g}$ be a sense-preserving harmonic mapping f satisfying (1.2) and let $\nu(z)$ be its second Beltrami coefficient with $\|\nu\|_\infty = \sup_{z\in\mathbb{D}}|\nu(z)| < 1$. Then, for every $|\lambda| \le 1$, $f_\lambda = h + \lambda\bar{g}$ has a continuous and homeomorphic extension $\tilde{f}_\lambda = \tilde{h} + \lambda\bar{\tilde{g}}$ to $\overline{\mathbb{D}}$ and the mapping*

$$F_\lambda(z) = \begin{cases} \tilde{h}(z) + \lambda\bar{\tilde{g}}(z) & \text{if } |z| \le 1, \\ h\left(\dfrac{1}{\bar{z}}\right) + \lambda\overline{g\left(\dfrac{1}{\bar{z}}\right)} + U_\lambda\left(\dfrac{1}{\bar{z}}\right) & \text{if } |z| > 1, \end{cases}$$

is a homeomorphic extension of the complex plane $\mathbb{C}$ onto itself. Moreover,

$$|\mu_{F_\lambda}(z)| \le \frac{k - (1 - \|\nu\|_\infty)|\nu^*(1/\bar{z})| + |\lambda|\|\nu\|_\infty|z|}{|z| - k|\lambda|\|\nu\|_\infty - (1 - \|\nu\|_\infty)|\nu^*(1/\bar{z})|}, \quad z \in \mathbb{C}\backslash\overline{\mathbb{D}},$$

where

$$U_\lambda(z) = \frac{h'(z)(1 - |z|^2)}{\bar{z}} + \lambda\frac{\overline{g'(z)}(1 - |z|^2)}{z}, \quad z \in \mathbb{D}\backslash\{0\}.$$

Furthermore, if

$$k < \frac{1 - \|\nu\|_\infty}{1 + \|\nu\|_\infty}, \tag{1.4}$$

then the family of mappings $F_\lambda(z)$ are a K-quasiconformal mapping of the complex plane $\mathbb{C}$ with

$$K = \frac{1 + k_1}{1 - k_1}, \quad k_1 = \frac{k + |\lambda|\|\nu\|_\infty}{1 - k|\lambda|\|\nu\|_\infty}.$$

By a complex parameter $\lambda \in \overline{\mathbb{D}}$, Theorem 1.1 builds a bridge between Theorems A and C (see Remark 2.1).

The class of harmonic Teichmüller mappings is a subclass of harmonic mappings that is closely related to minimal surfaces with a constant gaussian curvature (see [5, Theorem 1]). Chuaqui *et al.* [6] also showed that a harmonic Möbius transformation always sends circles to ellipses. One can see [4] for more properties for harmonic Teichmüller mappings. As the second result of this paper, we give a quasiconformal extension theorem of a harmonic Teichmüller mapping and give an asymptotically sharp estimate of its maximal dilatation.

THEOREM 1.2. *Let f be a sense-preserving harmonic mapping in the unit disk $\mathbb{D}$ with a representation $f = h + \alpha\bar{h}$, where h is a locally univalent analytic function in $\mathbb{D}$ and α is a constant with $|\alpha| < 1$. Assume that*

$$|P_h(z)(1 - |z|^2)| \le k < 1, \quad z \in \mathbb{D}. \tag{1.5}$$

Then f is a harmonic Teichmüller mappings of $\mathbb{D}$ and has a continuous and homeomorphic extension $\tilde{f}$ to $\overline{\mathbb{D}}$. Moreover, the mapping

$$F(z) = \begin{cases} \tilde{f}(z) & \text{if } |z| \le 1, \\ f\left(\dfrac{1}{\bar{z}}\right) + U_\alpha\left(\dfrac{1}{\bar{z}}\right) & \text{if } |z| > 1, \end{cases} \tag{1.6}$$

is a K-quasiconformal mapping of the complex plane $\mathbb{C}$ with

$$K = \frac{1 + k_1}{1 - k_1}, \quad k_1 = \frac{|\alpha| + k}{1 + |\alpha|k},$$

where

$$U_\alpha(z) = \frac{h'(z)(1 - |z|^2)}{\bar{z}} + \frac{\alpha\overline{h'(z)}(1 - |z|^2)}{z}, \quad z \in \mathbb{D}\backslash\{0\}.$$

The maximal dilatation estimate of the quasiconformal extension F is asymptotically sharp in k, and extremal mappings are of the form $f(z) = az + b\bar{z}$, where a and b are two nonvanishing constants.

2. Proof of Theorem 1.1

PROOF. For every $\lambda \in \overline{\mathbb{D}}$, let $f_\lambda = h + \lambda\bar{g}$. Write $\nu_\lambda = \bar{\lambda}\nu$ and $\nu_\lambda^* = (\nu_\lambda'(1 - |z|^2)/(1 - |\nu_\lambda|^2))$. Utilizing the triangle inequality,

$$\begin{aligned} |P_{f_\lambda}|(1 - |z|^2) + |\nu_\lambda^*| &= \left|\frac{h''}{h'} - \frac{\overline{\nu_\lambda}\nu_\lambda'}{1 - |\nu_\lambda|^2}\right| + |\nu_\lambda^*| \\ &\le \left|\frac{h''}{h'} - \frac{\bar{\nu}\nu'}{1 - |\nu|^2}\right|(1 - |z|^2) + \left|\frac{\bar{\nu}\nu'}{1 - |\nu|^2} - \frac{\overline{\nu_\lambda}\nu_\lambda'}{1 - |\nu_\lambda|^2}\right|(1 - |z|^2) + |\nu_\lambda^*| \\ &= |P_f|(1 - |z|^2) + \frac{|\nu||\nu'|(1 - |\lambda|^2)(1 - |z|^2)}{(1 - |\nu|^2)(1 - |\lambda|^2|\nu|^2)} + \frac{|\lambda||\nu'|(1 - |z|^2)}{1 - |\lambda|^2|\nu|^2} \\ &= |P_f|(1 - |z|^2) + \frac{|\lambda| + |\nu|}{1 + |\lambda||\nu|}\,\frac{|\nu'|(1 - |z|^2)}{1 - |\nu|^2} \\ &\le |P_f|(1 - |z|^2) + |\nu^*|. \end{aligned}$$

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The fact that $(1-|\lambda|)(1-|\nu|) \geq 0$ implies the above second inequality. Hence, for all $\lambda \in \overline{\mathbb{D}}$, it follows that

$$|P_{f_\lambda}|(1-|z|^2) + |\nu_\lambda^*| \leq k, \quad 0 \leq k < 1,$$

if f satisfies the assumption (1.2).

If the second Beltrami coefficient ν of f satisfies that $\|\nu\|_\infty < 1$, then, for each $\lambda \in \overline{\mathbb{D}}$, by [7, Theorem 1], the mapping f_λ can be continuously extended to a homeomorphism of $\overline{\mathbb{D}}$ and we write it by $\tilde{f}_\lambda$. Moreover, f_λ can be extended to a homeomorphism of the complex plane $\mathbb{C}$ as

$$F_\lambda(z) = \begin{cases} \tilde{f}_\lambda(z) & \text{if } |z| \leq 1, \\ h\Big(\dfrac{1}{\bar{z}}\Big) + \lambda\overline{g\Big(\dfrac{1}{\bar{z}}\Big)} + U_\lambda\Big(\dfrac{1}{\bar{z}}\Big) & \text{if } |z| > 1, \end{cases}$$

where

$$U_\lambda(z) = \frac{h'(z)(1-|z|^2)}{\bar{z}} + \lambda\frac{\overline{g'(z)}(1-|z|^2)}{z}, \quad z \in \mathbb{D}\backslash\{0\}.$$

Next, we will estimate the maximal dilatation of the mapping F_λ. When $|z| < 1$, by the assumption that $f = h + \bar{g}$ satisfies that $|\mu_f| = |g'/h'| \leq \|\nu\|_\infty < 1$ in $\mathbb{D}$,

$$|\mu_{F_\lambda}(z)| = |\nu_{f_\lambda}(z)| = |\lambda||\nu(z)| \leq |\lambda|\|\nu\|_\infty < 1, \quad z \in \mathbb{D}.$$

When $|z| > 1$, we set $w = 1/\bar{z}$ and obtain that

$$\begin{aligned} |\mu_{F_\lambda}(z)| &= \left|\frac{(F_\lambda)_{\bar{z}}}{(F_\lambda)_z}\right| = \left|\frac{(F_\lambda)_w w_{\bar{z}} + (F_\lambda)_{\bar{w}}\overline{w_z}}{(F_\lambda)_w w_z + (F_\lambda)_{\bar{w}}\overline{w_{\bar{z}}}}\right| = \left|\frac{-(F_\lambda)_w \frac{1}{\bar{z}^2}}{-(F_\lambda)_{\bar{w}} \frac{1}{\bar{z}^2}}\right| \\ &= \left|\frac{(F_\lambda)_w}{(F_\lambda)_{\bar{w}}}\right| = \left|\frac{h'(w) + U_w(w)}{\lambda\overline{g'(w)} + U_{\bar{w}}(w)}\right| \\ &\leq \frac{\left|w\frac{h''(w)}{h'(w)}\right|(1-|w|^2) + |\lambda|\|\nu\|_\infty}{1-|\lambda|\left|\frac{wg''(w)}{h'(w)}\right|(1-|w|^2)}. \end{aligned} \tag{2.1}$$

Since f satisfies (1.2), we get

$$\left|\frac{h''(w)}{h'(w)} - \frac{\overline{\nu(w)}\nu'(w)}{1-|\nu(w)|^2}\right|(1-|w|^2) \leq k - |\nu^*(w)|, \quad w \in \mathbb{D},$$

which implies that

$$\begin{aligned} \left|\frac{wh''(w)}{h'(w)}\right|(1-|w|^2) &\leq k|w| - |\nu^*(w)||w| + \left|\frac{w\overline{\nu(w)}\nu'(w)}{1-|\nu(w)|^2}\right|(1-|w|^w) \\ &\leq k|w| - |\nu^*(w)||w| + |\nu^*(w)|\|\nu\|_\infty|w| \\ &= k|w| + (\|\nu\|_\infty - 1)|\nu^*(w)||w|. \end{aligned} \tag{2.2}$$

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Moreover, $g'' = (\nu h')' = \nu' h' + \nu h''$ in the unit disk. Hence, we obtain

$$\begin{aligned}\frac{g''(w)}{h'(w)}(1-|w|^2) &= \Big(\nu(w)\frac{h''(w)}{h'(w)} + \nu'(w)\Big)(1-|w|^2)\\ &= \nu(w)\frac{h''(w)}{h'(w)}(1-|w|^2) + \frac{\nu'(w)(1-|w|^2)(1-|\nu(w)|^2)}{1-|\nu(w)|^2}\\ &= \nu(w)P_f(w)(1-|w|^2) + \frac{\nu'(w)(1-|w|^2)}{1-|\nu(w)|^2}, \quad w \in \mathbb{D}.\end{aligned}$$

So

$$\begin{aligned}\Big|w\frac{g''(w)}{h'(w)}\Big|(1-|w|^2) &\le |w\nu(w)||P_f(w)|(1-|w|^2) + |w\nu^*(w)|\\ &\le |w|\|\nu\|_\infty(k-|\nu^*(w)|) + |w||\nu^*(w)| \qquad (2.3)\\ &= k|w|\|\nu\|_\infty + (1-\|\nu\|_\infty)|\nu^*(w)||w|.\end{aligned}$$

By (2.2) and (2.3), we deduce from (2.1) that

$$|\mu_{F_\lambda}(z)| \le \frac{k-(1-\|\nu\|_\infty)|\nu^*(1/\bar{z})| + |\lambda|\|\nu\|_\infty|z|}{|z|-k|\lambda|\|\nu\|_\infty - (1-\|\nu\|_\infty)|\nu^*(1/\bar{z})|}, \quad z \in \mathbb{C}\backslash\overline{\mathbb{D}}.$$

Let $|\nu^*(1/\bar{z})| = x$ and define

$$\rho(x) = \frac{k-(1-\|\nu\|_\infty)x + |\lambda|\|\nu\|_\infty|z|}{|z|-k|\lambda|\|\nu\|_\infty - (1-\|\nu\|_\infty)x}.$$

The relation (1.4) implies that the function $\rho(x)$ is decreasing in $[0, k]$ with respect to x. Therefore

$$|\mu_{F_\lambda}(z)| \le \frac{k+|\lambda|\|\nu\|_\infty|z|}{|z|-k|\lambda|\|\nu\|_\infty} \le \frac{k+|\lambda|\|\nu\|_\infty}{1-k|\lambda|\|\nu\|_\infty} = k_1$$

holds for all $|z| > 1$. Since the assumption (1.4) implies that $\|\nu\|_\infty < ((1-k)/(1+k))$, the inequality

$$\|\nu_\lambda\|_\infty = |\lambda|\|\nu\|_\infty < \frac{1-k}{1+k}$$

holds for all $\lambda \in \overline{D}$, which implies that $k_1 < 1$. So $F_\lambda(z)$ is a quasiconformal mapping in $\mathbb{C}\backslash\overline{\mathbb{D}}$. Since $|\mu_{F_\lambda}(z)| \le |\lambda|\|\mu_f\|_\infty < 1$ for $|z| < 1$, $F_\lambda(z)$ is also a quasiconformal mapping in $\mathbb{D}$. By [9, Ch. I, Lemma 6.1], it follows that $F_\lambda(z)$ is a quasiconformal mapping in $\mathbb{C}$. Moreover, the fact that $|\lambda|\|\nu\|_\infty \le k_1 < 1$ shows that the mapping F_λ is K-quasiconformal in the complex plane $\mathbb{C}$ with the maximal dilatation

$$K = \frac{1+k+|\lambda|\|\nu\|_\infty(1-k)}{1-k-|\lambda|\|\nu\|_\infty(1+k)}.$$

This completes the proof of Theorem 1.1. □

REMARK 2.1. Two special cases, $\lambda = 0$ and $\lambda = 1$, in Theorem 1.1 are just Theorem A given by Ahlfors and Theorem C given by Hernández and Martín, respectively.

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3. Proof of Theorem 1.2

In order to give the proof of Theorem 1.2, we first need the following lemma.

LEMMA 3.1. *Let $a \in \mathbb{D}$ and $T(z) = (z + |a|)/(1 + |a|z)$. Then $T(z)$ is a Möbius transformation of the unit disk $\mathbb{D}$ onto itself and*

$$\frac{||a| - |z||}{1 - |a||z|} \le |T(z)| \le \frac{|a| + |z|}{1 + |a||z|}. \tag{3.1}$$

PROOF. Let $z = re^{i\theta} \in \mathbb{D}$. Then

$$|T(z)|^2 = \frac{|a|^2 + 2|a|r\cos\theta + r^2}{1 + 2|a|r\cos\theta + r^2|a|^2}.$$

Since the function $f(t) = (C_1 + t)/(C_2 + t)$ with $C_1 < C_2$ is monotonically increasing in t, it follows that

$$\frac{||a| - r|}{1 - |a|r} \le |T(z)| \le \frac{|a| + r}{1 + |a|r},$$

that is, (3.1) holds for a given $a \in \mathbb{D}$. □

PROOF OF THEOREM 1.2. Assume that f is a harmonic mapping of the unit disk $\mathbb{D}$ with a representation $f(z) = h(z) + \alpha\overline{h(z)}$, where h is a locally univalent analytic function and α is a constant satisfying that $|\alpha| < 1$. Then it follows that

$$\nu^* = \frac{\nu'(1 - |z|^2)}{1 - |\nu|^2} = 0$$

and

$$P_f = \frac{\partial}{\partial z}\log|J_f| = \frac{\partial}{\partial z}\log((1 - |\alpha|^2)|h'(z)|^2) = \frac{h''(z)}{h'(z)} = P_h. \tag{3.2}$$

By (1.5), the classical result due to Becker [2] for a locally univalent analytic function shows that $h(z)$ is univalent and can be extended to a continuous and injective mapping $\tilde{h}(z)$ in the closed unit disk. Hence, for any α with $|\alpha| < 1$, $f = h + \alpha\bar{h}$ is a harmonic Teichmüller mapping of $\mathbb{D}$.

Taking $g = \bar{\alpha}h$ in the proof of [7, Theorem 1] and by (3.2), $|P_f(1 - |z|^2)| \le k < 1$. Hence $f = h + \alpha\bar{h}$ can be extended to a homeomorphism of the complex plane $\mathbb{C}$. Furthermore, the homeomorphism can be constructed as

$$H(z) = \begin{cases} \tilde{h}(z) + \alpha\tilde{f}(z) & \text{if } |z| \le 1, \\ f\left(\frac{1}{\bar{z}}\right) + U_\alpha\left(\frac{1}{\bar{z}}\right) & \text{if } |z| > 1, \end{cases}$$

where

$$U_\alpha(z) = \frac{h'(z)(1 - |z|^2)}{\bar{z}} + \frac{\alpha\overline{h'(z)}(1 - |z|^2)}{z}, \quad z \in \mathbb{D},\ |\alpha| < 1.$$

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Next we will estimate the maximal dilatation of H and then show that H is a quasiconformal mapping in the complex plane $\mathbb{C}$. By the assumption that $|\alpha| < 1$,

$$|\mu_H(z)| = |\alpha| < 1 \quad \text{when } |z| < 1.$$

When $|z| > 1$, we set $w = 1/\bar{z}$ and obtain that

$$\begin{aligned}|\mu_H(z)| &= \left|\frac{h'(w) + U_w(w)}{\alpha\overline{h'(w)} + U_{\overline{w}}(w)}\right| = \left|\frac{w^2h''(w)(1-|w|^2) - \alpha\overline{wg'(w)}}{\bar{\alpha}w^2g''(w)(1-|w|^2) - \overline{wh'(w)}}\right| \\ &= \left|\frac{\alpha\overline{wh'(w)} - w^2h''(w)(1-|w|^2)}{\overline{wh'(w)} - \bar{\alpha}w^2h''(w)(1-|w|^2)}\right| = \left|\frac{\alpha + \frac{w^2h''(w)(|w|^2-1)}{\overline{wh'(w)}}}{1 + \bar{\alpha}\frac{w^2h''(w)(|w|^2-1)}{\overline{wh'(w)}}}\right|.\end{aligned}$$

Let $v(w) = w^2h''(w)(|w|^2-1)/\overline{wh'(w)}$. By (1.5),

$$|v(w)| \le k, \quad w \in \mathbb{D}. \tag{3.3}$$

Set $\lambda = \bar{\alpha}/|\alpha|$. Then it follows that

$$|\mu_H(z)| = \left|\frac{\alpha + v(w)}{1 + \bar{\alpha}v(w)}\right| = \left|\frac{\lambda\alpha + \lambda v(w)}{1 + \bar{\lambda}\bar{\alpha}\lambda v(w)}\right| = \left|\frac{|\alpha| + \lambda v(w)}{1 + |\alpha|\lambda v(w)}\right|, \quad z \in \mathbb{C}\backslash\overline{\mathbb{D}}.$$

By Lemma 3.1 and (3.3), we obtain the inequality

$$|\mu_H(z)| \le \frac{|\alpha| + |v(w)|}{1 + |\alpha v(w)|} \le \frac{|\alpha| + k}{1 + |\alpha|k} = k_1, \quad z \in \mathbb{C}\backslash\overline{\mathbb{D}}. \tag{3.4}$$

So $H(z)$ is a quasiconformal mapping in $z \in \mathbb{C}\backslash\overline{\mathbb{D}}$. Since $|\mu_H(z)| = |\alpha| < 1$ for $|z| < 1$, $H(z)$ is also a quasiconformal mapping in $\mathbb{D}$. By [9, Ch. I, Lemma 6.1], we know $H(z)$ is a quasiconformal mapping in the complex plane $\mathbb{C}$. Moreover, it follows that $|\alpha| \le k_1 < 1$. Therefore, H is a K-quasiconformal mapping in the complex plane $\mathbb{C}$ with

$$K = \frac{(1+k)(1+|\alpha|)}{(1-k)(1-|\alpha|)}.$$

Take $f = az + b/\bar{a}(\overline{az})$ with $|b| < |a|$. Then f satisfies (1.5) with $k = 0$, so its quasiconformal extension given by (1.6) is just of the form $f(z) = az + b\bar{z}$, $z \in \mathbb{C}$ and its maximal dilatation is equal to $k_1 = |\bar{b}/a| = |\alpha| = (K-1)/(K+1)$, which shows that the estimate (3.4) is asymptotically sharp in k. This completes the proof of Theorem 1.2. □

Remark 3.2. For the class of harmonic Teichmüller mappings f, the method in Theorem 1.1 gives the estimate of the maximal dilatation of its quasiconformal extension H as

$$|\mu_H(z)| \le \frac{k + |\alpha|}{1 - k|\alpha|}, \quad z \in \mathbb{C}\backslash\overline{\mathbb{D}}.$$

Hence, for the class of harmonic Teichmüller mappings, the maximal dilatation estimate of their quasiconformal extensions H given by Theorem 1.2 is better than the one given by Theorem 1.1.

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