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Luzin-type Holomorphic Approximation on Closed Subsets of Open Riemann Surfaces

In loving memory: André Boivin, our student and respectively supervisor.

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Abstract. It is known that if E is a closed subset of an open Riemann surface R and f is a holomorphic function on a neighbourhood of E, then it is "usually" not possible to approximate f uniformly by functions holomorphic on all of R. We show, however, that for every open Riemann surface R and every closed subset $E \subset R$, there is closed subset $F \subset E$ that approximates E extremely well, such that every function holomorphic on F can be approximated much better than uniformly by functions holomorphic on R.

1 Introduction

Undergraduate students are often first introduced to Riemann surfaces via so-called *concrete* Riemann surfaces, meaning surfaces constructed with paper, scissors, and paste. *Abstract* Riemann surfaces, defined as manifolds, are usually encountered later in their studies.

A remarkable theorem of Gunning and Narasimhan [6] essentially asserts that every abstract non-compact Riemann surface can be represented as a concrete Riemann surface *having no branch points*.

Precisely, it says the following. For every open Riemann surface R, there exists a holomorphic mapping ρ of R into the complex plane that is a local homeomorphism; the mapping ρ induces a complex structure on R that is the initial complex structure R, since ρ is locally biholomorphic. We call ρ a spreading of R over $\mathbb{C}$. We denote Lebesgue measure in $\mathbb{C} = \mathbb{R}^2$ by λ and the measure on R induced by ρ and λ by μ, in the sense that if $X = \bigcup X_n$ is the disjoint union of $X_n, n = 1, \dots$ and each X_n is contained in a chart where ρ is injective, then $\mu(X) = \sum \lambda(\rho(X_n))$. One could also say that $\mu(X)$ is the Lebesgue measure of the projection $\rho(X)$ "counting multiplicities". A subset E of a Riemann surface R is said to be *bounded* if the closure $\overline{E}$ in R is compact.

If a function can be approximated uniformly by holomorphic functions on a set E, then that function must necessarily be in the class $A(E)$ of continuous functions on E that are holomorphic on the interior E^0. Let us say that a closed set E in an open

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Riemann surface R is a set of *uniform approximation* if, for every $f \in A(E)$ and every *number* $\epsilon > 0$, there is a function g holomorphic on R such that $|f(p) - g(p)| < \epsilon$, for all $p \in E$. Similarly, we say that E is a set of *tangential approximation* if, for every $f \in A(E)$ and every *continuous function* $\epsilon > 0$, there is a function g holomorphic on R such that $|f(p) - g(p)| < \epsilon(p)$, for all $p \in E$. A theorem of Carleman [3], which deserves to be better known, asserts that the real line is a set of tangential approximation in $\mathbb{C}$. For this reason, sets of tangential approximation are often called *sets of Carleman approximation.*

For the case where R is a planar domain G, Arakelian [1] gave a complete topological characterization of closed subsets $E \subset G$ for which E is a set of uniform approximation by functions holomorphic on G. Let us denote by $R^* = R \cup \{*\}$ the one-point compactification of an open Riemann surface R. Arakelian's theorem states that E is a set of uniform approximation in G if and only if $G^* \setminus E$ is connected and locally connected.

Gauthier and Hengartner [5] showed that the topological conditions of Arakelian are still necessary in order for a closed set E in an open Riemann surface R to be a set of uniform approximation. That is, $R^* \setminus E$ must be connected and locally connected. In the same paper an example was given to show that these topological conditions of Arakelian, although necessary, are not sufficient to guarantee that E be a set of uniform approximation in R. Thus, in passing from planar domains to open Riemann surfaces, Arakelian's topological conditions no longer give a characterization of closed sets of uniform approximation. In fact, Scheinberg [8], showed that no topological conditions whatsoever could characterize sets of uniform approximation on Riemann surfaces.

If a closed set E in an open Riemann surface R is a set of tangential approximation, then of course it must *a fortiori* be a set of uniform approximation, and so $R^* \setminus E$ must be connected and locally connected. A further condition, now called the long island condition, was introduced by the first author [4]. A closed subset $E \subset R$ is said to satisfy the *long island condition* if for every compact set $K \subset R$, there exists a compact set $Q \subset R$ such that every component of the interior of E that meets K is contained in Q. If E is a closed set of uniform approximation in $\mathbb{C}$, then it was shown in [4] that the long island condition is necessary in order for E to be a set of tangential approximation. Nersessian [7] showed, that, in fact a closed set E of uniform approximation in a plane domain G is a set of tangential approximation in G if and only if the long island condition is satisfied. A closed strip in $\mathbb{C}$ of strictly positive width is a set of uniform approximation but not of tangential approximation. If the width becomes zero, then a straight line is a set of tangential approximation (and a fortiori of uniform approximation). At the end of this paper, the authors construct a Riemann Surface R where the real line is a set of tangential approximation (and a fortiori of uniform approximation) but a "strip" around it in R is not even a set of uniform approximation. See Example 3.3.

Thus, for planar domains, we have a characterization of closed subsets of uniform approximation and also a characterization of closed subsets of tangential approximation. The problem of characterizing closed sets of uniform approximation on open

Riemann surfaces is open; however, Boivin extended Nersessian's result to open Riemann surfaces, thus giving a characterization of closed sets of tangential approximation in open Riemann surfaces. Here is Boivin's theorem.

Theorem 1.1 ([2]) *Let E be a proper closed subset of an open Riemann surface R; then the following are equivalent:*

(i) *E is a set of tangential approximation;*
(ii) *$R^* \setminus E$ is connected and locally connected and E satisfies the long island condition;*
(iii) *E is a set of uniform approximation that satisfies the long island condition.*

Our principal result is the following Luzin-type theorem, which, loosely speaking, asserts that for an arbitrary open Riemann surface R, an arbitrary (proper) closed set E in R, an arbitrary function $f \in A(E)$, and an arbitrary $\epsilon > 0$, although there is practically no chance that there exists a function g holomorphic on R such that $|f - g| < \epsilon$, nevertheless, we can always find a closed subset F of E which is most of E (in the sense that $E \setminus F$ is small and becomes smaller at arbitrary speed as we approach the ideal boundary point $*$) and on F such approximations are possible, in fact, with arbitrary speed. That is, $\epsilon(p)$ can decrease to zero with arbitrary speed as p tends to the ideal boundary. The precise statement is the following theorem.

Theorem 1.2 *Let R be an arbitrary open Riemann surface and ρ be a spreading of R over $\mathbb{C}$. Let E be a closed subset of R. For every positive sequence δ_n, and every regular smooth exhaustion $\{K_n\}$ of R, there exists a closed subset F of E such that F is a set of tangential approximation in R and*

$$\mu((E \setminus F) \setminus K_n) < \delta_n, \quad n = 1, 2, \dots.$$

Paraphrasing the "100%" conjecture for the Riemann Hypothesis and denoting a proposition regarding a point p by $P(p)$, we say that the proposition $P(p)$ is true for 100% of the points in a set $E \subset R$ if, for every smooth exhaustion $\{K_n\}$ of R, we have

$$\lim_{n\to\infty} \frac{\mu\{p \in E \cap K_n : P(p)\}}{\mu(E \cap K_n)} = 1.$$

If E is bounded, this notion is equivalent to saying that the property holds almost everywhere on E.

The following corollary is an easy consequence of Theorem 1.2.

Corollary 1.3 *If $\mu(E) = \infty$, then for every $f \in A(E)$ and for every positive $\epsilon \in C(E)$, there exists $g \in H(R)$ such that, for 100% of the points p in E, we have $|f(p) - g(p)| < \epsilon(p)$.*

The preceding corollary does not hold for arbitrary closed sets E.

First of all, we note that, in case E is bounded, to say that $|f(p)-g(p)| < \epsilon(p)$ holds for 100% of the values of E is equivalent to saying that $|f(p) - g(p)| < \epsilon(p)$ a.e. on E. Since f, g, and ϵ are continuous, this implies that $|f(p) - g(p)| \le \epsilon(p)$ everywhere on $\overline{E^0}$. Now let $R = \mathbb{C}$ and let E be the closed annulus $(1 \le |z| \le 2)$. Suppose, to obtain a contradiction, that for every $f \in A(E)$ and every $\epsilon > 0$, there is an entire function

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(equivalently a polynomial g) such that $|f(z) - g(z)| < \epsilon$, for 100% of the points of E. We have seen that this implies that $|f(z) - g(z)| \le \epsilon$ for every point of $\overline{E^0} = E$. We have shown that every function in $A(E)$ is the uniform limit of polynomials. But this is well known to be false. This contradiction confirms our claim that the hypothesis that E be unbounded cannot be dropped.

A sequence g_n of almost everywhere finite measureable functions on a measureable set E is said to converge in measure to an almost everywhere finite measureable function f, if for each $\epsilon > 0$,

$$\mu\{p \in E : |g_n(p) - f(p)| > \epsilon\} \longrightarrow 0, \quad \text{as} \quad n \to \infty.$$

For the next corollary, we do not need to assume that the measure of E is infinite.

Corollary 1.4 *For every measureable subset $E \subset R$ and for every complex measureable function f on E, there exists a sequence $g_n \in H(R)$ such that $g_n \to f$ in measure.*

These results are new, even for the case where R is the complex plane $\mathbb{C}$. We could state similar results for approximation by meromorphic functions on Riemann surfaces, but that is a topic for another paper. This note is concerned only with approximation by holomorphic functions.

In the following section we prove Theorem 1.2, and in the last section we briefly consider some so-called Myrberg surfaces, which are the most important source of examples where approximation fails. See Example 3.3.

2 Proof of Theorem 1.2

Proof Fix a regular smooth exhaustion $\{K_n\}$ of R, with $K_0 = \emptyset$, and let $\{\delta_n\}$ be a sequence of positive numbers, which we may assume decreases strictly to zero. Since $\{K_n\}$ is a regular exhaustion, for each $n = 1, 2, \dots$, the open set $A_n = (K_{n+1}^0 \setminus K_{n-1})$ has only finitely many components $U_{n,j}$, $j = 1, 2, \dots, j_n$.

Claim 1. For each component $U_{n,j}$ of each A_n, we can assume that $E^c \cap U_{n,j}$ is nonempty. To see this, we form a closed subset E_1 of E as follows. For each $n = 1, 2, \dots$, and each $j = 1, 2, \dots, j_n$, we construct an open subset $V_{n,j}$ of $U_{n,j}$ so small that $\mu(V_{n,j}) < \delta_{n+1}/j_n 2^{n+2}$. Now, set

$$E_1 = E \setminus \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{j_n} V_{n,j}.$$

If we can show that E_1 satisfies the conclusion of the theorem, then it follows that E also satisfies the conclusion of the theorem. Thus, we assume that E^c meets each component $U_{n,j}$ of each A_n.

Claim 2. For each component $U_{n,j}$ of each A_n, we can assume that $E^c \cap U_{n,j}$ is connected. To see this, we note that the open set $E^c \cap U_j$ has at most countably many components, and so there is a countable collection $J_{n,j}$ of compact smooth Jordan arcs in $U_{n,j}$ that connect all the components of $E^c \cap U_{n,j}$. For each such arc α, we have $\mu(\alpha) = 0$, since α is smooth. We can surround α by a closed Jordan domain G_α

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contained in $U_{n,j}$ so small that, setting

$$H_{n,j} = \bigcup \{G_\alpha : \alpha \in J_{n,j}\}, \quad H_n = \bigcup_{j=1}^{j_n} H_{n,j},$$

we have

$$\mu(H_{n,j}) < \frac{\delta_{n+1}}{j_n 2^{n+2}}, \quad \mu(H_n) < \frac{\delta_{n+1}}{2^{n+2}}.$$

Now replace E by the closed subset $E_2 = E \setminus \bigcup_{n=1}^{\infty} H_n$. If E_2 satisfies the conclusion of the theorem, so does E, and for each component $U_{n,j}$ of each A_n, we have that $E_2^c \cap U_{n,j}$ is connected. We therefore assume that E itself has this property.

For each $n = 1, 2, \dots$, and each $j = 1, 2, \dots j_n$, choose a point $p_{n,j}$ in $E^c \cap U_{n,j}$. For each $j = 1, 2, \dots, j_n$ and each $k = 1, 2, \dots, k_{n+1}$, we say that $(n, j) < (n, k)$ if $p_{n,j}$ and $p_{n+1,k}$ are in the same component of $R \setminus K_{n-1}$. For each $(n, j) < (n, k)$, let $\beta_{j,k}$ be a smooth arc in $A_n \cup A_{n+1}$ from $p_{n,j}$ to $p_{n+1,k}$. Since there are finitely many such arcs, for fixed n, and each arc has μ-measure zero, we can surround each such arc by a Jordan domain $G_{n,j,k}$ in $A_n \cup A_{n+1}$, such that, for fixed n, setting

$$G_n = \bigcup \{G_{n,j,k} : (n, j) < (n, k)\},$$

we have

$$\mu(G_n) < \frac{\delta_{n+1}}{2^{n+2}}.$$

Now set $E_3 = E \setminus \bigcup_n G_n$. It is enough to show that E_3 satisfies the conclusion of the theorem.

We claim that $R^* \setminus E_3$ is locally connected at $*$. It is sufficient to show that for each n the set $(R^* \setminus E_3) \setminus K_{n-1}$ is connected. It is sufficient to show that for each $p \in (R \setminus E_3) \setminus K_{n-1}$, the component C_p of $(R \setminus E_3) \setminus K_{n-1}$ containing p is unbounded. The point p is contained in some $U_{n,j}$, and since $p \notin E_3$, it is in $E^c \cap U_{n,j}$ or in G_n. But G_n connects $E^c \cap U_{n,j}$ to some $E^c \cap U_{n+1,k}$. In any case, C_p will contain some $E^c \cap U_{n+1,k}$. Since $E^c \cap U_{n+1,k}$ is connected to some $E^c \cap U_{n+2,\ell}$ by G_{n+1}, it follows by induction that the component C_p is unbounded. Thus, $(R^* \setminus E_3) \setminus K_{n-1}$ is a connected neighbourhood of $*$. As n varies, these form a neighbourhood basis of $*$ in $R^* \setminus E_3$, so $R^* \setminus E_3$ is locally connected at $*$. Since $R^* \setminus E_3$ is clearly locally connected at each point of $R \setminus E_3$, it is locally connected. We have shown in passing that each point of $R \setminus E_3$ is connected to $*$, so $R^* \setminus E_3$ is not only locally connected but also connected.

There remains to perform one last modification to obtain the long island condition. For each n, let $Q_{n,j}$ be a regular exhaustion of K_n^0 and choose j so large that, setting $W_n = K_n^0 \setminus Q_{n,j}$, we have

$$\mu(K_n^0 \setminus Q_{n,j}) < \frac{\delta_{n+1}}{2^{n+2}}.$$

Finally, set $F = E_3 \setminus \bigcup_{n=1}^{\infty} W_n$. Then $R^* \setminus F$ continues to be connected and locally connected, and, moreover, since K_n is smoothly bounded,

$$\mu((E_3 \setminus F) \setminus K_n) = \mu((E_3 \setminus F) \setminus K_n^0) < \delta_n, \quad n = 1, 2, \dots.$$

By Theorem 1.1, F is a set of tangential approximation. ∎

Proof of Corollary 1.3 Fix $f \in A(E)$ and $\epsilon(p)$ a positive continuous function on E, which we may assume goes to zero as $p \to *$. Let K_m be a regular exhaustion of R such that $K_1 \cap E \neq \varnothing$, and choose a sequence $\{\delta_m\}$ of positive numbers.

By Theorem 1.2, there exists a closed subset F of E such that F is a set of tangential approximation and

$$\mu((E \setminus F) \setminus K_m) < \delta_m,$$

so in particular, $\mu(E \setminus F) < \infty$.

Let $g \in H(R)$ such that $|f(p) - g(p)| < \epsilon(p)$ for $p \in F$ and denote by $P(p)$ the proposition that $|f(p) - g(p)| < \epsilon(p)$. Then we have

$$\begin{aligned}\lim_{n\to\infty} \frac{\mu(E \cap K_n : P(p))}{\mu(E \cap K_n)} &= \lim_{n\to\infty} \frac{\mu(E \cap K_n) - \mu\{p \in E \cap K_n : \sim P(p)\}}{\mu(E \cap K_n)} \\ &\geq 1 - \lim_{n\to\infty} \frac{\mu(E \setminus F)}{\mu(E \cap K_n)} = 1.\end{aligned}$$

■

Proof of Corollary 1.4 Let $G_k \nearrow R$ be a a regular exhaustion by smoothly bounded open sets and put $A_k = E \cap (G_k \setminus \overline{G}_{k-1})$. For fixed $(k, n) \in \mathbb{N} \times \mathbb{N}$, by Luzin's Theorem, there exists a compact set $K_{k,n} \subset A_k$, with $\mu(A_k \setminus K_{k,n}) < 1/(2^k n)$, such that f restricted to $K_{k,n}$ is continuous.

We claim that we can assume $K^0_{k,n} = \varnothing$. First of all, there is a finite union

$$L_{k,n} = \bigcup \{ L_{k,n,j} : j = 1, \ldots, J(k,n) \}$$

of disjoint closed squares $L_{k,n,j} \subset K^0_{k,n}$, such that $\mu(L_{k,n})$ approximates $\mu(K^0_{k,n})$ as well as we please. Here, when we say that $L_{k,n,j}$ is a closed square on the Riemann surface R, we mean that ρ maps $L_{k,n,j}$ homeomorphically onto a square $I \times I \subset \mathbb{R}^2 = \mathbb{C}$, where I is a closed interval in $\mathbb{R}$. We can construct a Cantor-type set $B \subset I$, whose 1-dimensional measure approximates the length of I as well as we please. Then $B \times B$, is a "Cantor-square" whose measure approximates that of $\rho(L_{k,n,j})$ as well as we please. Thus, $Q_{k,n,j} = \rho^{-1}(B \times B)$ is a compact nowhere dense subset of $L_{k,n,j}$ such that $\mu(Q_{k,n,j})$ approximates $\mu(L_{k,n,j})$ as well as we please. Consequently, the union

$$Q_{k,n} = \bigcup \{ Q_{k,n,j} : j = 1, \ldots, J(k,n) \}$$

is a compact nowhere dense subset of $K^0_{k,n}$ whose measure approximates the measure of $K^0_{k,n}$ as well as we please. Now set $M_{k,n} = Q_{k,n} \cup \partial K_{k,n}$. Since $\mu(Q_{k,n})$ is a good approximation of $\mu(K^0_{k,n})$, $\mu(M_{k,n})$ is an equally good approximation of $\mu(K_{k,n})$. Since the compact set $M_{k,n}$ has empty interior, this proves our claim. Thus, we assume that $K_{k,n}$ has empty interior. Let $E_n = \bigcup_k K_{k,n}$ and let f_n be the restriction of f to E_n. Since $f_n \in C(E_n)$, and $E^0_n = \varnothing$, we have $f_n \in A(E_n)$. By Theorem 1.2, there is a closed subset $F_n \subset E_n$ with $\mu(E_n \setminus F_n) < 1/n$ and a function $g_n \in H(R)$ such that $|f_n - g_n| < 1/n$ on F_n. Since $f = f_n$ on E_n and $\mu(E \setminus E_n) < 1/n$, this completes the proof. ■

3 A Myrberg Surface where Approximation Fails

We use the expression *Myrberg surface* loosely to refer to a Riemann surface obtained by taking two copies of the complex plane having identical slits and joining these two slit planes along the slits in the usual way.

Definition 3.1 A sequence of points $\{z_n\}$ inside the unit disc is said to satisfy the *Blaschke condition* when $\Sigma_n(1-|z_n|) < \infty$. The sequence $\{z_n\}$ is called a *Blaschke sequence*.

For $\lambda \geq 0$, we denote $S_\lambda = \{x+iy : |y| \leq \lambda\}$. The function $\theta(z) = w$, where

$$\theta(z) = \frac{e^z - 1}{e^z + 1}, \quad \theta^{-1}(w) = \ln\Big(\frac{1+w}{1-w}\Big), \quad \ln 1 = 0,$$

maps the strip $S_{\pi/2}$ conformally onto the unit disc. In general, if the sequence $\{x_n\}$ is increasing to ∞, then

$$\sum_{n=1}^{\infty}(1-|\theta(x_n)|) < +\infty \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty}\frac{1}{e^{x_n}} < +\infty.$$

In particular, if the sequence X grows linearly, more precisely, if $x_n \geq a + nx_0$, for some $x_0 \geq 0$ and $a > 0$, then $u_n = \theta(x_n)$, $n = 1, 2, \ldots$, is a Blaschke sequence. Indeed,

$$\sum_{n=1}^{\infty}(1-|u_n|) = \sum_{n=1}^{\infty}\frac{2}{e^{x_n}+1} \leq 2\sum_{n=1}^{\infty}\frac{1}{e^{na}} < \infty.$$

It follows that if $\liminf(x_{n+1}-x_n) > 0$, then $\{\theta(x_n)\}$ is a Blaschke sequence. The converse does not hold. Indeed, let $u_n = 1 - 1/n^2$; then $\sum(1-|u_n|) = \sum 1/n^2 < +\infty$, which is the Blaschke condition, while

$$x_{n+1} - x_n = \theta^{-1}\Big(1 - \frac{1}{(n+1)^2}\Big) - \theta^{-1}\Big(1 - \frac{1}{n^2}\Big) = \ln\Big(\frac{2n^2+4n+1}{2n^2-1}\Big) \longrightarrow 0.$$

To recapitulate, if $\liminf(x_{n+1}-x_n) > 0$, then $\{\theta(x_n)\}$ is a Blaschke sequence, but we can have $\lim(x_{n+1}-x_n) = 0$ for a Blaschke sequence $\{\theta(x_n)\}$. Equivalently, if $\{\theta(x_n)\}$ is not a Blaschke sequence (and hence $S_{\pi/2}$ is not a set of approximation), then $\lim(x_{n+1}-x_n) = 0$, but it is possible to also have $\lim(x_{n+1}-x_n) = 0$, with $\{\theta(x_n)\}$ a Blaschke sequence.

Consider $x_n = \delta \ln n$. Then

$$\sum \frac{1}{e^{x_n}} = \sum \frac{1}{n^\delta},$$

so $\{u_n\}$ is a Blaschke sequence if and only if $\delta > 1$. For $\delta \leq 1$ and $x_n = \delta \ln n$, $E_{\pi/2}$ is not a set of approximation. Notice that for every $\delta > 0$, the sequence $\{\delta \ln n\}$ is "tight" in both senses $x_{n+1} - x_n \to 0$ and $x_{n+1}/x_n \to 1$.

The following lemma due to Scheinberg [9] gives us a Blaschke condition for a strip.

Lemma 3.2 *Let x_n be a sequence of distinct real numbers such that $|x_n| \to \infty$ and $0 \leq \lambda < \infty$ and let θ be a conformal map that sends $S_\lambda = \{x+iy, |y| \leq \lambda\}$ to the unit*

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disc, i.e., $u_n = \theta^{-1}(x_n)$. *Then*

$$\sum_{n=1}^{\infty}(1-|u_n|) = \infty$$

if and only if

$$\sum_{n=1}^{\infty} a^{-|x_n|} < \infty, \quad \textit{where} \quad a = \exp\Big(\frac{\pi}{2\lambda}\Big).$$

This lemma allows us to construct the following example, where approximation fails.

Example 3.3 There exists a Riemann surface R of infinite genus, formed by joining two copies of the complex plane with slits on the real axis, such that all closed "strips over" the real axis are not sets of holomorphic approximation, while the set "over" the real axis itself is a set of tangential approximation.

Proof For $\lambda \geq 0$, we denote $S_\lambda = \{x + iy : |y| \leq \lambda\}$, and for $n = 1, 2, \dots,$ let θ_n be the conformal map of $S_{1/n}$ onto the unit disc, which maps $-\infty, 0, +\infty$ to $-1, 0, +1$, respectively Let $\{x_j\}$ be an increasing sequence of positive numbers tending to infinity such that for each n, the sequence $\{\theta_n(x_j)\}$ is not a Blaschke sequence. To obtain such a sequence, for each n let $\{x_{n,j}\}$ be a sequence of distinct real numbers tending to $+\infty$ with $x_{n,j} \geq n$, such that $\{\theta_n(x_{n,j})\}$ is not a Blaschke sequence. We can assume that these sequences are disjoint from each other. Now we let $\{x_j\}$ be any sequence formed by combining all of the sequences $\{x_{n,j}\}$ into a single sequence.

We take two copies of the complex plane, remove the intervals (x_{2j-1}, x_{2j}) and join the slit planes together in the usual way to form a Riemann surface $R = R_X$, where X signifies the dependence on the sequence $X = \{x_j\}_j$. Let π be the projection map from R to $\mathbb{C}$ and put $E = E_\lambda := \pi^{-1}(S_\lambda)$. Let us show that $R^* \smallsetminus E$ is connected. $R \smallsetminus E$ has four connected components; each one is an open complex half-plane. None of these components is contained in any compact subset of R. Since every open neighbourhood of $*$ in R^* contains the complement of some compact subset of R, every neighbourhood of $*$ in R^* intersects each component of $R \smallsetminus E$. Suppose $R^* \smallsetminus E$ is disconnected. Then it is the union of non-empty disjoint sets A and B, open in $R^* \smallsetminus E$, whose union is $R^* \smallsetminus E$. We may suppose that A contains $*$. Since A is an open neighbourhood of $*$ in $R^* \smallsetminus E$, it is of the form $A = U \cap (R^* \smallsetminus E)$, where U is an open neighbourhood of $*$ in R^*. We can assume that $U = R^* \smallsetminus K$, where K is a compact subset of R. Thus, $A = R^* \smallsetminus (K \cup E)$. Suppose some component H of $R \smallsetminus E$ does not intersect A. Then $H \subset (K \cup E)$, which is precluded. Therefore, A intersects each component of $R \smallsetminus E$. But $B \cup (R \smallsetminus E)$ is non-empty so some component H of $R \smallsetminus E$ also intersects B. This shows that the half-plane H is the union of two non-empty disjoint open sets: $A \cap H$, and $B \cap H$. So H is disconnected, a contradiction. Therefore, $R^* \smallsetminus E$ is connected.

Now let us show that $R^* \smallsetminus E$ is locally connected. Obviously, it is locally connected at points of $R \smallsetminus E$. To show it is locally connected at infinity, let H_1, H_2, H_3, H_4 be the half-planes composing $R \smallsetminus E$. For $n = 1, 2, \dots,$ set

$$U_n = \{z : |z| > n\}, \quad H_{j,n} = \pi^{-1}(U_n) \cap H_j, \quad \text{and} \quad V_n = \bigcup_{j=1}^{4} H_{j,n} \cup \{*\}.$$

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Then $\{V_n\}_n$ is a neighbourhood basis of $*$, all of whose members are connected, so $R^* \setminus E$ is locally connected at infinity. Thus, $R^* \setminus E$ is locally connected.

The set E_0 has empty interior, so by Theorem 1.1 it is a set of holomorphic tangential approximation.

Fix $\lambda > 0$, let α be a point that is not in S_λ, and consider $\pi^{-1}(\alpha) = \{p_1, p_2\}$. Let f be a meromorphic function on R which has a pole at p_1 and only at p_1.

Suppose, to obtain a contradiction, that there exists a holomorphic function F on R such that $|F - f| \le \varepsilon$ on E_λ. The function $g := F - f$ is meromorphic on R, has a pole at p_1, is holomorphic elsewhere, and is bounded on E_λ. Denote by X the set of values of the sequence $\{x_j\}$. Then $\widetilde{X} = \pi^{-1}(X)$ is the set of branch points of R. Let $\rho: R \setminus \widetilde{X} \to R \setminus \widetilde{X}$ be the involution, mapping each point of $R \setminus \widetilde{X}$ to the corresponding point on the other sheet having the same projection on $\mathbb{C}$. Set $g_1(p) = (g(p) - g(\rho(p)))^2$, for $p \in R \setminus \widetilde{X}$. Then $G = g_1 \circ \pi^{-1}$ is a well-defined holomorphic function on $\mathbb{C} \setminus (X \cup \{\alpha\})$, which is bounded on $S_\lambda \setminus X$. Riemann's theorem on removable singularities implies that G extends holomorphically on $\mathbb{C} \setminus \{\alpha\}$ and vanishes at each point of the sequence $\{x_j\}$. Now by using the Blaschke condition, G is identically zero on S_λ, *i.e.*, $g(p) = g(\rho(p))$, for $p \in E_\lambda \setminus \widetilde{X}$. By the uniqueness of meromorphic continuation, we obtain that $g(p) = g(\rho(p))$, for $p \in R \setminus \widetilde{X}$. In particular, g, and hence, f has a pole at p_1, which is a contradiction. Thus, E_λ is not a set of approximation. ■

The referee has remarked that the proof of the above example can be simplified, but that the proof we furnish yields interesting quantitative information (Lemma 3.3)
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