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SHIFTS ON TYPE II_1 FACTORS

GEOFFREY L. PRICE

1. Introduction. A shift on a unital C^*-algebra $\mathscr{A}$ is a *-endomorphism α of $\mathscr{A}$ which fixes the identity and has the property that the intersection of the ranges of α^n for $n = 1, 2, 3, \ldots$ consists only of multiples of the identity. In **[4]** R. T. Powers introduced the notion of a shift on a C^*-algebra and considered both discrete and continuous one-parameter semi-groups of shifts. In this paper we focus on discrete shifts. We use a construction of Powers to obtain shifts on certain unital AF C^*-algebras. These are defined by constructing a set $\{u_i : i = 1, 2, \ldots\}$ of self-adjoint unitary operators which pairwise either commute or anticommute. Setting $\alpha(u_i) = u_{i+1}$ determines an endomorphism on the group algebra generated by the u_i's. This algebra is called a binary shift algebra. By passing to the (unique) C^*-algebra completion we obtain an AF-algebra $\mathscr{A}$ on which α defines a shift.

In this paper we give necessary and sufficient conditions for binary shift algebras constructed as above to have a unique faithful trace, Theorem 2.3. In this case the weak operator closure of $\pi(\mathscr{A})$, where π is the GNS representation associated with the trace, is the unique hyperfinite II_1 factor R. As in **[4]** α induces a shift on R with Jones' index $[R:\alpha(R)] = 2$, see **[3]**.

We say that a shift on the factor R that is induced via a binary shift algebra, as above, is a binary shift on R. Conversely, one may consider a general shift α on R of index 2 and ask whether α is a binary shift. We show in Theorem 5.11 that this is not always the case, even if α is regular (i.e., the normalizer $N(\alpha)$ of α,

$$N(\alpha) = \{U \in R_U : U\alpha^k(R)U^{-1} = \alpha^k(R), k \in \mathbf{N}\},$$

generates the whole algebra). Moreover, under the assumptions that α is regular and that $N(\alpha)$ consists of elements whose squares are scalar multiples of the identity, we obtain necessary and sufficient conditions for α to be a binary shift, in Theorem 4.5. Our condition on $N(\alpha)$ may not be as restrictive as it seems, since this condition holds automatically in the case where the subfactor $\alpha^2(R)$ has non-trivial relative commutant (Theorem 3.3) and may possibly hold in general.

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2. Factor condition for the shift. Our notation will be consistent with Section 3 of [**4**]. In particular, for any subset S of $\mathbf{N}$ we form, as in Definition 3.8 of [**4**], the binary shift algebra $B(S)$ generated by elements u_i, $i \in \mathbf{N}$, satisfying

(1.1) $\quad u_i^* = u_i$,

(1.2) $\quad u_i^2 = I$,

(1.3) $\quad u_iu_j = u_ju_i \quad$ if $|i - j| \notin S$, and

(1.4) $\quad u_iu_j + u_ju_i = 0 \quad$ if $|i - j| \in S$.

Also, for finite subsets $Q = \{i_1, i_2, \dots, i_n\}$ of $\mathbf{N}$ with $i_1 < \dots < i_n$ we associate the word

$$\Gamma(Q) = u_{i_1}u_{i_2} \dots u_{i_n}$$

in $B(S)$. The shift α is then the homomorphism of $B(S)$ defined on generators by $\alpha(u_i) = u_{i+1}$. The shift extends to a homomorphism on $A(S)$, the (unique) C^*-algebraic completion of $B(S)$.

To each anti-commutation set S in $\mathbf{N}$ we have the corresponding signature function σ_S defined on the integers by $\sigma_S(i) = 1$ if $|i| \in S$ and $\sigma_S(i) = 0$, otherwise. Using this notation (1) may be replaced with the equivalent list of conditions

(2.1) $\quad u_i^* = u_i$,

(2.2) $\quad u_i^2 = I$,

(2.3) $\quad u_iu_j = (-1)^{\sigma_S(i-j)}u_ju_i$.

Denote by $\Sigma(S)$ the *signature sequence*

$$(\dots, \sigma_S(-1), \sigma_S(0), \sigma_S(1), \dots)$$

of S: of course, $\Sigma(S)$ is symmetric about the entry $\sigma_S(0)$.

A set S is called primary [**4**, Definition 3.7] if it is the anti-commutation set of a binary shift on the hyperfinite II_1 factor R. In [**4**] Powers has obtained the following characterizations of primary sets ([**4**, Theorem 3.9]):

THEOREM P. *Let S be a subset of $\mathbf{N}$. The following conditions are equivalent*:

(i) *S is primary*

(ii) *$B(S)$ is simple*

(iii) *$B(S)$ has center consisting of scalar multiples of I*

(iv) *There is a unique trace on $B(S)$*

(v) *For each non-empty finite set Q of positive integers, there is a positive integer k such that in $B(S)$,*

$$u_k\Gamma(Q) = -\Gamma(Q)u_k.$$

In [**4**] Powers determined that there are uncountably many subsets S which are primary sets. Since S is a conjugacy invariant there are uncountably many non-conjugate binary shifts of the factor R, [**4**, Theorem 3.6 and Theorem 3.10]. We sharpen these results by giving a precise characterization below of the primary sets. We need the following straightforward result.

Definition 2.1. If $Q = \{i_1, \ldots, i_n\}$ then the *length* of $\Gamma(Q)$ is $i_n - i_1 + 1$. The identity I has length 0.

LEMMA 2.2. *Suppose $B(S)$ has non-trivial center. Then there exists a unique word $\Gamma(Q)$ of minimal length in the center with $1 \in Q$.*

Proof. By [**4**, Theorem 3.9] there is a non-trivial word

$$\Gamma(Q') = u_{i_1} \ldots u_{i_n}$$

in the center. Choose Q' so that $\Gamma(Q')$ has minimal length among all such words. If $i_1 > 1$, then for $Q'' = \{i_1 - 1, \ldots, i_n - 1\}$,

$$\alpha(\Gamma(Q'')u_i) = \Gamma(Q')u_{i+1} = u_{i+1}\Gamma(Q') = \alpha(u_i\Gamma(Q'')),$$

so $\Gamma(Q'')$ is also in the center. We may then continue to backshift until we obtain an element $\Gamma(Q)$ of minimal length in the center with $1 \in Q$. If $\Gamma(Q_1)$ is another such word, then

$$\pm\Gamma(Q_1)\Gamma(Q) = \Gamma(Q_1 \Delta Q)$$

($Q_1 \Delta Q$ is the symmetric difference) is a word in the center shorter than $\Gamma(Q)$, which cannot be unless $Q_1 = Q$.

THEOREM 2.3. *Let S be a subset of $\mathbf{N}$. S is primary if and only if its signature sequence $\Sigma(S)$ is not periodic.*

Proof. Rewrite $\Sigma(S)$ as $(\ldots, a_{-1}, a_0, a_1, \ldots)$ with

$$a_j = \sigma_S(j), \quad j \in \mathbf{Z}.$$

Suppose $\Sigma(S)$ is periodic with period length n. We verify that a non-trivial word $\Gamma(Q)$ lies in the center of $B(S)$. Consider the homogeneous linear system of equations in $n + 1$ variables $x_0, \ldots, x_n$ over the field $\mathscr{F} = \mathbf{Z}/2\mathbf{Z}$:

$$(3) \quad \begin{array}{lllll}
a_0x_0 + & a_1x_1 + & a_2x_2 & + \ldots + & a_nx_n = 0 \\
a_1x_0 + & a_0x_1 + & a_1x_2 & + \ldots + & a_{n-1}x_n = 0 \\
a_2x_0 + & a_1x_1 + & a_0x_2 & + \ldots + & a_{n-2}x_n = 0 \\
\vdots & & & & \\
a_nx_0 + & a_{n-1}x_1 + & a_{n-2}x_2 & + \ldots + & a_0x_n = 0 \\
a_{n+1}x_0 + & a_nx_1 + & a_{n-1}x_2 & + \ldots + & a_1x_n = 0 \\
\vdots & & & &
\end{array}$$

Using the periodicity $a_j = a_{j+n}$ as well as the symmetry $a_{-k} = a_k$ one observes that the $(n + j)$th equation is identical to the jth equation, for all j, so the system reduces to n equations in $n + 1$ unknowns. Let

$$(x_0, \dots, x_n) = (k_0, \dots, k_n)$$

be a non-trivial solution, and let

$$u = u_1^{k_0} u_2^{k_1} \dots u_{n+1}^{k_n}.$$

Repeated use of (2.3) gives

$$u_k u = (-1)^{c_k} u u_k,$$

where c_k is the left side of the kth equation in (3). Since $c_k = 0$ u commutes with each u_k, so u is in the center of $B(S)$.

For the other direction suppose S is not primary. Then there exists a word

$$u = u_1^{k_0} u_2^{k_1} \dots u_{n+1}^{k_n}, \quad k_j \in \mathscr{F},$$

satisfying the conclusion of Lemma 2.2, so we may assume that $k_0 = 1$ and $k_n = 1$. Using (2.3) repeatedly, the equations $uu_k = u_k u$, $k \in \mathbf{N}$, imply $K = [k_0, \dots, k_n]^T$ is a non-trivial solution to the linear system $AX = 0$, where $X = [x_0, \dots, x_n]^T$, and A is the matrix

$$\text{(4)} \qquad \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \dots & a_n \\ a_1 & a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_2 & a_1 & a_0 & a_1 & \dots & a_{n-2} \\ \vdots & & & & & \\ a_n & a_{n-1} & & & \dots & a_0 \\ a_{n+1} & a_n & & & \dots & a_1 \\ \vdots & \vdots & & & & \vdots \end{bmatrix}.$$

Indeed, if $L = [l_0, \dots, l_n]^T$ is any solution to the system, then

$$w = u_1^{l_0} u_2^{l_1} \dots u_{n+1}^{l_n}$$

commutes with the generators u_k of $B(S)$ and w lies in the center. By the uniqueness of u, however, either $w = u$ or $w = I$ (in which case L is the trivial solution). This implies that the system $AX = 0$ has only one non-trivial solution over $\mathscr{F}$, so A must have rank n.

In fact the first n rows have rank n over $\mathscr{F}$. Let A_j be the jth row of A, $j \in \mathbf{N}$. Our assertion follows from the identities

$$\text{(5)} \qquad k_0 A_{n+j} + k_1 A_{n+j-1} + \dots + k_n A_j = [0, \dots, 0],$$

for $j \in \mathbf{N}$. The case $j = 2$ should suffice as an illustration. The first row entry of the left side of (5) is $k_0 a_{n+1} + k_1 a_n + \dots + k_n a_1$, which coincides with the inner product of A_{n+1} with the solution vector K of $AX = 0$; the second entry of the left side is $k_0 a_n + k_1 a_{n-1} + \dots + k_n a_0$, which coincides with the inner product of A_n with K, also giving 0, and so on, until the last entry, $k_0a_1 + k_1a_0 + \ldots + k_na_{n-1}$, which agrees with the inner product of row A_2 with K.

Replace the system $AX = 0$ with the equivalent system $A'X = 0$, where A' consists of the first $n + 1$ rows of A. It follows from the symmetry of A' that if $K = [k_0, k_1, \ldots, k_n]^T$ is a solution, so is

$$K_0 = [k_n, k_{n-1}, \ldots, k_0]^T.$$

Then $K = K_0$, since A' admits only one non-trivial solution over $\mathscr{F}$. Hence if $(A_0)_j$ is the row vector obtained from A_j by reversing the order of the entries then $(A_0)_j$ has inner product 0 with K also. It now follows that $B_j \cdot K = 0, j \in \mathbf{Z}$, where B_j is the row vector $[a_j, a_{j+1}, \ldots, a_{j+n}], j \in \mathbf{Z}$. Hence

$$D_{j+1}^T = C(D_j^T),$$

where $D_j = [a_{j+1}, \ldots, a_{j+n}], j \in \mathbf{Z}$, and C is the $n \times n$ matrix

$$\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 1 \\ k_0 & k_1 & k_2 & \ldots & k_{n-1} \end{bmatrix}.$$

C is invertible over $\mathscr{F}$, so $C^m = I$ for some m and therefore $D_{j+m} = D_j$, all $j \in \mathbf{Z}$, so that the signature sequence $\Sigma(S)$ is periodic.

The following is a consequence of the uniqueness result in Lemma 2.2, and shows that the center of $B(S)$ is an invariant subalgebra for the shift.

COROLLARY 2.4. *Suppose S is not primary. Let $u = \Gamma(Q)$ be the unique central element of minimal length with $1 \in Q$. The center of $B(S)$ is generated by the shifts $\alpha^p(u)$ of u, $p \in \mathbf{N} \cup \{0\}$.*

Proof. With

$$u = u_1^{k_0}u_2^{k_1} \ldots u_{n+1}^{k_n}$$

(with $k_0 = 1 = k_n$) the equations $u_ju = uu_j, j \in \mathbf{N}$,are by (2.3) equivalent to the vanishing of the inner products $E_{j-1} \cdot K$ over $\mathscr{F}$, where

$$E_j = [a_j, a_{j-1}, \ldots, a_{j-n}].$$

Using (2.3),

$$u_j\alpha^p(u) = \alpha^p(u)u_j$$

if and only if $E_{j-p-1} \cdot K = 0$, which holds since $\Sigma(S)$ is periodic. Hence the algebra generated by the shifts of u lies in the center.

Conversely, using the identities $u_j w u_j = \pm w$, for any word w, any element of the center must be a linear combination of words, each of which lies in the center. Suppose $w = \Gamma(Q)$ is such a word, with $Q = \{i_1, i_2, \ldots, i_r\}$. Since w is no shorter than u, the central element

$$w_1 = \alpha^{i_1 - 1}(u)w$$

must have length shorter than w. Then w_1 could have length 0, in which case

$$w = \pm\alpha^{i_1 - 1}(u),$$

and we are done, or w_1 could have length no shorter than that of u. Repeat the process above to obtain a central word

$$w_2 = \alpha^{j_2}(u)\alpha^{i_1 - 1}(u)w$$

of length shorter than w_1. This procedure ends when we finally obtain

$$\pm I = \alpha^{j_q}(u) \ldots \alpha^{j_2}(u)\alpha^{i_1 - 1}(u)w.$$

By (2.2), w is in the algebra generated by the shifts of u.

3. The normalizer for general shifts. Let $B(S)$ be the binary shift algebra corresponding to a primary subset $S \subseteq \mathbf{N}$. Let π be the cyclic *-representation of $B(S)$ induced by its unique normalized trace (see [**4**, Theorem 3.9]). Then $\pi(B(S))''$ is the hyperfinite II_1 factor R, and from [**4**] the shift α on $B(S)$ extends to a shift on R with

$$[R:\alpha(R)] = 2 \quad \text{and} \quad \bigcap_{n \geqq 1} \alpha^n(R) = \{cI:c \in \mathbf{C}\}.$$

Conversely, one may ask whether any shift α on R of index 2 and

$$\bigcap_{n \geqq 1} \alpha^n(R) = \{cI:c \in \mathbf{C}\}$$

arises as the completion of a binary shift algebra $B(S)$. We show in Theorem 5.11 that this is not always the case, even under the additional assumption that α is regular, i.e., that $N(\alpha)$ generates R (recall from [**4**] that $N(\alpha) = \{u \in R: u \text{ is unitary and } u\alpha^n(R)u^* = \alpha^n(R), \text{ all } n \in \mathbf{N}\}$).

Another question arises regarding $N(\alpha)$ itself. An easy consequence of [**4**, Theorem 3.3] is that the square of any unitary in the normalizer of α on $\pi(B(S))''$ is a scalar multiple of the identity. Is this true for any shift of index 2 on R? Although we do not know the answer in general, we have a partial answer below. We thank R. T. Powers for suggesting some improvements to our original proof of the following result.

THEOREM 3.1. *Let α be a shift on R with $[R:\alpha(R)] = 2$ such that $\alpha^2(R)$ has non-trivial relative commutant N. Then there exists a self-adjoint unitary u which generates N and lies in $N(\alpha)$.*

Proof. We have

$$[R:\alpha^2(R)] = [R:\alpha(R)][\alpha(R):\alpha^2(R)] = 4,$$

[**3**, Proposition 2.1.8]. Since $\alpha^2(R)' \cap R$ is non-trivial the Jones local index theory [**3**, Lemma 2.2.2] establishes that a self-adjoint unitary u generates the relative commutant. To show $u \in N(\alpha)$, we need only to verify that

$$u\alpha(R)u = \alpha(R).$$

Let θ be the period 2 automorphism fixing $\alpha(R)$, ([**3**, Corollary 3.4.3] or [**2**, Theorem 1]). Since

$$\theta(u)\alpha^2(x)\theta(u) = \theta(u\alpha^2(x)u) = \alpha^2(x),$$

for $x \in R$,

$$\theta(u) \in \alpha^2(R)' \cap R.$$

Hence $\theta(u) = aI + bu$, for some $a, b \in \mathbf{C}$. But $I = \theta(u)^2$ implies $a = 0$, $b = \pm 1$. In fact, $b = -1$. For suppose $b = 1$, then θ fixes the von Neumann algebra M generated by u and $\alpha(R)$. But u is not in $\alpha(R)$; otherwise $u = \alpha(v) \in \alpha^2(R)'$, so that v lies in $\alpha(R)' \cap R$, which is trivial by [**3**, Lemma 2.2.2]. Hence $M = R$ and θ fixes R, a contradiction, giving $\theta(u) = -u$. Moreover, observe that any $x \in R$ has the form $\alpha(x_0) + \alpha(x_1)u$ so that $\theta(x) = x$ if and only if $x \in \alpha(R)$. But for $y \in R$,

$$\theta(u\alpha(y)u) = u\alpha(y)u,$$

so $u\alpha(y)u \in \alpha(R)$, so that $u\alpha(y)u = \alpha(\gamma(y))$ for some period two automorphism γ of R. Hence $u \in N(\alpha)$.

COROLLARY 3.2. *If $\alpha^2(R)$ has non-trivial relative commutant N generated by the hermitian unitary u, then*

$$u\alpha(y)u = \alpha(\theta(y)), \quad \textit{all } y \in R.$$

Proof. From the proof of the theorem there is a period 2 automorphism γ of R satisfying $u\alpha(y)u = \alpha(\gamma(y))$. But

$$\alpha^2(y) = u\alpha^2(y)u = \alpha(\gamma(\alpha(y))),$$

so γ fixes $\alpha(R)$. Hence $\gamma = \theta$.

THEOREM 3.3. *Let α be a shift on R with $[R:\alpha(R)] = 2$. Suppose $\alpha^2(R)$ has non-trivial relative commutant N. Then any $v \in N(\alpha)$ has square equal to a scalar multiple of the identity.*

Proof. We proceed along the lines of the proof of [**4**, Theorem 3.3]. If $v \in N(\alpha)$, $\theta(v) = \pm v$, for if γ is the automorphism satisfying

$$v\alpha(y)v^* = \alpha(\gamma(y)), \quad y \in R,$$

$$\theta(v)\alpha(y)\theta(v^*) = \alpha(\gamma(y)).$$

Hence

$$v\theta(v^*) \in \alpha(R)' \cap R = \{\lambda I : \lambda \in \mathbf{C}\},$$

so $v = \lambda\theta(v)$. Since θ has period 2, $\lambda = \pm 1$.

Let u be the hermitian unitary generating N, (Theorem 3.1). By the proof of Theorem 3.1, $\theta(u) = -u$. If $\theta(v) = -v$, set $v_1 = uv$; otherwise, set $v_1 = v$. Now

$$v_1 \in \alpha(R) \cap N(\alpha),$$

so $\alpha^{-1}(v_1)$ lies in $N(\alpha)$ also. If

$$\theta(\alpha^{-1}(v_1)) = -\alpha^{-1}(v_1),$$

set $v_2 = \alpha(u)v_1$; otherwise set $v_2 = v_1$. In either case,

$$v_2 \in \alpha^2(R) \cap N(\alpha),$$

so that $\alpha^{-2}(v_2) \in N(\alpha)$. If $\theta(\alpha^{-2}(v_2)) = -\alpha^{-2}(v_2)$, set

$$v_3 = \alpha^2(u)v_2;$$

otherwise set $v_3 = v_2$. Then

$$v_3 \in \alpha^3(R) \cap N(a).$$

Continuing as above, we get for each $n \geqq 0$ a unitary

$$v_{n+1} \in \alpha^{n+1}(R) \cap N(\alpha)$$

and elements $k_j \in \{0, 1\}$ such that

$$v = u^{k_0}\alpha(u^{k_1})\alpha^2(u^{k_2}) \ldots \alpha^n(u^{k_n})v_{n+1}.$$

By Corollary 3.2,

$$\alpha^j(u)\alpha^{j+1}(u) = -\alpha^{j+1}(u)\alpha^j(u), \quad j \in \mathbf{N} \cup \{0\}.$$

Using these identities, along with $\alpha^j(u) \in \alpha^{j+2}(R)'$, one computes

$$v^2 = \pm\alpha^n(u^{k_n})v_{n+1}\alpha^n(u^{k_n})v_{n+1}.$$

Hence v^2 lies in $\alpha^n(R)$ for all n, so that v^2 is a scalar multiple of the identity.

COROLLARY 3.4. *Let α be as above. Then $uv = \pm vu$ for any u, $v \in N(\alpha)$.*

Proof. By the theorem u and v are scalar multiples of hermitian operators, so for the proof we may assume $u^2 = I = v^2$. In this case,

$$(uv)^2 = \lambda I = (vu)^2 = ((uv)^2)^* = \bar{\lambda}I,$$

so $\lambda = \pm 1$, and $uv = \pm vu$.

4. A characterization of binary shifts. Let α be a shift of index 2 on the hyperfinite II_1 factor R, with trace tr. In this section we adopt the following standing assumptions: that α is regular, i.e., $N(\alpha)$ generates R, and that any $u \in N(\alpha)$ has square a multiple of the identity. We shall often use the result, which follows from the proof of Corollary 3.4, that $uv = \pm vu$ for any $u, v \in N(\alpha)$. The theorem below gives necessary and sufficient conditions for α to be a binary shift on R. First we introduce some useful notation.

Definition 4.1. Let H be a subset of $N(\alpha)$. Then $M(H)$ is the von Neumann sub-algebra of R generated by the elements of H and their shifts.

Since we are assuming α to be regular, $R = M(N(\alpha))$.

Definition 4.2. Let $H \subseteq N(\alpha)$. Then $\widetilde{H}$ is the subgroup of $N(\alpha)$ generated by the elements of H, their shifts, and the (modulus 1) scalar multiples of the identity.

LEMMA 4.3. *Let $u \in N(\alpha)$, $H = \{u\}$, and suppose $\widetilde{H}$ has finite (group) index in $N(\alpha)$. Then $M(\{u\})$ is a subfactor of R, and $[R:M(\{u\})]$ is equal to the group index $[N(\alpha):\widetilde{H}]$.*

Proof. Let $M = M(\{u\})$. By [**4**, Theorem 3.3] the normalizer of α in M consists of (modulus 1) scalar multiples of words in u and its shifts, and therefore coincides with $\widetilde{H}$. If M is not a factor, there is by Corollary 2.4 an element $v \in \widetilde{H}$ such that the center Z of $\widetilde{H}$ is generated by v and its shifts. In particular, $[Z]$ has infinite order, where $[Z]$ is the set of equivalence classes of elements of Z identified if they differ by a scalar multiple of I.

Let

$$\coprod_{i=1}^{n} \widetilde{H}u_i$$

be the decomposition of $N(\alpha)$ into cosets of $\widetilde{H}$, with $u_1 = I$. Let Z_i be the subgroup of Z consisting of elements which commute with $u_1, \ldots, u_i$. $Z_1 = Z$, so $[Z_1]$ is infinite. Suppose $[Z_j]$ is infinite, for some j, $1 \leqq j \leqq n - 1$. Let

$$A_{j+1} = \{w \in Z_j : wu_{j+1}w^* = -u_{j+1}\}.$$

$[Z_j]$ is the disjoint union of $[A_{j+1}]$ and $[Z_{j+1}]$, so if $[A_{j+1}]$ is finite $[Z_{j+1}]$ is infinite. If $[A_{j+1}]$ is infinite then so is the subset $[\{ww' : w, w' \in A_{j+1}\}]$ of $[Z_{j+1}]$, so in either case $[Z_{j+1}]$ has infinite order. In particular, $[Z_n]$ is infinite, so there exists a non-trivial unitary z commuting with all of $N(\alpha)$, and therefore with R, a contradiction. Hence M is a subfactor of R.

To verify the index equation we observe that if $w \in N(\alpha)$ is not a scalar multiple of I, then $\mathrm{tr}(w) = 0$. For, if $v \in N(\alpha)$,

$$vwv^* = \pm w,$$

by the proof of Corollary 3.4. In fact, since R is a factor and α is regular, $vwv^* = -w$, for some $v \in N(\alpha)$, so

$$\mathrm{tr}(w) = \mathrm{tr}(vwv^*) = \mathrm{tr}(-w).$$

Hence $\mathrm{tr}(w) = 0$ for any w in $\widetilde{H}u_i$, $i > 1$. Let $V_i = \overline{Mu_i}$ in $L^2(R, \mathrm{tr})$. Since V_i is generated as a subspace by $\widetilde{H}u_i$, the V_i are equivalent, orthogonal, and span $L^2(R, \mathrm{tr})$. The remainder of the argument now follows exactly as in the proof of [**3**, Example 2.3.2].

The following notation and observations will be useful in proving the theorem below. As before, let $\mathcal{F}$ be the field $\{0, 1\}$ and $\mathcal{F}[t]$ the ring of polynomials over $\mathcal{F}$.

Definition 4.4. Let $\langle\ ,\ \rangle : N(\alpha) \times \mathcal{F}[t] \to N(\alpha)$ be the mapping given by

$$\langle w, p\rangle = w^{k_0}\alpha(w^{k_1}) \dots \alpha^n(w^{k_n}),$$

where $p(t)$ is the polynomial

$$k_0 + k_1 t + \dots + k_n t^n.$$

The following properties are easily verified:

$$(6.1) \qquad \langle w, p\rangle\langle w, q\rangle = \pm\langle w, p + q\rangle$$

$$(6.2) \qquad \langle \langle w, p\rangle, q\rangle = \pm\langle w, pq\rangle$$

$$(6.3) \qquad \langle w, p\rangle\langle w', p\rangle = \pm\langle ww', p\rangle.$$

The idea for generating the sequence $\{v_k\}$ in the following proof is due to R. T. Powers.

THEOREM 4.5. *Let α be a shift on R of index 2. Suppose α is regular and that any $u \in N(\alpha)$ has square a scalar multiple of I. Then α is a binary shift if and only if, for all $u \in N(\alpha)$, with u not a scalar multiple of I, $[R:M(\{u\})] < \infty$.*

Proof. If α is a binary shift there is a $v \in N(\alpha)$ such that $R = M(\{v\})$. If $u \in N(\alpha)$, then by [**4**, Theorem 3.3] u has the form

$$\lambda v^{k_0}\alpha(v^{k_1}) \dots \alpha^n(v^{k_n})$$

for some scalar λ and $k_j \in \{0, 1\}$. If $k_n = 1$, one verifies that

$$[N(\alpha):\{u\}^{\sim}] = [\{v\}^{\sim}, \{u\}^{\sim}] = 2^n,$$

so by the preceding lemma $[R:M(\{u\})] = 2^n < \infty$.

Now suppose $[R:M(\{u\})] < \infty$ for all $u \in N(\alpha)$ not a scalar multiple of I. Fix $u \in N(\alpha)$ with $\theta(u) = -u$ (we may do so by employing an argument similar to the first paragraph of the proof of Theorem 3.3). If

$[R:M(\{u\})] = 1$, then

$$[N(\alpha):\{u\}^{\sim}] = 1,$$

so $\{u\}^{\sim} = N(\alpha)$, $M(\{u\}) = R$, and we are done. Otherwise there exists a v_0 in $N(\alpha)$ but not in $\{u\}^{\sim}$, so

$$[\{u, v_0\}^{\sim}:\{u\}^{\sim}] > 1.$$

We may assume $\theta(v_0) = -v_0$.

We show there exists a $w \in N(\alpha)$ such that

$$M(\{w\}) \supseteq M(\{u, v_0\}).$$

First, since $\theta(uv_0) = uv_0$ there is an element $v_1 \in N(\alpha)$ such that

$$\theta(v_1) = -v_1 \quad \text{and} \quad \alpha^{j_1}(v_1) = uv_0,$$

for some $j_1 \in \mathbf{N}$. Since $v_0 = u\alpha^{j_1}(v_1)$,

$$\{v_1, u\}^{\sim} \supseteq \{v_0, u\}^{\sim}.$$

Similarly there are $v_2 \in N(\alpha)$ with $\theta(v_2) = -v_2$ and $j_2 \in \mathbf{N}$ such that

$$\alpha^{j_2}(v_2) = uv_1.$$

Continuing we obtain an ascending sequence of groups $H_k = \{v_k, u\}^{\sim}$. Since

$$[N(\alpha):H_k] \leqq [N(\alpha):\{u\}^{\sim}] = [R:M(\{u\})] < \infty,$$

$H_k = H_{k+1}$ for some k. Hence $v_{k+1} \in H_k$.

For simplicity write $v = v_k$, $j = j_{k+1}$, so $v_{k+1} = \alpha^{-j}(uv)$, and write $H = \{u, v\}^{\sim}$. Then $\alpha^{-j}(uv) \in H$, so there are $\lambda_0 \in \mathbf{C}$, and $p_0(t)$, $q_0(t)$ in $\mathscr{F}(t)$ so that

$$\alpha^{-j}(uv) = \lambda_0 \langle u, p_0\rangle\langle v, q_0\rangle.$$

Taking α^j of both sides of this equation and rewriting, there are $\lambda \in \mathbf{C}$ and $p(t)$, $q(t)$ in $F[t]$ so that

$$\langle u, p\rangle = \lambda\langle v, q\rangle, \quad \text{or} \tag{7}$$

$$\langle u, p\rangle \cong \langle v, q\rangle \tag{8}$$

where $\cong$ indicates equality up to a (modulus 1) scalar multiple of the identity.

We shall show by induction on $(\deg(p) + \deg(q))$ that there is a w in $N(\alpha)$ such that $\langle w, q\rangle \cong u$, $\langle w, p\rangle \cong v$. If $\deg(p) + \deg(q) = 0$, set $w = u$. Suppose $\deg(p) + \deg(q) > 0$. Let

$$p(t) = a_0 + a_1t + \dots a_mt^m, \quad q(t) = b_0 + b_1t + \dots + b_mt^m.$$

Since $\theta(u) = -u$ and $\theta(v) = -v$, then $a_0 = b_0 = 1$, so that $\langle u, p\rangle \cong \langle v, q\rangle$ implies

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$$uv \cong \alpha(u)^{a_1}\alpha(v)^{b_1} \ldots \alpha^m(u)^{a_m}\alpha^m(v)^{b_m}.$$

Let $k \geqq 1$ be the first index for which $a_k \neq 0$ or $b_k \neq 0$; then clearly $uv \in \alpha^k(R)$, so $\alpha^{-k}(uv) \in N(\alpha)$. Using (6) the following equations are easily shown to be equivalent:

(9.1) $\langle u, p(t) \rangle \cong \langle v, q(t) \rangle$

(9.2) $\langle u, p(t) \rangle\langle u, q(t) \rangle \cong \langle v, q(t) \rangle\langle u, q(t) \rangle$

(9.3) $\langle u, p(t) + q(t) \rangle \cong \langle uv, q(t) \rangle$

(9.4) $\alpha^{-k}(\langle u, p(t) + q(t) \rangle) \cong \alpha^{-k}(\langle uv, q(t) \rangle)$

(9.5) $\langle u, (p(t) + q(t))/t^k\rangle \cong \langle\alpha^{-k}(uv), q(t) \rangle.$

By the induction step there is a $w \in N(\alpha)$ such that

$$\langle w, q(t) \rangle \cong u \quad\text{and}\quad \langle w, (p(t) + q(t))/t^k\rangle \cong \alpha^{-k}(uv).$$

Shifting the latter equation by α^k, we get

$$\langle w, p(t) + q(t) \rangle \cong uv, \quad\text{or}$$

$$\langle w, p(t) \rangle\langle w, q(t) \rangle \cong uv, \quad\text{or}$$

(since $\langle w, q\rangle \cong u$),

$$\langle w, p\rangle \cong v.$$

This completes the induction.

Hence we have

$$M(\{w\}) \supseteq M(\{v, u\}) \supseteq M(\{v_0, u\}),$$

and

$$[R:M(\{w\})] = [N(\alpha):\{w\}^{\sim}] \leqq [N(\alpha):\{u, v_0\}^{\sim}]$$
$$< [N(\alpha):\{u\}^{\sim}] = [R:M(\{u\})].$$

Continuing this procedure, if necessary, we shall obtain an element w' in $N(\alpha)$ with

$$[N(\alpha):\{w'\}^{\sim}] = 1,$$

so $\{w'\}^{\sim} = N(\alpha)$, and since α is regular, $R = M(\{w'\})$.

5. A non-binary shift of R. In this section we give an example of a shift α of index 2 on the hyperfinite II_1 factor R which is not binary. The basic construction, which we present below, is to view R as the completion of an inductive limit of binary shift algebras. Our example will be regular and such that the square of any unitary in the normalizer is a scalar multiple of the identity. Hence by the preceding theorem there is an element u in $N(\alpha)$

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such that $[R:M(\{u\})] = \infty$.

Let $\{u_{ij}:i, j \in \mathbf{N}\}$ be a set of elements satisfying

(10.1) $u_{ij}^* = u_{ij}$,

(10.2) $(u_{ij})^2 = I$,

(10.3) $u_{ij}u_{kl} = \pm u_{kl}u_{ij}$.

Each pair of elements u_{ij}, u_{pq} either commutes or anti-commutes. We shall prescribe which below.

Let α be the transformation on the elements u_{ij} defined by

$$\alpha(u_{ij}) = u_{i,j+1}.$$

Then α extends to a homomorphism on the group of words in the u_{ij}, also denoted by α.

We impose the condition

$$u_{k1} = u_{k+1,1}\alpha(u_{k+1,1}) \quad \text{for all } k \in \mathbf{N}.$$

Shifting by α^{j-1} we obtain, for all $k, j \in \mathbf{N}$,

(11) $\quad u_{kj} = u_{k+1,j}\alpha(u_{k+1,j})$.

In other words, the relations (11) are compatible with α. Therefore, if B_k, $k \in \mathbf{N}$, is the algebra generated by all words in the elements u_{ij} with $1 \leqq i \leqq k$ and $j \in \mathbf{N}$, B_k is invariant under the shift and $B_1 \subseteq B_2 \subseteq \dots$.

Now we define the commutation rules for the u_{ij}. Begin by fixing $S = S_1 \subseteq \mathbf{N}$ such that the signature sequence

$$\Sigma(S) = (\dots, \sigma_S(-1), \sigma_S(0), \sigma_S(1), \dots)$$

is not periodic. Impose the relations

(12) $\quad u_{1k}u_{1j} = (-1)^{\sigma_S(k-j)}u_{1j}u_{1k}$.

Then by Theorem 2.3, S is primary, so by [**4**, Theorem 3.9] there is a unique tracial state on B_1, tr_1. In fact, tr_1 is faithful. This follows from the proof of [**4**, Theorem 3.9], since it is shown there that $\mathrm{tr}_1(w) = 0$ for a word w in the u_{1j}, so that if

$$y = \sum_{i=1}^{n} c_i w_i$$

with the c_i scalars and the w_i distinct words in the u_{1j},

$$\mathrm{tr}_1(y^*y) = \sum_{i=1}^{n} |c_i|^2.$$

We now associate a subset $T = S_2$ of $\mathbf{N}$ and a signature sequence

$\Sigma(T)$ with the elements u_{2j}, $j \in \mathbf{N}$. T may not be chosen arbitrarily, of course, because of (11). In fact, if

$$\Sigma(S) = (\ldots, a_{-1}, a_0, a_1, \ldots)$$

and

$$\Sigma(T) = (\ldots, b_{-1}, b_0, b_1, \ldots),$$

then from

$$u_{1j} = u_{2j}\alpha(u_{2j}) \quad \text{and} \quad u_{11}u_{1j} = (-1)^{a_{j-1}}u_{1j}u_{11}$$

we must have

$$\begin{aligned} u_{21}\alpha(u_{21})u_{2j}\alpha(u_{2j}) &= (-1)^{b_{j-2}+b_{j-1}+b_{j-1}+b_j}u_{2j}\alpha(u_{2j})u_{21}\alpha(u_{21}) \\ &= (-1)^{b_{j-2}+b_j}u_{2j}\alpha(u_{2j})u_{21}\alpha(u_{21}) \\ &= (-1)^{a_{j-1}}u_{2j}\alpha(u_{2j})u_{21}\alpha(u_{21}), \end{aligned}$$

so $b_{j-2} + b_j = a_{j-1}, j \in \mathbf{N}$. (Here we are viewing the a_j and b_j as elements of the field $\{0, 1\} = \mathscr{F}$.) Since $b_{-j} = b_j$ and $a_j = a_{-j}$ for all $j \in \mathbf{N}$ we therefore obtain $b_{j-2} + b_j = a_{j-1}$, all $j \in \mathbf{Z}$, or

(13) $\qquad b_{j-1} + b_{j+1} = a_j, \quad j \in \mathbf{Z}.$

We must have $b_0 = 0$, since $b_0 = \sigma_T(0)$. The entry b_1 may be chosen arbitrarily, and for $n > 1$, $b_n = a_{n-1} + b_{n-2}$. We show that $\Sigma(T)$ is not periodic. For if $\Sigma(T)$ is periodic, so is the sequence

$$(\ldots, b'_{-1}, b'_0, b'_1, \ldots),$$

where $b'_j = b_{j-2}$. The sum of the sequences,

$$(\ldots, b_{-1} + b'_{-1}, b_0 + b'_0, b_1 + b'_1, \ldots)$$

is therefore also periodic. But this is the sequence

$$\Sigma(S) = (\ldots, a_{-1}, a_0, a_1, \ldots),$$

which is not periodic, a contradiction.

Since the signature sequence $\Sigma(S_2) = \Sigma(T)$ associated with the u_{2j}'s is not periodic, there is a unique tracial state tr_2 on B_2. We may argue, as with tr_1, that tr_2 is faithful.

Proceeding as above we may obtain inductively from $S_k \subseteq \mathbf{N}$ a subset S_{k+1} of $\mathbf{N}$ and corresponding sequence $\Sigma(S_{k+1})$ satisfying

(14) $\qquad \sigma_{S_{k+1}}(j + 1) + \sigma_{S_{k+1}}(j - 1) = \sigma_{S_k}(j).$

Each of these sequences is not periodic, so from Theorem 2.3 and [**4**, Theorem 3.9] there is a unique (faithful) tracial state tr_k on B_k. By uniqueness

$$\mathrm{tr}_{k|B_j} = \mathrm{tr}_j \quad \text{for } j \leqq k.$$

Let $\mathscr{A}$ denote the unique C^*-algebra completion of $\cup_{k \geqq 1} B_k$. We make a few observations about $\mathscr{A}$. First, it is clear that $\mathscr{A}$ is an AF-algebra, since for each k, B_k is the binary shift algebra $B(S_k)$, and $B_1 \subseteq B_2 \subseteq \dots$. Secondly, $\mathscr{A}$ is equipped with a unique tracial state tr, since there is a unique tracial state on

$$B = \bigcup_{k \geqq 1} B_k.$$

Finally, tr is faithful, since for each $k \in \mathbf{N}$, tr_k is faithful (see [**1**, Lemma 3.1]).

We summarize our results below.

THEOREM 5.1. *Let* $\{u_{ij}:i, j \in \mathbf{N}\}$ *be a set of elements satisfying* (10) *and* (11). *Then there exists a sequence of primary sets* S_k, $k \in \mathbf{N}$, *such that the relations*

$$u_{kj}u_{kl} = (-1)^{\sigma_{S_k}(l-j)}u_{kl}u_{kj}$$

are consistent with (11). *The AF-algebra* $\mathscr{A}$ *generated by the* u_{ij} *has a unique* (*faithful*) *trace* tr.

Remark. Let $w \in B$ be a word in the u_{ij}, $w \neq \pm I$. Then $\mathrm{tr}(w) = 0$. In fact, there is a word u in the u_{ij} such that $uwu^* = -w$. For, using (11) if necessary, there is a k so that w may be brought into the form

$$\lambda u_{kj_1} \dots u_{kj_n}.$$

Since the set S_k is primary the assertions follow by applying [**4**, Theorem 3.9] to $B_k = B(S_k)$.

Since tr is faithful, in what follows we shall identify $\mathscr{A}$ with its image $\pi(\mathscr{A})$ in the GNS representation π associated with tr. Then $\mathscr{A}''$ is the hyperfinite II_1 factor R. By continuity, α extends to a *-endomorphism of R. By Proposition 5.10 below, α is a shift on R.

THEOREM 5.2. $[R:\alpha(R)] = 2$.

Proof. By repeated application of (11) we have for $k > 1$,

$$\begin{aligned} u_{k1} &= u_{k-1,1}\alpha(u_{k1}) \\ &= u_{k-2,1}\alpha(u_{k-1,1})\alpha(u_{k1}) \\ &= u_{11}\alpha(u_{21}) \dots \alpha(u_{k1}). \end{aligned}$$

Hence R is generated by u_{11} and $\alpha(R)$. Moreover by the preceding remark, if $w \in \alpha(R) \cap B$ is a word then $\mathrm{tr}(wu_{11}) = 0$. Hence $L^2(R, \mathrm{tr})$ decomposes into the two orthogonal equivalent subspaces $\overline{\alpha(R)}$, $\overline{\alpha(R)u_{11}}$, so $[R:\alpha(R)] = 2$ (cf. [**3**, Example 2.3.2.]).

COROLLARY 5.3. *Let* θ *be the unique period* 2 *automorphism fixing* $\alpha(R)$. *Then* $\theta(u_{k1}) = -u_{k1}$, *for all* $k \in \mathbf{N}$.

Proof. From the preceding proof every $x \in R$ has a unique decomposition of the form $x = \alpha(x') + \alpha(x'')u_{11}$, so

$$\theta(x) = \alpha(x') - \alpha(x'')u_{11}.$$

Hence $\theta(u_{11}) = -u_{11}$. The general result follows from the equation

$$u_{k1} = \pm u_{11}\alpha(u_{21}) \ldots \alpha(u_{k1}), \quad \text{for } k > 1.$$

Let $w \in B$ be a word. Then $wuw^* = \pm u$ for any other word u in B. Since $\alpha^j(R)$ is generated by elements of the form $u_{k,j+l}$, clearly $w \in N(\alpha)$. (Hence α is a regular shift.) Conversely, we shall show that any element $w \in N(a)$ is a word in B. This will follow from the series of lemmas below. The first lemma is proved exactly as in the proof of [**4**, Lemma 3.3], so we omit the proof.

LEMMA 5.4. $\theta(w) = \pm w$ *for any* $w \in N(\alpha)$. *Let* $u = u_{n1}$, *for some n. Then for any* $m \in \mathbf{N}$, w *may be written uniquely in the form*

$$u^{k_0}\alpha(u^{k_1}) \ldots \alpha^m(u^{k_m})\alpha^{m+1}(w_{m+1}),$$

for some w_{m+1} *in* R.

Definition 5.5. Let $\| \ \|_2$ denote the norm on R given by

$$\|x\|_2 = \operatorname{tr}(x^*x)^{1/2}, \quad x \in R.$$

Definition 5.6. Let $\Phi:R \to \alpha(R)$ denote the conditional expectation $(I + \theta)/2$.

LEMMA 5.7. *Let* $w \in N(\alpha)$, *and let* $v' \in B_n$ *for some n. Suppose w has the form*

$$w = u^{k_0}\alpha(u^{k_1}) \ldots \alpha^m(u^{k_m})\alpha^{m+1}(w_{m+1}),$$

where $u = u_{n1}$.

Then there is a $v \in B_n$ *of the form*

$$v = u^{k_0}\alpha(u^{k_1}) \ldots \alpha^m(u^{k_m})\alpha^{m+1}(v_{m+1}),$$

such that $\|w - v\|_2 \leqq \|w - v'\|_2$.

Proof. If $\theta(w) = w$, replace v' with $v_0 = \Phi(v')$. Then

$$\|w - v_0\|_2 = \|\Phi(w - v')\|_2 \leqq \|w - v'\|_2.$$

If $\theta(w) = -w$, replace v' with $v_0 = u\Phi(uv')$, in which case

$$\|w - v_0\|_2 = \|u\Phi(uw) - u\Phi(uv')\|_2 = \|\Phi(uw - uv')\|_2$$
$$\leqq \|w - v'\|_2.$$

In either case, if $w = u^{k_0}\alpha(w_1)$, then $v_0 = u^{k_0}\alpha(v_1)$, for some v_1 in B_n. We have

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$$\|w_1 - v_1\|_2 = \|w - v_0\|_2 \leqq \|w - v'\|_2, \quad \text{and}$$

$$w_1 = u^{k_1}\alpha(w_2).$$

Proceeding as above we may replace v_1 with an element of the form $u^{k_1}\alpha(v_2)$ with v_2 in B_n such that

$$\|w_1 - u^{k_1}\alpha(v_2)\|_2 \leqq \|w_1 - v_1\|_2,$$

whence

$$\|w - u^{k_0}\alpha(u^{k_1})\alpha^2(v_2)\|_2 \leqq \|w - v'\|_2.$$

Continuing this process m steps yields the result.

LEMMA 5.8. *Let $w \in N(\alpha)$. Let $n \in \mathbf{N}$ be sufficiently large so that for some $v' \in B_n$, $\|w - v'\|_2 < 1$. Let $u = u_{n1}$, and let*

$$N = \sup(\{q \in N: w \text{ has the form } u^{k_0} \ldots \alpha^m(u^{k_m})\alpha^{m+1}(w_{m+1}),$$

$$m \geqq q \text{ and } k_m = 1\}).$$

Then $N < \infty$.

Proof. By hypothesis there is a $p \in \mathbf{N}$ such that for some y' in the algebra generated by $u, \alpha(u), \ldots, \alpha^p(u)$,

$$\|w - y'\|_2 < 1.$$

Suppose $N = \infty$; then there is an $m > p$ such that w has the form above with $k_m = 1$. From the proof of the preceding lemma there is a y in B_n of the form

$$au^{k_0} \ldots \alpha^p(u^{k_p})$$

with $a \in \mathbf{C}$ such that

$$\|w - y\|_2 \leqq \|w - y'\|_2 < 1.$$

Let

$$w' = \alpha^{p+1}(u^{k_{p+1}}) \ldots \alpha^m(u^{k_m})\alpha^{m+1}(w_{m+1}).$$

Then $\|w - y\|_2 < 1$ implies $\|w' - aI\|_2 < 1$. But

$$\|w' - aI\|_2^2 = 1 + |a|^2 - 2\,\mathrm{Re}(\mathrm{tr}(\bar{a}w')),$$

and $\mathrm{tr}(w') = 0$; for if j is the first index greater than p for which $k_j = 1$,

$$\mathrm{tr}(w') = \mathrm{tr}(\alpha^{-j}(w')) = \mathrm{tr}(\theta(\alpha^{-j}(w')))$$

$$= -\mathrm{tr}(\alpha^{-j}(w')) = -\mathrm{tr}(w').$$

Hence

$$\|w' - a\|_2^2 = 1 + |a|^2 > 1,$$

a contradiction. Thus N is finite.

For the sake of completeness we include the following result. This result implies that α is regular and that every element in $N(\alpha)$ has square a scalar multiple of the identity.

THEOREM 5.9. *If $w \in N(\alpha)$, then there is a positive integer n such that w is a scalar multiple of a word in B_n.*

Proof. Let $n \in \mathbf{N}$ be sufficiently large so that

$$\|w - y\|_2 < 1, \quad \text{for some } y \in B_n.$$

By the previous lemma there is a maximum $m \in \mathbf{N}$ such that w has the form

$$u^{k_0} \dots \alpha^m(u^{k_m})\alpha^{m+1}(w_{m+1})$$

with $k_m = 1$, where $u = u_{n1}$. Since u and its shifts lie in $N(\alpha)$, $\alpha^{m+1}(w_{m+1})$ lies in $N(\alpha)$, hence so does w_{m+1}. Hence

$$\theta(w_{m+1}) = \pm w_{m+1},$$

by Lemma 5.4. If $\theta(w_{m+1}) = -w_{m+1}$, then

$$w_{m+2} = uw_{m+1} \in \alpha(R),$$

so that $w_{m+1} = uw_{m+2}$. But this contradicts the maximality of m, so $\theta(w_{m+1}) = w_{m+1}$, or equivalently, $w_{m+1} \in \alpha(R)$. Similarly, $\alpha^{-1}(w_{m+1})$ lies in $N(\alpha)$ and is fixed by θ, by the maximality of m, so

$$\alpha^{-1}(w_{m+1}) \in \alpha(R), \quad \text{or} \quad w_{m+1} \in \alpha^2(R).$$

Continuing, we have

$$w_{m+1} \in \bigcap_{n \geqq 1} \alpha^n(R).$$

But then w_{m+1} is a scalar multiple of I, by the following proposition.

PROPOSITION 5.10.

$$\bigcap_{n \geqq 1} \alpha^n(R) = \{cI : c \in \mathbf{C}\}.$$

Proof. Suppose w is a unitary element of $\cap_{n \geqq 1}\alpha^n(R)$. If $1 > \epsilon > 0$, there are $k, p \in \mathbf{N}$ such that for some v in the unit ball of the finite-dimensional algebra A generated by $u_{k1}, \alpha(u_{k1}), \dots, \alpha^{p-1}(u_{k1})$,

$$\|w - v\|_2 < \epsilon.$$

Similarly, since $w \in \alpha^p(R)$ there exist $l \geqq k$, $q \geqq p$, such that for some v' in the unit ball of the finite-dimensional algebra A' generated by $\alpha^p(u_{l1}), \dots, \alpha^q(u_{l1})$,

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$$\|w - v'\|_2 < \epsilon$$

also. Now let u be any word in A, $u \neq \pm I$, and let $u' \neq \pm I$ be a word in A'; then

$$\mathrm{tr}(uu') = 0.$$

For, if u has the form

$$\alpha^j(u_{k1})\alpha^{j+1}(u_{k1}{}^{a_{j+1}}) \ldots \alpha^{p-1}(u_{k1}{}^{a_{p-1}}),$$

then repeated application of (11) transforms the expression above into one involving shifts of u_{l1},

$$u = \alpha^j(u_{l1})\alpha^{j+1}(u_{l1}{}^{b_{j+1}}) \ldots .$$

Then uu' is a non-trivial word in B_l (since $j \leqq p - 1$) so $\mathrm{tr}(uu') = 0$ by the Remark following Theorem 5.1.

Write

$$v = aI + \sum_{i=1}^{m} a_i v_i,$$

where a, $a_i \in \mathbf{C}$ and the $v_i \neq \pm I$ are distinct words in A (i.e., $v_i v_j \neq \pm I$, for $i \neq j$), and

$$v' = a'I + \sum_{i=1}^{m'} a_i' v_i',$$

where v_i' are distinct words in A'. Then

$$\|I - v^*v'\|_2 = \|w^*w - v^*v'\|_2 < 2\epsilon.$$

Therefore, $|1 - \mathrm{tr}(v^*v')| < 2\epsilon$, so $|1 - \bar{a}a'| < 2\epsilon$. Since

$$1 \geqq \|v\|^2 \geqq \|v\|_2^2 = |a|^2 + \sum_{i=1}^{m} |a_i|^2,$$

and similarly,

$$1 \geqq |a'|^2 + \sum_{i=1}^{m'} |a_i'|^2,$$

we must have $1 - |a| < 2\epsilon$. Therefore

$$\|v - aI\|_2^2 = \sum_{i=1}^{m} |a_i|^2 < 4\epsilon, \quad \text{and}$$

$$\|w - aI\|_2 \leqq \|w - v\|_2 + \|v - aI\|_2 < 2\epsilon + 2\epsilon^{1/2}.$$

Hence for $1 > \epsilon > 0$, there is $a_\epsilon \in \mathbf{C}$ such that

$$1 > |a_\epsilon| > 1 - 2\epsilon \quad \text{and} \quad \|w - a_\epsilon I\|_2 < 2\epsilon + 2\epsilon^{1/2}.$$

Let a_0 be a limit point of the a_ϵ, as ϵ approaches 0. Then

$$\|w - a_0 I\|_2 = 0.$$

By the faithfulness of tr, $w = a_0 I$.

THEOREM 5.11. *Let α be the shift of index 2 on the hyperfinite II_1 factor R constructed above. Then α is not a binary shift.*

Proof. Using the notation of the previous section, $B''_n = M(\{u_{n1}\})$, for $n \in \mathbf{N}$. Since

$$u_{n1} = u_{n+1,1}\alpha(u_{n+1,1})$$

from (11), the first paragraph of the proof of Theorem 4.5 shows that

$$[B''_{n+1}:B''_n] = [M(\{u_{n+1,1}\}):M(\{u_{n,1}\})] = 2.$$

Therefore,

$$[R:M(\{u_{11}\})] \geqq [M(\{u_{n+1}, 1\}):M(\{u_{11}\})] = 2^n,$$

from the multiplicativity of the index, [**3**, Proposition 2.1.8]. Hence

$$[R:M(\{u_{11}\})] = \infty.$$

Applying Theorem 4.5, α cannot be a binary shift.

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United States Naval Academy,
Annapolis, Maryland