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ON A NON-CONVEX HYPERBOLIC DIFFERENTIAL INCLUSION

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We prove the existence of a solution $u(.,.;\alpha,\beta)$ of the Darboux problem $u_{xy}\in F(x,y,u)$, $u(x,0)=\alpha(x)$, $u(0,y)=\beta(y)$, which is continuous with respect to (α,β). We assume that F is Lipschitzean with respect to u but not necessarily convex valued.

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1. Introduction and main result

Let $I=[0,1]$, $Q=I\times I$ and denote by $\mathscr{L}$ the σ-algebra of the Lebesgue measurable subsets of Q. Denote by 2^{R^n} the family of all closed nonempty subsets of R^n and by $\mathscr{B}(R^n)$ the family of all Borel subsets of R^n. For $x\in R^n$ and A, $B\in 2^{R^n}$ we denote by $d(x,A)$ the usual point-to-set distance from x to A and by $h(A,B)$ the Hausdorff pseudo-distance from A to B.

By $C(Q,R^n)$ (resp. $L^1(Q,R^n)$) we denote the Banach space of all continuous (resp. Bochner integrable) functions $u:Q\to R^n$ with the norm $\|u\|_\infty=\sup\{\|u(x,y)\|:(x,y)\in Q\}$ (resp. $\|u\|_1=\int_0^1\int_0^1\|u(x,y)\|\,dx\,dy$), where $\|.\|$ is the norm in R^n.

Recall that a subset K of $L^1(Q,R^n)$ is said to be *decomposable* ([**9**]) if for every u, $v\in K$ and $A\in\mathscr{L}$ we have $u\chi_A+v\chi_{Q\setminus A}\in K$, where χ_A stands for the characteristic function of A. We denote by $\mathscr{D}$ the family of all decomposable closed nonempty subsets of $L^1(Q,R^n)$.

Let $F:Q\times R^n\to 2^{R^n}$ be a multivalued map. Recall that F is called $\mathscr{L}\otimes\mathscr{B}(R^n)$-measurable if for any closed subset C of R^n we have that $\{(x,y,z)\in Q\times R^n: F(x,y,z)\cap C\neq\emptyset\}\in\mathscr{L}\otimes\mathscr{B}(R^n)$.

We associate to F: $Q\times R^n\to 2^{R^n}$ the Darboux problem

$$(D_{\alpha\beta})\qquad\qquad u_{xy}\in F(x,y,u),\quad u(x,0)=\alpha(x),\quad u(0,y)=\beta(y),$$

where α, β are two continuous functions from I into R^n with $\alpha(0)=\beta(0)$.

Definition. $u(.,.;\alpha,\beta)\in C(Q,R^n)$ is said to be a *solution* of the Darboux problem $(D_{\alpha\beta})$ if there exists $v(.,.;\alpha,\beta)\in L^1(Q,R^n)$ such that

(i) $v(x,y;\alpha,\beta)\in F(x,y,u(x,y;\alpha,\beta))$ a.e. in Q,

(ii) $u(x,y;\alpha,\beta)=\alpha(x)+\beta(y)-\alpha(0)+\int_0^x\int_0^y v(\xi,\eta;\alpha,\beta)\,d\xi\,d\eta$, for every $(x,y)\in Q$.

Note that the function $v(.,.;\alpha,\beta)$ which corresponds to $u(.,.;\alpha,\beta)$ in the above definition is unique (a.e.). Consider the Banach space

$$\mathbf{S}=\{(\alpha,\beta)\in C(I,R^n)\times C(I,R^n):\alpha(0)=\beta(0)\}$$

endowed with the norm

$$\|(\alpha,\beta)\|=\|\alpha\|_\infty+\|\beta\|_\infty,$$

and, for (α,β) in $\mathbf{S}$, we denote by $\mathbf{T}(\alpha,\beta)$ the set of all solutions of the problem $(D_{\alpha\beta})$. The aim of this note is to prove the following:

Theorem. *Let $F:Q\times R^n\to 2^{R^n}$ satisfy the following assumptions:*

(H_1) *F is $\mathcal{L}\otimes\mathcal{B}(R^n)$-measurable,*

(H_2) *there exists $L>0$ such that $h(F(x,y,u),\ F(x,y,v))\leqq L\|u-v\|$ for all u, $v\in R^n$, a.e. in Q,*

(H_3) *there exists a function $\delta\in L^1(Q,R)$ such that $d(0,F(x,y,0))\leqq\delta(x,y)$ a.e. in Q.*

Then there exists $u:Q\times\mathbf{S}\to R^n$ such that

(i) *$u(.,.;\alpha,\beta)\in\mathbf{T}(\alpha,\beta)$ for every $(\alpha,\beta)\in\mathbf{S}$*

(ii) *$(\alpha,\beta)\to u(.,.;\alpha,\beta)$ is continuous from* $\mathbf{S}$ *to $C(Q,R^n)$.*

In other words we prove the existence of a global solution $u(.,.;\alpha,\beta)$ of the problem $(D_{\alpha\beta})$ depending continuously on (α,β) in the space $\mathbf{S}$.

This result is a natural extension of the well posedness property (i.e., existence of a unique solution depending continuously on the initial data) of the Darboux problems defined by Lipschitzean single-valued maps (see [3]). We obtain the solution by a completeness argument without assumptions on the convexity or boundedness of the values of F.

Filippov has obtained in [7] the existence of solutions to an ordinary differential inclusion $x'\in F(t,x)$ defined by a multifunction F Lipschitzian with respect to x, without assumptions on the convexity or boundedness of the values $F(t,x)$ by using a successive approximation process.

Following an idea in [4] we extend this process to Darboux problems and we do it continuously with respect to (α,β) in the space $\mathbf{S}$ by using a result on the existence of a continuous selection from multifunctions with decomposable values, proved in [8] and extended in [2].

The construction in the proof of our theorem works for the case when F is Lipschitzian in u, but the assumption (H_2) is not only a technical one. We shall give an example showing that if (H_2) is relaxed, allowing F to be merely continuous then the conclusion of the theorem is in general no longer true.

However the Lipschitz property of F is not necessary for the existence of solutions. If F is upper semicontinuous with compact convex values then the existence of local and global solutions has been obtained in [11] and [12], by using the Kakutani-Ky Fan fixed point theorem. The convexity assumption is essential in this case. To avoid the convexity assumption we have to increase the regularity of F. If F is a Carathéodory function which is compact not necessarily convex valued then there exists a solution of the Darboux problem and this fact has been proved in [13] by using a continuous selection argument and the Schauder fixed point theorem. Qualitative properties and the structure of the set of solutions of Darboux problems has been studied in [5] and [6].

Remark finally that another extension of the well posedness property of a Darboux problem defined by a multifunction Lipschitzian in u, lower semicontinuous with respect to a parameter, expressed in terms of lower semicontinuous dependence of the set of all solutions of the problem on the initial data and parameter is given in [10].

2. Proof of the main result

In the following two lemmas S is a separable metric space. Let X be a Banach space and $G: S \to 2^X$ be a multifunction. Recall that G is said to be *lower semicontinuous* (l.s.c.) if for every closed subset C of X the set $\{s \in S: G(s) \subset C\}$ is closed in S.

Lemma 1 ([4, Proposition 2.1]). *Assume $F_*: Q \times S \to 2^{R^n}$ to be $\mathscr{L} \otimes \mathscr{B}(S)$-measurable, l.s.c. with respect to $s \in S$. Then the map $s \to G_*(s)$ given by*

$$G_*(s) = \{v \in L^1(Q, R^n): v(x,y) \in F_*(x,y,s) \text{ a.e. in } Q\}, \quad s \in S,$$

is l.s.c. with decomposable closed nonempty values if and only if there exists a continuous function $\sigma: S \to L^1(Q, R)$ such that $d(0, F_*(x,y,s)) \leqq \sigma(s)(x,y)$ a.e. in Q.

Lemma 2 ([4, Proposition 2.2]). *Let $G: S \to \mathscr{D}$ be a l.s.c. multifunction and let $\phi: S \to L^1(Q, R^n)$ and $\psi: S \to L^1(Q, R)$ be continuous maps. If for every $s \in S$ the set*

$$H(\xi) = cl\{v \in G(s): \|v(x,y) - \phi(s)(x,y)\| < \psi(s)(x,y) \text{ a.e. in } Q\} \tag{2.1}$$

is nonempty then the map $H: S \to \mathscr{D}$ defined by (2.1) *admits a continuous selection.*

We note that the second lemma is a direct consequence of Proposition 4 and Theorem 3 in [2] (see also [8]).

Proof of the theorem. Fix $\varepsilon > 0$ and set $\varepsilon_n = \varepsilon/2^{n+1}$, $n \in N$. For $(\alpha, \beta) \in \mathbf{S}$ define $u_0(.,.;\alpha,\beta): Q \to R^n$ by $u_0(x,y;\alpha,\beta) = \alpha(x) + \beta(y) - \alpha(0)$ and observe that for all $(x,y) \in Q$ we have

$$\|u_0(x,y;\alpha_1,\beta_2) - u_0(x,y;\alpha_2,\beta_2)\| \leqq \|\alpha_1(x) - \alpha_2(x)\| + \|\beta_1(y) - \beta_2(y)\| + \|\alpha_1(0) - \alpha_2(0)\|$$

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$$\leqq 2\|(\alpha_1,\beta_1)-(\alpha_1,\beta_2)\|.$$

This implies that $(\alpha,\beta)\to u_0(.,.;\alpha,\beta)$ is continuous from $\mathbf{S}$ to $C(Q,R^n)$. Setting $\sigma(\alpha,\beta)(x,y)=\delta(x,y)+L\|u_0(x,y;\alpha,\beta)\|$ we obtain that σ is a continuous map from $\mathbf{S}$ to $L^1(Q,R)$ and

$$d(0,F(x,y,u_0(x,y;\alpha,\beta)))\leqq\sigma(\alpha,\beta)(x,y) \text{ a.e. in } Q. \tag{2.2}$$

For $(\alpha,\beta)\in\mathbf{S}$, define

$$G_0(\alpha,\beta)=\{v\in L^1(Q,X): v(x,y)\in F(x,y,u_0(x,y;\alpha,\beta)) \text{ a.e. in } Q\},$$

and

$$H_0(\alpha,\beta)=cl\{v\in G_0(\alpha,\beta): \|v(x,y)\|<\sigma(\alpha,\beta)(x,y)+\varepsilon_0 \text{ a.e. in } Q\}.$$

Then, by (2.2) and Lemma 1, it follows that G_0 is l.s.c. from $\mathbf{S}$ into $\mathcal{D}$ and, by (2.2), $H_0(\alpha,\beta)\neq\emptyset$ for each $(\alpha,\beta)\in\mathbf{S}$. Therefore by Lemma 2, there exists $h_0:\mathbf{S}\to L^1(Q,R^n)$, which is a continuous selection of H_0. Set $v_0(x,y;\alpha,\beta)=h_0(\alpha,\beta)(x,y)$ and observe that $v_0(x,y;\alpha,\beta)\in F(x,y,\ u_0(x,y;\alpha,\beta))$ and $\|v_0(x,y)\|\leqq\sigma(\alpha,\beta)(x,y)+\varepsilon_0$, for a.e. $(x,y)\in Q$. Define

$$u_1(x,y;\alpha,\beta)=u_0(x,y;\alpha,\beta)+\int_0^x\int_0^y v_0(\xi,\eta;\alpha,\beta)\,d\xi\,d\eta,$$

and, for $n\geqq 1$, set

$$\sigma_n(\alpha,\beta)(x,y)=L^{n-1}\left[\int_0^x\int_0^y\frac{(x-\xi)^{n-1}}{(n-1)!}\frac{(y-\eta)^{n-1}}{(n-1)!}\sigma(\alpha,\beta)(\xi,\eta)\,d\xi\,d\eta+\left(\sum_{i=0}^{n}\varepsilon_i\right)\frac{(x+y)^n}{n!}\right]. \tag{2.3}$$

Then, for every $(x,y)\in Q\backslash\{0,0\}$, we have

$$\|u_1(x,y;\alpha,\beta)-u_0(x,y;\alpha,\beta)\|\leqq\int_0^x\int_0^y\|v_0(\xi,\eta;\alpha,\beta)\|\,d\xi\,d\eta\leqq\int_0^x\int_0^y\sigma(\alpha,\beta)(\xi,\eta)\,d\xi\,d\eta+\varepsilon_0(x+y)$$

$$<\sigma_1(\alpha,\beta)(x,y),$$

and so

$$d(v_0(x,y;\alpha,\beta),F(x,y,u_1(x,y;\alpha,\beta))\leqq L\|u_1(x,y;\alpha,\beta)-u_0(x,y;\alpha,\beta)\|<L\sigma_1(\alpha,\beta)(x,y).$$

We claim that there exist two sequences $\{v_n(x,y;\alpha,\beta)\}_{n\in N}$ and $\{u_n(x,y;\alpha,\beta)\}_{n\in N}$ such that for each $n\geqq 1$ we have:

(a) $(\alpha,\beta)\to v_n(.,.;\alpha,\beta)$ is continuous from $\mathbf{S}$ to $L^1(Q,R^n)$.

(b) $v_n(x,y;\alpha,\beta)\in F(x,y,u_n(x,y;\alpha,\beta))$ for any $(\alpha,\beta)\in \mathrm{S}$ and a.e. $(x,y)\in Q$.

(c) $\|v_n(x,y;\alpha,\beta)-v_{n-1}(x,y;\alpha,\beta)\|\leqq L\sigma_n(\alpha,\beta)(x,y)$ a.e. in Q.

(d) $u_n(x,y;\alpha,\beta)=u_0(x,y;\alpha,\beta)+\int_0^x\int_0^y v_{n-1}(\xi,\eta;\alpha,\beta)\,d\xi\,d\eta$.

Suppose we have constructed $v_1,\dots,v_n$ and $u_1,\dots,u_n$ satisfying (a)–(d). Then define

$$u_{n+1}(x,y;\alpha,\beta)=u_0(x,y;\alpha,\beta)+\int_0^x\int_0^y v_n(\xi,\eta;\alpha,\beta)\,d\xi\,d\eta.$$

Let $(x,y)\in Q\backslash\{(0,0)\}$. Using (c) we have

$$\|u_{n+1}(x,y;\alpha,\beta)-u_n(x,y;\alpha,\beta)\|\leqq\int_0^x\int_0^y\|v_n(\xi,\eta;\alpha,\beta)-v_{n-1}(\xi,\eta;\alpha,\beta)\|\,d\xi\,d\eta.$$

$$\leqq L\int_0^x\int_0^y\sigma_n(\alpha,\beta)(\xi,\eta)\,d\xi\,d\eta=L^n\int_0^x\int_0^y\sigma(\alpha,\beta)(\xi,\eta)\left(\int_\xi^x\frac{(x-u)^{n-1}}{(n-1)!}du\int_\eta^y\frac{(y-v)^{n-1}}{(n-1)!}dv\right)d\xi\,d\eta$$

$$+L^n\left(\sum_{i=0}^n\varepsilon_i\right)\int_0^x\int_0^y\frac{(\xi-\eta)^n}{n!}d\xi\,d\eta=L^n\int_0^x\int_0^y\frac{(x-\xi)^n}{n!}\frac{(y-\eta)^n}{n!}\sigma(\alpha,\beta)(\xi,\eta)\,d\xi\,d\eta$$

$$+\frac{L^n}{n!}\left(\sum_{i=0}^n\varepsilon_i\right)\frac{(x+y)^{n+2}-x^{n+2}-y^{n+2}}{(n+1)(n+2)}\leqq L^n\left[\int_0^x\int_0^y\frac{(x-\xi)^n}{n!}\frac{(y-\eta)^n}{n!}\sigma(\alpha,\beta)(\xi,\eta)\,d\xi\,d\eta\right.$$

$$\left.+\left(\sum_{i=0}^n\varepsilon_i\right)\frac{(x+y)^{n+1}}{(n+1)!}\right]<\sigma_{n+1}(\alpha,\beta)(x,y), \tag{2.4}$$

Then, by virtue of (2.4) and of the assumption (H_2), it follows that

$$d(v_n(x,y;\alpha,\beta),F(x,y,u_{n+1}(x,y;\alpha,\beta)))\leqq L\|u_{n+1}(x,y;\alpha,\beta)-u_n(x,y;\alpha,\beta)\|$$

$$<L\sigma_{n+1}(\alpha,\beta)(x,y), \tag{2.5}$$

Since σ is continuous from S to $L^1(Q,R)$, by (2.3) it follows that also σ_n is continuous from S to $L^1(Q,R)$. Therefore, by (2.5) and Lemma 1, we have that the multivalued map G_{n+1} defined by

$$G_{n+1}(\alpha,\beta)=\{v\in L^1(Q,X): v(x,y)\in F(x,y,u_{n+1}(x,y;\alpha,\beta))\text{ a.e. in }Q\}$$

is l.s.c. from S to $\mathscr{D}$. Moreover, by (2.5), it follows that

$$H_{n+1}(\alpha,\beta)=cl\{v\in G_{n+1}(\alpha,\beta):\|v(x,y)-v_n(x,y;\alpha,\beta)\|<L\sigma_{n+1}(\alpha,\beta)(x,y)\ \text{a.e. in } Q\}$$

is nonempty. Then, by Lemma 2, there exists $h_{n+1}:\mathbf{S}\to L^1(Q,R^n)$ a continuous selection of H_{n+1}. Set $v_{n+1}(x,y;\alpha,\beta)=h_{n+1}(\alpha,\beta)(x,y)$ and observe that v_{n+1} satisfies the properties (a)–(d). By (c) and the computations in (2.4) it follows that

$$\|v_n(.,.;\alpha,\beta)-v_{n-1}(.,.;\alpha,\beta)\|_1\leqq\frac{L^n}{n!}\|\sigma(\alpha,\beta)\|_1+\varepsilon\frac{[2L]^n}{n!}. \tag{2.6}$$

and

$$\|u_{n+1}(.,.;\alpha,\beta)-u_n(.,.;\alpha,\beta)\|_\infty\leqq\|v_{n+1}(.,.;\alpha,\beta)-v_{n-1}(.,.;\alpha,\beta)\|_1$$
$$\leqq\frac{L^n}{n!}\|\sigma(\alpha,\beta)\|_1+\varepsilon\frac{[2L]^n}{n!}. \tag{2.7}$$

Therefore $\{u_n(.,.;\alpha,\beta)\}_{n\in N}$ and $\{v_n(.,.;\alpha,\beta\}_{n\in N}$ are Cauchy sequences in $C(Q,R^n)$ and $L^1(Q,R^n)$, respectively. Moreover since $(\alpha,\beta)\to\|\sigma(\alpha,\beta)\|_1$ is continuous, it is locally bounded; hence the Cauchy condition is satisfied locally uniformly with respect to (α,β). Let $u(.,.;\alpha,\beta)\in C(Q,R^n)$ and $v(.,.;\alpha,\beta)\in L^1(Q,R^n)$ be the limit of $\{u_n(x,y;\alpha,\beta)\}$ and $\{v_n(.,.;\alpha,\beta)\}$ respectively. Then $(\alpha,\beta)\to u(.,.;\alpha,\beta)$ is continuous from $\mathbf{S}$ to $C(Q,X)$ and $(\alpha,\beta)\to v(.,.;\alpha,\beta)$ is continuous from $\mathbf{S}$ to $L^1(Q,R^n)$. Letting $n\to\infty$ in (d) we obtain that

$$u(x,y;\alpha,\beta)=u_0(x,y;\alpha,\beta)+\int_0^x\int_0^y v(\xi,\eta;\alpha,\beta)\,d\xi\,d\eta\quad\text{for any } (x,y)\in Q. \tag{2.8}$$

Furthermore, since

$$d(v_n(x,y;\alpha,\beta),F(x,y,u(x,y;\alpha,\beta)))\leqq L\|u_{n+1}(x,y;\alpha,\beta)-u(x,y;\alpha,\beta)\|$$

and F has closed values, letting $n\to\infty$ we have

$$v(x,y;\alpha,\beta)\in F(x,y,u(x,y;\alpha,\beta))\quad\text{a.e. in } Q. \tag{2.9}$$

By (2.8) and (2.9) it follows that $u(.,.;\alpha,\beta)$ is a solution of $(D_{\alpha\beta})$, which completes the proof.

Remark 1. Theorem 1 remains true (with the same proof) if R^n is replaced by a separable Banach space X and F is a multifunction from $Q\times X$ to the closed nonempty subsets of X satisfying the assumptions $(H_1)-(H_3)$.

Remark 2. If the assumption (H_2) is relaxed, allowing F to be merely continuous then the conclusion of the theorem is in general no longer true. To see this consider the Darboux problem

$$(D_{\alpha,\beta}) \qquad u_{xy}=\sqrt[3]{u}, \quad u(x,0)=\alpha(x), \quad u(0,y)=\beta(y), \quad (x,y)\in Q$$

Remark that $f(u)=\sqrt[3]{u}$ is continuous but not Lipschitzean in a neighbourhood of 0 and, for $\alpha_0(x)=0=\beta_0(y)$, the problem (D_{α_0,β_0}) admits as solutions:

$$u_0^+(x,y)=\left(\frac{2}{3}\right)^3 x^{3/2}y^{3/2} \quad \text{and} \quad u_0^-(x,y)=-\left(\frac{2}{3}\right)^3 x^{3/2}y^{3/2}.$$

Let

$$\alpha_n^+(x)=\left(\frac{2}{3\sqrt{n}}\right)^3 x^{3/2}, \quad \alpha_n^-(x)=-\left(\frac{2}{3\sqrt{n}}\right)^3 x^{3/2}, \quad \beta_n^+(y)=0=\beta_n^-(y).$$

Then

$$(\alpha_n^+,\beta_n^+),(\alpha_n^-,\beta_n^-)\in \mathbf{S} \quad \text{and} \quad \|(\alpha_n^+,\beta_n^+)\|=\|(\alpha_n^-,\beta_n^-)\|=\left(\frac{2}{3\sqrt{n}}\right)^3,$$

therefore (α_n^+,β_n^+) and (α_n^-,β_n^-) converge to $(\alpha_0,\beta_0)=(0,0)$ in the space $\mathbf{S}$.

On the other hand the unique solution of the Darboux problem $(D_{\alpha_n^+,\beta_n^+})$ (resp. of $(D_{\alpha_n^-,\beta_n^-})$) is given by

$$u_n^+(x,y)=\left(\frac{2}{3}\right)^3 x^{3/2}\left(\frac{1}{n}+y\right)^{3/2}$$

$$\left(\text{resp. } u_n^-(x,y)=-\left(\frac{2}{3}\right)^3 x^{3/2}\left(\frac{1}{n}+y\right)^{3/2}\right).$$

which for $n\to\infty$ converges to u_0^+ (resp. u_0^-).

Suppose that there exists $r:\mathbf{S}\to C(Q,X)$ a continuous selection of the solution map $(\alpha,\beta)\to\mathbf{T}(\alpha,\beta)$. Then, for $n\to\infty$, we have that $r((\alpha_n^+,\beta_n^+))=u_n^+$ converges to u_0^+ and $r((\alpha_n^-,\beta_n^-))=u_n^-$ converges to u_0^-. This is a contradiction to the continuity of r.

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