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Fredholm theory for arbitrary measure spaces

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The classical formulae for Fredholm integral equations, including expansions in terms of eigenfunctions such as Mercer's Theorem are extended to square-integrable kernels on an arbitrary measure space.

1. Introduction

We shall generalise the results of Withers [8] on Fredholm integral equations for $n \times n$ matrix kernels from Lebesgue measure on R^p to an arbitrary measure space (Ω, A, μ). Our statement of Mercer's Theorem in §3 requires a topology on Ω and for the first time is extended to kernels which have an infinite number of both positive and negative eigenvalues. That such an extension was possible for Ω a locally compact space and positive semi-definite kernels was noted by Gohberg and Kreĭn [2], p. 113.

Partial results had been obtained for special cases in R^p by Kneser, Lichtenstein and Günther; (see Smirnov, [5], §49). We shall assume throughout that $L_2(\Omega, A, \mu)$ is separable. This condition is discussed in §4.

Some of the basic results in §2 are known to hold for completely continuous linear operators on a Hilbert space; see, for example, Gohberg and Kreĭn [2], pp. 28, 111, 168, and 1.2, p. 207. The same authors ([2], p. 123) consider extensions to "matrix measures":

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$$(N_0 f_0)(x) = \int \underset{n\times n}{N_0(x, y)} \underset{n\times n}{d\mu_0(y)} \underset{n\times 1}{f_0(y)} \;, \quad N_0 \text{ symmetric.}$$

However by choosing a measure μ dominating the elements of μ_0, this case easily reduces to the case we consider, provided $T = \left(\frac{d\mu_0}{d\mu}\right)^{\frac{1}{2}}$ has an inverse almost everywhere (μ) in Ω, for then

$$f_0 - \lambda N_0 f_0 = g_0 \Leftrightarrow f - \lambda N f = g \;,$$

where

$$(Nf)(x) = \int N(x, y) f(y) d\mu(y) \;,$$

$$N(x, y) = T^*(x) N_0(x, y) T(y)$$

is symmetric, that is, $N^* = N$,

$$f = Tf_0 \;, \quad g = Tg_0 \;.$$

Finally recall that the solution of $f - \lambda N f = g$ requires that condition (1) below on N merely be satisfied by N^m for some $m \geq 1$, since

$$(1-\lambda N)^{-1} \cdot g = \left(1-\lambda N^m\right)^{-1} \cdot \left(1 + \lambda N + \ldots + \lambda^{m-1} N^{m-1}\right) \cdot g \;.$$

(For example for Ω a bounded set in R^p and $N(x, y) = |x-y|^{-\alpha}$, we need not $\alpha < p/2$ but $\alpha < p$; see Pogorzelski [3], p. 79.)

2. Some general results

We shall use the notation of Withers [8], including the definition of eigenvalues $\{\lambda_i\}_1^\infty$ and eigenfunctions $\{\phi_i\}_1^\infty$, and the use of script to denote the operator associated with a kernel, with the understanding that Ω is now a general set and integration with respect to Lebesgue measure over Ω occurring in [8] is replaced by integration with respect to μ. Then Fredholm's Theorems and the results of §2, §4 of [8] carry over. (An alternative proof is given by the method of Smithies [6].)

We assume throughout that N is a complex measurable $n \times n$ function on Ω^2 satisfying

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$$0 < \iint \|N(x,\ y)\|^2 d\mu(x) d\mu(y) < \infty . \tag{1}$$

This condition ensures the existence of solutions to

$$N\phi = \lambda^{-1}\phi\ ,\quad N^*\psi = \bar{\lambda}^{-1}\psi\ ,\quad 0 < \int |\phi|^2 d\mu < \infty\ ,\quad 0 < \int |\psi|^2 d\mu < \infty$$

whenever λ is an eigenvalue of N, that is a zero of

$$\hat{D}(\lambda) = \exp\left\{- \int_0^\lambda d\lambda \int \operatorname{trace}\bigl(N(x,\ x,\ \lambda) - N(x,\ x)\bigr) d\mu(x)\right\} . \tag{2}$$

Thus if

$$\sum \int |N_{ii}(x,\ x)| d\mu(x) < \infty\ , \tag{3}$$

then

$$D(\lambda) = \exp\left\{-\lambda \int \operatorname{trace} N(x,\ x) d\mu(x)\right\}\hat{D}(\lambda) . \tag{4}$$

Hence forward we shall assume that N is symmetric:

$$N^*(y,\ x) = N(x,\ y) \quad \text{in} \quad \Omega^2 . \tag{5}$$

One can show that when N is bounded and $\mu(\Omega) < \infty$ then

$$\hat{D}(\lambda) = \prod_1^\infty \bigl(1-\lambda/\lambda_j\bigr)\exp\{\lambda/\lambda_j\}\ , \tag{6}$$

so that by (2),

$$\int \operatorname{trace} N_p(x,\ x) d\mu(x) = \sum_1^\infty \lambda_j^{-p}\ ,\quad p = 2,\ 3,\ \ldots\ , \tag{7}$$

where N_p is the kernel of N^p ; (*cf.* [3], p. 178, where b should be zero).

In §3 we shall obtain (6) and (7) under different conditions.

Let Ω_0 be any measurable subset of Ω such that $\mu\bigl(\Omega - \Omega_0\bigr) = 0$. Note that $f - \lambda Nf = g$ *defines* f in $\Omega - \Omega_0$ in terms of g in $\Omega - \Omega_0$ and f in Ω_0, so that we need only solve for f in Ω_0.

For greater completeness we add the following generalisations of the

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Hilbert-Schmidt Theorem and the mean-convergence theorems.

Suppose N satisfies (1), (5), and

$$\sup_{\Omega_0} \int \|N(x, y)\|^2 d\mu(x) < \infty . \tag{8}$$

Suppose also h is a complex measurable $n \times 1$ function on Ω such that $\int |h|^2 = \int |h(x)|^2 d\mu(x) < \infty$, and $f = Nh$. Then the Fourier series $\sum_1^\infty \phi_i(x) \int \phi_i{}^* f$ is absolutely and uniformly convergent in Ω_0 and equals $f(x)$ almost everywhere (μ) in Ω . When (1), (5) hold then for $p = 1, 2, \ldots ,$

$$\iint \left\| \sum_1^q \phi_i(x)\phi_i(y)^*/\lambda_i{}^p - N_p(x, y)\right\|^2 d\mu(x)d\mu(y) =$$

$$= \int \operatorname{trace} N_{2p}(x, x)d\mu(x) - \sum_1^q \lambda_i{}^{-2p} \to 0$$

as $q \to \infty$ by (7).

A stronger result is given by the following lemma needed for Mercer's Theorem. (See, for example, [3], p. 130, for the method of proof.)

LEMMA. *When* N *satisfies* (1), (5), *and*

$$\sum_1^\infty |\phi_i(x)|^2/|\lambda_i| \quad \textit{converges uniformly in } \Omega_0 \tag{9}$$

then

$$\sum_1^\infty \phi_i(x)\phi_i(y)^*/\lambda_i = N(x, y) \quad \textit{almost everywhere } (\mu\times\mu) \textit{ in } \Omega^2 , \tag{10}$$

and the convergence is (elementwise) absolute and uniform.

Formula (10) is the function form of the formula for Hermitian matrices: $N = U \wedge U^*$ where $UU^* = 1$ and $\wedge$ is diagonal.

3. Mercer's Theorem

Suppose (Ω, T) is a topological space. Taking any Ω_0 as in §2,

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let $T_0 = \{\tau \cap \Omega_0 : \tau \in T\}$.

THEOREM 1. *Suppose* N *satisfies* (1), (5), *and*

(11) *for* x *in* Ω_0, *for some* $m \geq 1$,

$$\operatorname{trace}\left(N_{2m}(x, x) - 2\operatorname{Re} N_{2m}(x, y) + N_{2m}(y, y)\right)$$

is continuous at $y = x$.

Then $\{\phi_i\}$ *are continuous in* Ω_0 . *Suppose also that*

(12) N *has only a finite number of negative eigenvalues,*

(13) $$\sup_{\Omega_0} \operatorname{trace} N(x, x) < \infty ,$$

(14) N *is continuous in* $\Omega_0{}^2$.

Then (10) *holds.*

NOTES (i) *If* Ω_0 *is the support of* μ , *in the sense that*

(15) *every* $T \neq \phi$ *in* Ω_0 *contains a measurable set of positive measure,*

then (10) *and* (14) *imply*

(16) $$\sum_1^\infty \lambda_i{}^{-1}\phi_i(x)\phi_i(y)^* \equiv N(x, y) \quad \text{in} \quad \Omega_0{}^2 .$$

In particular (15) *holds for* $\Omega \subset R^p$ *if* Ω_0 *consists of the points of increase of* μ .

(ii) *Condition* (14) *may be weakened to*

(14') $\operatorname{Re} N_{ii}$ *is continuous at each* (x, x) *in* Ω_0 , $1 \leq i \leq n$.

However (14'), (13) (and (11) if N has negative eigenvalues), imply (14) via the inequality for positive semidefinite N :

(17) $$\begin{aligned}\|N(x, y) - N(x', y')\|^2 &\leq \\ &\leq \operatorname{trace} N(x, x)\operatorname{trace}\left(N(y, y) + N(y', y') - 2\operatorname{Re} N(y, y')\right) \\ &\quad + \operatorname{trace} N(y', y')\operatorname{trace}\left(N(x, x) + N(x', x') - 2\operatorname{Re} N(x, x')\right) .\end{aligned}$$

In particular (8), (9) of Withers [8] imply N is continuous in Ω^2

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so that by Note (i) the conclusion there and (12), (13) of [8] hold everywhere in Ω^2 .

(iii) *Suppose* $\mu(\Omega) < \infty$, (Ω_0, T_0) *compact. Then* (14) *ensures that conditions* (1), (13) *hold; also the inequality*

$$\int \|N(x, s)-N(y, s)\|^2 d\mu(s) \le \mu(\Omega) \sup_{s\in\Omega_0} \|N(x, s)-N(y, s)\|^2$$

implies that (11) *holds when* (Ω_0, T_0) *is metrisable.*

(iv) *As* m *increases,* (11) *becomes weaker.*

Proof. This is virtually unaltered from that of Theorem 1 of [8], consisting of proving that (9) holds, by showing that

$$N(x, x) - \sum_1^q \lambda_i^{-1}|\phi_i(x)|^2$$

is real and non-negative where $\{\lambda_1, \ldots, \lambda_q\}$ includes all negative eigenvalues.

When trace $N(x, x)$ is bounded away from zero, (13) may be removed.

COROLLARY 1. *Suppose* N *satisfies the conditions of the theorem with* (13) *replaced by*

(18) $$0 < \text{trace } N(x, x) \text{ in } \Omega_0 .$$

Let $N_0(x, y) = K(x)N(x, y)K(y)$ *where* $K(x) = (\text{trace } N(x, x))^{-\frac{1}{2}}$. *Let* $d\mu_0(x) = \text{trace } N(x, x)d\mu(x)$, *and* $\phi_{0i}(x) = \phi_i(x)K(x)$. *Then* (10) *holds with* $N, \mu, \{\phi_i\}$ *replaced by* $N_0, \mu_0, \{\phi_{0i}\}$. *Hence if also*

(19) $$0 < \inf_{\Omega_0} \text{trace } N(x, x) ,$$

then (5) *holds.*

Proof. (14) implies $K(x) > 0$, so that

$$\lambda N\phi = \phi \iff \lambda \int N_0(x, y)\phi_0(y)d\mu_0(y) = \phi_0(x) ,$$

where $\phi_0 = K\cdot\phi$. Also $N_{0m}(x, y) = K(x)N_m(x, y)K(y)$. In some

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applications a version of the theorem is required when there are an infinite number of both positive and negative eigenvalues. For this purpose it is convenient to introduce the notion of the positive and negative parts of N.

DEFINITION. By the 'positive' and negative' parts of N we mean $N^+ = \frac{1}{2}(P+N)$, $N^- = \frac{1}{2}(P-N)$ where P is the kernel associated with the unique non-negative square-root of the operator N^2; (see, for example, [4], p. 265). N^+, N^- can also be characterised as the almost everywhere - unique positive semi-definite kernels satisfying (5) such that almost everywhere $(\mu\times\mu)$ $N = N^+ - N^-$ and N^+ is orthogonal to N^-: $N^+N^- = N^-N^+ = 0$. Continuity of N need not imply continuity of N^+, N^-; (see the example in [2], p. 118).

COROLLARY 2. *Suppose N satisfies* (1), (5), (11). *If N^+, N^- satisfy* (14), *and either* (13) *or* (19), *then* (10) *holds, and*

$$N^+(x, y) = \sum_{\lambda_i>0} \phi_i(x)\phi_i(y)^*/\lambda_i \quad \textit{almost everywhere in } \Omega^2,$$

$$N^-(x, y) = -\sum_{\lambda_i<0} \phi_i(x)\phi_i(y)^*/\lambda_i \quad \textit{almost everywhere in } \Omega^2,$$

the convergence being absolute and uniform.

Proof. By (11), $\{\phi_i\}$ are continuous. Also (1) implies (1) for N^+, N^-. Now, $\{\phi_i, \lambda_i > 0\}$ and $\{\phi_i, \lambda_i < 0\}$ are the eigenfunctions and eigenvalues of N^+, N^- to which we apply Theorem 1.

As in [8] we have the following expansions for the iterates and the resolvent.

COROLLARY 3. *Under the conditions of Theorem* 1 *or Corollary* 2 *for* $j \geq 1$,

(20) $$N_j(x, y) = \sum \lambda_i^{-j}\phi_i(x)\phi_i(y)^* \quad \textit{almost everywhere in } \Omega_0^{\,2},$$

(21) $$N(x, y, \lambda) = \sum (\lambda_i-\lambda)^{-1}\phi_i(x)\phi_i(y)^* \quad \textit{almost everywhere in } \Omega_0^{\,2},$$

if λ is not an eigenvalue.

The convergence in (20) and (21) is absolute and uniform in $\Omega_0^{\,2}$. If

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also (15) holds, then we may delete "almost everywhere" so that (7) and (6) hold.

Hence (15) and $\int \operatorname{tr} N^{+}(x, x) d\mu(x) < \infty$, $\int \operatorname{tr} N^{-}(x, x) d\mu(x) < \infty$ imply

$$D(\lambda) = \prod_{1}^{\infty} \left(1-\lambda/\lambda_j\right) .$$

We may compare these conditions with those of

THEOREM 2. *If* (1), (5), (8), (11) *for* $m = 1$, *and* (15) *hold, then*

(22) $$N_j(x, y) \equiv \sum \lambda_i^{-j} \phi_i(x) \phi_i(y)^* \quad in \quad \Omega_0^{\,2} , \quad j \geq 2 ,$$

convergence being absolute and uniform, and hence (7) *and* (6) *hold.*

Proof. By (11) for $m = 1$, the right hand side of (22) is continuous in y for all x in Ω_0 .

For $j \geq 1$ let $E_j = \int \|N_j(s, x)\|^2 d\mu(s)$. Then $E_{j+1} \leq E_j \iint \|N\|^2$, so that by (8), E_j is bounded, $j \geq 1$. For $j \geq 2$,

$$\|N_j(y, x) - N_j(y', x)\|^2 = \left\| \int \left(N(y, s) - N(y', s)\right) N_{j-1}(s, x) d\mu(s) \right\|^2 \leq$$

$$\leq \int \|N(y, s) - N(y', s)\|^2 d\mu(s) \cdot E_{j-1} .$$

Hence by (11) for $m = 1$, the left hand side of (22) is continuous in y for all x in Ω_0 .

Hence by (6) of Withers [8], and (15), (22) holds.

Finally we give another mean-convergence theorem (*cf.* [3], p. 138)

THEOREM 3. *If* (1), (2), (8), (15) *hold and*

(23) *for some* x *in* Ω_0 *and some* $m \geq 1$, Re trace $N_{2m}(x, y)$ *and* trace $N_{2m}(y, y)$ *are continuous at* $y = x$,

then for x, m *as in* (23),

$$(24) \quad \int \left\| N_m(x, y) - \sum_1^q \phi_i(x)\phi_i(y)^*\lambda_i^{-m} \right\|^2 d\mu(y) =$$

$$= \text{trace } N_{2m}(x, x) - \sum_1^q |\phi_i(x)|^2 \lambda_i^{-2m} \to 0 \text{ as } q \to \infty .$$

Proof. $\{\phi_i\}$ are continuous at x by (23) so that by the generalised form of (6) of [8] and (23), $\text{Re trace } N_{2m}(x, y) - \text{Re} \sum_1^\infty \phi_i(y)^*\phi_i(x)\lambda_i^{-2m}$ is continuous at $y = x$ and equals 0 almost everywhere (μ). Hence (15) implies (24).

4. Separability

We have assumed throughout that

$$L_2 = \left\{ f : (\Omega, A, \mu) \to C^n, \ f \ \mu\text{-measurable}, \ \int |f|^2 d\mu < \infty \right\}$$

is separable. Here we note some conditions for this to be so.

THEOREM 4. *If either*

(a) $X = (\Omega, T)$ *is a locally compact, separable and metrisable space,* A *contains the compact sets, and* $\mu(K) < \infty$ *for* K *compact; or*

(b) $\Omega \in A$, A *is the* σ*-algebra generated by some countable subset of* A, *and* μ *is* σ*-finite;*

then L_2 *is separable.*

Proof. *(a)* follows from (13.11.6) of Dieudonné [1]. *(b)* follows from Problems 5, 6, p. 381 of Taylor [7].

By *(b)* we have

COROLLARY 4. *If* $\Omega \subset C^p$,

$$A = \{\Omega \cap B : B \text{ a Lebesgue-measurable set in } C^p\} ,$$

$\psi : C^p \to R^+$ *a Lebesgue-measurable function,* $d\mu(x) = \psi(x)dx$, *then* L_2 *is separable.*

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