

Apolar Schemes of Algebraic Forms

Jaydeep Chipalkatti

Abstract. This is a note on the classical Waring’s problem for algebraic forms. Fix integers (n, d, r, s) , and let Λ be a general r -dimensional subspace of degree d homogeneous polynomials in $n+1$ variables. Let \mathcal{A} denote the variety of s -sided polar polyhedra of Λ . We carry out a case-by-case study of the structure of \mathcal{A} for several specific values of (n, d, r, s) . In the first batch of examples, \mathcal{A} is shown to be a rational variety. In the second batch, \mathcal{A} is a finite set of which we calculate the cardinality.

1 Introduction

We begin with a classical example to illustrate the theme of this paper. Let F_1, F_2 be general quadratic forms in variables x_0, \dots, x_n , with coefficients in \mathbf{C} . It is then possible to diagonalize the F_i simultaneously (see [16, Chapter 22]), *i.e.*, one can find linear forms L_1, \dots, L_{n+1} such that

$$F_i = c_{i1}L_1^2 + \dots + c_{i(n+1)}L_{n+1}^2$$

for $i = 1, 2$ and some constants $c_{ij} \in \mathbf{C}$. Moreover, up to rescaling there is a unique choice for the set $\{L_1, \dots, L_{n+1}\}$. This result naturally leads to similar questions about forms of higher degree, where much less is known in general.

Now assume that F_1, \dots, F_r are forms of degree d in x_0, \dots, x_n . Let $Z = \{L_1, \dots, L_s\}$ be a collection of linear forms in the x_i , such that it is possible to write

$$F_i = c_{i1}L_1^d + \dots + c_{is}L_s^d, \quad 1 \leq i \leq r,$$

for some constants $c_{ij} \in \mathbf{C}$. In nineteenth century terminology (introduced by Reye), Z is then called a polar s -hedron (*polar s -seit*) of the $\{F_i\}$. It corresponds to a collection of hyperplanes in \mathbb{P}^n which stands in some geometric relation to the system of hypersurfaces defined by the F_i . The precise nature of this relation is very sensitive to the values (n, d, r, s) , but in any event it is invariant under the automorphisms of \mathbb{P}^n .

For instance, in the example above, let Π_i be the hyperplane defined by $L_i = 0$. Then the $n+1$ points

$$P_k = \Pi_1 \cap \dots \cap \hat{\Pi}_k \cap \dots \cap \Pi_{n+1}, \quad (k = 1, \dots, n+1)$$

are exactly the vertices of the singular quadrics belonging to the pencil

$$\{F_1 + \lambda F_2 = 0\}_{\lambda \in \mathbb{P}^1}.$$

Received by the editors March 7, 2004.
 AMS subject classification: 14N05, 14N15.
 Keywords: Waring’s problem, apolarity, polar polyhedron.
 ©Canadian Mathematical Society 2006.

1.1 Summary of Results

Fix degree d forms $\{F_1, \dots, F_r\}$ as above. Then the polar s -hedra of this collection move in an algebraic family, denoted by \mathcal{A} . (See Definition 2.5 *et seq.* for the precise statement.) In this note we deduce results about the birational structure of \mathcal{A} for several specific quadruples (n, d, r, s) , in each case assuming that the F_i are chosen generally. A parameter count shows that the dimension of the variety \mathcal{A} is “expected” to be $s(n+r) - r\binom{n+d}{d}$ (more on this in Section 2 below). For the quadruples

$$(2, 4, 2, 8), (2, 3, 4, 7), (3, 2, 6, 7) (2, 3, 7, 8),$$

we show that \mathcal{A} is a rational variety of expected dimension. For the quadruples

$$(2, 3, 8, 8), (3, 2, 7, 7), (2, 4, 3, 9), (3, 3, 2, 8), (2, 3, 3, 6),$$

the variety \mathcal{A} is expected to be (and is) a finite set of points; in each case we determine its cardinality. The calculation for $(2, 3, 3, 6)$ was done by Franz London over a century ago; we have given a more rigorous and modern version of his proof.

Along the way, we deduce some miscellaneous results for the quadruples

$$(2, 3, 2, 6), (2, 4, 2, 8), (3, 2, 3, 9).$$

For instance, the result for $(2, 3, 2, 6)$ says the following: let F_1, F_2 be two general ternary cubics and E a smooth planar cubic curve apolar to F_1, F_2 (in the sense explained below). Then E passes through exactly three sextuples in \mathcal{A} .

In each of the cases above, there is a specific feature of the free resolution of s general points in \mathbb{P}^n which is exploited to deduce the answer. Hence we require a technical condition on the polar s -hedra, namely that they be “resolution-general” (in the sense of Definition 2.4). Although the specific technique used depends on the case at hand, two general themes are identifiable: the geometry of associated points and intersection theory on symmetric products of elliptic curves. I do not know of any technique which would apply uniformly to all (n, d, r, s) .

This subject is broadly referred to as “reduction to canonical form” or “Waring’s problem for algebraic forms”—see [2, 14, 19] for an introduction and further references. The paper [22] is an excellent compendium of known results about the structure of \mathcal{A} when $r = 1$. For a discussion of ternary cubics (the case $n = 2, d = 3$), see [21, 23].

2 Preliminaries

In this section we establish notation and describe the basic set-up of apolarity. The proofs may be found in [19], also see [6, 7, 14, 18].

The base field is \mathbb{C} . Let V be an $(n + 1)$ -dimensional \mathbb{C} -vector space and consider the symmetric algebras

$$R = \bigoplus_{i \geq 0} \text{Sym}^i V^*, \quad S = \bigoplus_{j \geq 0} \text{Sym}^j V.$$

If $\underline{u} = \{u_0, \dots, u_n\}$ and $\underline{x} = \{x_0, \dots, x_n\}$ are dual bases of V^* and V respectively, then

$$R = \mathbb{C}[u_0, \dots, u_n], \quad S = \mathbb{C}[x_0, \dots, x_n].$$

There are internal product maps $R_i \otimes S_j \xrightarrow{f_{ij}} S_{j-i}$ (see e.g. [13, p. 476]), so S acquires the structure of a graded R -module. With the identification $u_\ell = \frac{\partial}{\partial x_\ell}$, the internal product can be seen as partial differentiation: if $\varphi \in R_i$ and $F \in S_j$, then $f_{ij}(\varphi \otimes F)$ is obtained by applying the differential operator $\varphi(\frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_n})$ to $F(x_0, \dots, x_n)$. We will write $\varphi \circ F$ for $f_{ij}(\varphi \otimes F)$.

Let $\Lambda \subseteq S_d$ be an r -dimensional subspace of degree d forms in the \underline{x} , defining a point in the Grassmannian $G(r, S_d)$. Let

$$(1) \quad \Lambda^\perp = \{\varphi \in R : \varphi \circ F = 0 \text{ for every } F \text{ in } \Lambda\}.$$

Then $\Lambda^\perp = \bigoplus_i \Lambda_i^\perp$ is a graded ideal in R , with $\Lambda_i^\perp = R_i$ for $i > d$. (It follows that the quotient R/Λ^\perp is an artin level algebra of socle degree d and type r , but we will not use this explicitly.)

For $i \leq d$, the codimension of Λ_i^\perp in R_i equals the dimension of the image of the internal product map

$$R_{d-i} \otimes \Lambda \rightarrow S_i$$

Hence

$$(2) \quad \dim \Lambda_i^\perp \geq \max\{0, \dim R_i - r \cdot \dim R_{d-i}\}.$$

Equality always holds for $i = d$, and it holds for all $i < d$ if Λ is a general point in $G(r, S_d)$.

We will commonly use geometric language in the sequel, e.g., if $n = 3$, then a point in $G(2, S_4)$ will be called a *pencil of planar quartics*.

Remark 2.1 If $\varphi \circ F = 0$, then φ, F were classically said to be *apolar to each other*; and sometimes the entire set-up is called *apolarity*. Of course, all of the above is subsumed in the statement that R, S are dual Hopf algebras such that all structure maps are $SL(V)$ -equivariant.

Henceforth we set $\mathbb{P}^n = \mathbb{P}S_1 = \text{Proj } R$. Usually $Z \subseteq \mathbb{P}^n$ will denote a closed subscheme with (saturated) ideal $I_Z \subseteq R$.

Definition 2.2 (cf. [19, Definition 5.1]) The scheme Z is said to be *apolar* to Λ , if $I_Z \subseteq \Lambda^\perp$.

The point of the definition is the following:

Theorem 2.3 (Reye) *If Z consists of s distinct points $\{L_1, \dots, L_s\} \subseteq \mathbb{P}^n$, then Z is apolar to Λ if and only if $\Lambda \subseteq \text{span}\{L_1^d, \dots, L_s^d\}$.*

We would like to consider the family of such Z , but for technical reasons, we single out those schemes whose ideals are well-behaved.

Definition 2.4 A (zero-dimensional) length s scheme $Z \subseteq \mathbb{P}^n$ will be called *resolution-general*, if the graded Betti numbers in the minimal resolution of I_Z are the same as those in the resolution of s general points.

For instance, a length 7 subscheme $Z \subseteq \mathbb{P}^2$ is resolution-general iff its minimal resolution looks like

$$0 \rightarrow R(-5) \oplus R(-4) \rightarrow R(-3)^3 \rightarrow R \rightarrow R/I_Z \rightarrow 0.$$

In particular, Z does not lie on a conic.

Definition 2.5 A zero-dimensional scheme $Z \subseteq \mathbb{P}^n$ will be called a *polar polyhedron* of Λ , if it is apolar to Λ and resolution-general.

Let $\text{Hilb}(s, \mathbb{P}^n)$ be the Hilbert scheme parametrising length s subschemes of \mathbb{P}^n . Let $\mathcal{A}(s, \Lambda)$ denote the set of polar s -hedra of Λ , it is then a constructible subset of $\text{Hilb}(s, \mathbb{P}^n)$. We will write \mathcal{A} for $\mathcal{A}(s, \Lambda)$ if no confusion is likely.

Remark 2.6 In the literature there is no unanimity on the definition of a “polar polyhedron”. In particular the approaches in [6] and [22] are different from ours and from each other. It is understood that if $Z = \{L_1, \dots, L_s\}$ are s general points, then morally Z should count as a polar s -hedron of any $\Lambda \subseteq \text{span}\{L_i^d\}$. However, it is not obvious which degenerations of Z should be allowed, and it seems that (within reason) we should tailor our definition to the specific problem at hand. Many of our results depend on a free resolution of I_Z , and hence “resolution-general” seems to be the most suitable notion. This issue never arises in [21], because there it is tacitly assumed that all geometric configurations are nondegenerate.

If $\mathcal{A}(s, \Lambda)$ is nonempty, so is $\mathcal{A}(t, \Lambda)$ for any $t > s$. It is the case that every Λ in $G(r, S_d)$ admits a polar $\binom{n+d}{d}$ -hedron. An elementary parameter count (see [2]) shows that a general Λ in $G(r, S_d)$ will admit a polar s -hedron only if

$$(3) \quad s \geq \frac{r \binom{n+d}{d}}{n+r}.$$

Definition 2.7 A quadruple (n, d, r, s) which satisfies (3) is said to be *nondegenerate*, if a general Λ admits a polar s -hedron.

A quadruple satisfying (3) is degenerate if the set $\{\Lambda : \mathcal{A}(s, \Lambda) \neq \emptyset\}$ fails to be dense in $G(r, S_d)$. Very few such examples are known (see [2] for the list), but none of them is without its geometric peculiarity. In general it is not trivial to prove that a particular quadruple is nondegenerate.

For $r = 1$, we have the following classification theorem by Alexander and Hirschowitz.

Theorem 2.8 (see [18]) *Assuming $r = 1$ and $d > 2$, the only degenerate cases are $(n, d, s) = (2, 4, 5), (3, 4, 9), (4, 4, 14)$ and $(4, 3, 7)$.*

For $r > 1$ we have the following results by Dionisi and Fontanari.

Theorem 2.9 *Assume $r > 1$. Then*

- (i) *for $n = 2$, the only degenerate quadruple is $(2, 3, 2, 5)$;*
- (ii) *there are no degenerate quadruples with $r \geq n + 1$.*

The proofs may be found in [4, 11] respectively. Part (i) was claimed by Terracini [26], but his proof is obscure.

If (n, d, r, s) is nondegenerate, then with a slight abuse of notation we will write \mathcal{A} for $\mathcal{A}(s, \Lambda)$, where Λ is understood to be a general point of $G(r, S_d)$. It has dimension $s(n + r) - r \binom{n+d}{d}$.

3 Associated Systems of Points

Recall ([9, p. 313]) that if Γ is a zero-dimensional Gorenstein scheme, then any closed subscheme $\Gamma' \subseteq \Gamma$ has a residual scheme $\Gamma'' \subseteq \Gamma$, such that

$$\deg \Gamma' + \deg \Gamma'' = \deg \Gamma.$$

In particular this applies if Γ is a (global) complete intersection in \mathbb{P}^n , which is the only case we will need.

Now let Λ denote a general pencil of planar quartics. Then $\mathcal{A} = \mathcal{A}(8, \Lambda)$ is 2-dimensional; we will show that it is rational. Every $Z \in \mathcal{A}$ has a Hilbert–Burch resolution

$$0 \rightarrow R(-5)^2 \xrightarrow{\mu} R(-4) \oplus R(-3)^2 \rightarrow R \rightarrow R/I_Z \rightarrow 0.$$

(See [3] for the basic theory behind the Hilbert–Burch theorem.) In particular $\dim(I_Z)_3 = 2$, so Z has an associated point $\alpha(Z)$, defined to be the residual intersection of cubics passing through Z . The matrix of the map μ has the form

$$(4) \quad M = \begin{bmatrix} \underline{2} & \underline{2} & \underline{1} \\ \underline{2} & \underline{2} & \underline{1} \end{bmatrix},$$

with the convention that \underline{j} stands for a degree j form.

Theorem 3.1 *Let Λ be a general pencil of planar quartics. Then the morphism $\alpha: \mathcal{A} \rightarrow \mathbb{P}^2$ admits a rational inverse, hence \mathcal{A} is a rational surface.*

Proof Fix a general point in the image of α , by change of coordinates we assume it to be $P = [0, 0, 1]$. We would like to show that there is a *unique* resolution general length 8 scheme Z with associated point P .

Now P is defined by the vanishing of the rightmost column in (4), hence, after row-operations, M can be brought into the form

$$M = \begin{bmatrix} q_1 & q_2 & u_0 \\ q_3 & q_4 & u_1 \end{bmatrix}, \quad q_i \in R_2.$$

We start with the 24-dimensional vector space of 2×2 matrices

$$V_1 = \left\{ N = \begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix} : q_i \in R_2 \right\}.$$

For $N \in V_1$, write

$$(5) \quad \theta_N = u_1q_1 - u_0q_3, \theta'_N = u_1q_2 - u_0q_4, \omega_N = q_1q_4 - q_2q_3,$$

and let J_N be the ideal generated by $\theta_N, \theta'_N, \omega_N$. Thus V_1 is a parameter space for all Hilbert–Burch matrices as above. For a dense open set of elements N in V_1 , the ideal J_N defines a planar length 8 scheme.

We let $\text{GL}_2(\mathbb{C})$ act on V_1 by right multiplication, *i.e.*, for $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \text{GL}_2$, and N as above,

$$(6) \quad Ng = \begin{bmatrix} q_1\alpha + q_2\gamma & q_1\beta + q_2\delta \\ q_3\alpha + q_4\gamma & q_3\beta + q_4\delta \end{bmatrix}$$

Define $V_2 = \{N \in V_1 : \theta_N, \theta'_N \in \Lambda_3^\perp\}$, which is a 12-dimensional subspace of V_1 . (If $F \in \Lambda$, then $\theta_N \circ F = \theta'_N \circ F = 0$ is a set of six linear equations. In all, V_2 is defined by 12 linear equations which are independent for a general Λ , hence $\dim V_2 = 12$.) Inside V_2 , there is a 6-dimensional subspace

$$V_3 = \left\{ \begin{bmatrix} au_0 & bu_0 \\ au_1 & bu_1 \end{bmatrix} : a, b \in R_1 \right\}.$$

(Since $\theta_N, \theta'_N = 0$ for $N \in V_3$, the containment $V_3 \subseteq V_2$ is clear.) Form the 6-dimensional space $W = V_2/V_3$. For $N \in V_2$, write $[N]$ for the corresponding point in the projective space $\mathbb{P}W \simeq \mathbb{P}^5$. Since $V_3 \subseteq V_2 \subseteq V_1$ are inclusions of GL_2 -modules, W is also a (right) GL_2 -module; in particular PGL_2 acts on $\mathbb{P}W$. The point of this construction lies in the following lemma:

Lemma 3.2

- (i) If $N, \tilde{N} \in V_2$ are such that $[N], [\tilde{N}]$ lie in the same PGL_2 -orbit of $\mathbb{P}W$, then $J_N = J_{\tilde{N}}$.
- (ii) Let $Z \in \alpha^{-1}(P)$. Consider two minimal resolutions of I_Z with corresponding Hilbert–Burch matrices M, \tilde{M} , and let N, \tilde{N} denote their leftmost 2×2 submatrices. Then $[N], [\tilde{N}]$ lie in the same PGL_2 -orbit in $\mathbb{P}W$.

Proof By straightforward calculation,

$$(7) \quad \theta_{Ng} = \alpha\theta_N + \gamma\theta'_N, \theta'_{Ng} = \beta\theta_N + \delta\theta'_N, \omega_{Ng} = \det(g)\omega_N,$$

so $J_N = J_{Ng}$. Let $Q = \begin{bmatrix} au_0 & bu_0 \\ au_1 & bu_1 \end{bmatrix} \in V_3$. Then

$$(8) \quad \theta_{N+Q} = \theta_N, \theta'_{N+Q} = \theta'_N, \omega_{N+Q} = \omega_N - a\theta'_N + b\theta_N,$$

so $J_{N+Q} = J_N$. This proves (i).

Any two minimal resolutions of I_Z are isomorphic (see [8, Section 20.1]), which translates into the statement that N and some GL_2 -translate of \tilde{N} must differ by an element of V_3 . This says that $[N], [\tilde{N}]$ must be in the same orbit, which is (ii). ■

Now define a subvariety $Y = \{[N] \in \mathbb{P}W : \omega_N \circ \Lambda = 0\} \subseteq \mathbb{P}W$. Formulae (8) imply that $\omega_{N+Q} \circ \Lambda = 0 \Leftrightarrow \omega_N \circ \Lambda = 0$ (since $\theta_N \circ \Lambda = \theta'_N \circ \Lambda = 0$), hence this definition is meaningful. The inclusion $Y \subseteq \mathbb{P}W$ is a PGL_2 -stable by formulae (7). By the previous lemma, each $Z \in \alpha^{-1}(P)$ defines an orbit $\Omega_Z \subseteq Y$. The PGL_2 -stabilizer of a point in Ω_Z is trivial, hence $\dim \Omega_Z = 3$. The union of $\{\Omega_Z\}_{Z \in \alpha^{-1}(P)}$ fills a dense open subset in Y . Hence it is enough to show that Y contains only one three-dimensional component, this will imply that $\alpha^{-1}(P)$ is singleton. Define

$$\Gamma_1 = \left\{ [N] \in \mathbb{P}W : N = \begin{bmatrix} q_1 & 0 \\ q_3 & 0 \end{bmatrix} \text{ for some } q_i \text{ and } \theta_N \circ \Lambda = 0 \right\},$$

$$\Gamma_2 = \left\{ [N] \in \mathbb{P}W : N = \begin{bmatrix} 0 & q_2 \\ 0 & q_4 \end{bmatrix} \text{ for some } q_i \text{ and } \theta'_N \circ \Lambda = 0 \right\},$$

each of which is a copy of \mathbb{P}^2 in Y . Define a birational map $h: \Gamma_1 \rightarrow \Gamma_2$ as follows. Let $[N] \in \Gamma_1$, then there is a 4-dimensional family of solutions (q_2, q_4) to the equations

$$\theta'_N \circ \Lambda = \omega_N \circ \Lambda = 0.$$

(This is so because q_2, q_4 together depend upon 12 parameters and there are 8 equations.) However, if (q_2, q_4) is one such solution, then $(q_2 + au_0, q_4 + au_1)$ is also one for any $a \in R_1$, and this accounts for all the solutions. Hence the class in $\mathbb{P}W$ of the matrix $\begin{bmatrix} 0 & q_2 \\ 0 & q_4 \end{bmatrix}$ is uniquely determined. We define $h([N])$ to be this class. (The reader should verify that this definition is independent of the choice of coset representative for $[N]$.) Now a general element in Y can be written as a sum $[N] + [h(N)]$ for $[N] \in \Gamma_1$, i.e., the ruled join of Γ_1, Γ_2 along h contains a dense open subset of Y . Since this join is irreducible (it is the image of the Segre imbedding $\mathbb{P}^2 \times \mathbb{P}^1 \subseteq \mathbb{P}^5$), we are done. ■

The argument for the following proposition is similar. As before, $(2, 3, 4, 7)$ is nondegenerate by Theorem 2.9.

Proposition 3.3 *Let Λ be a general web of planar cubics. Then $\mathcal{A}(7, \Lambda)$ is a rational surface.*

Proof The Hilbert–Burch matrix for $Z \in \mathcal{A}$ is

$$\begin{bmatrix} \underline{1} & \underline{1} & \underline{1} \\ \underline{2} & \underline{2} & \underline{2} \end{bmatrix}.$$

For a general Z , the linear forms in the top row are independent, hence after column operations we can assume the matrix to be

$$\begin{bmatrix} u_0 & u_1 & u_2 \\ q_0 & q_1 & q_2 \end{bmatrix}, \quad q_i \in R_2.$$

Let V_1 denote the 18-dimensional vector space $\{[q_0, q_1, q_2] : q_i \in R_2\}$ and V_2 the 3-dimensional subspace $\{[au_0, au_1, au_2] : a \in R_1\}$. Let $W = V_1/V_2$. Then the 12 equations $\{(u_jq_j - u_iq_i) \circ \Lambda = 0\}$ cut out a 2-plane in $\mathbb{P}W$ which is birational to \mathcal{A} . ■

Now let $(n, d, r, s) = (3, 2, 6, 7)$; we will show that \mathcal{A} is birational to the projective 3-space. The ideal of every $Z \in \mathcal{A}$ is generated by three quadrics and a cubic. The associated point $\alpha(Z)$ is defined to be the residual intersection of the quadrics through Z .

Proposition 3.4 *Let Λ be a general point of $G(6, S_2)$. Then the map $\alpha: \mathcal{A} \rightarrow \mathbb{P}^3$ is birational.*

Proof Let Z be a resolution-general scheme of length 7. It is apolar to Λ iff the three generating quadrics lie in Λ_2^\perp .

Let P be a general point of \mathbb{P}^3 , and let $W \subseteq \Lambda_2^\perp$ be the 3-dimensional subspace of forms vanishing at P . Then W defines a length 8 scheme Y . Now the residual scheme of P in Y is the only point of \mathcal{A} mapping to P . ■

Remark 3.5 The case $(2, 3, 7, 8)$ has a similar geometry, where \mathcal{A} is birational to \mathbb{P}^2 . For $(2, 3, 8, 8)$ (resp. $(3, 2, 7, 7)$), \mathcal{A} is a finite set consisting of 9 (resp. 8) points.

4 Symmetric Products of Elliptic Curves

For the examples in this section, the determination of \mathcal{A} reduces to an intersection-theoretic calculation on the symmetric product of an elliptic curve. If E is a smooth projective curve, then $E^{(m)}$ will denote its m -th symmetric product. This is a smooth projective variety whose points are naturally seen as effective degree m divisors on E .

Let Λ be a general net of planar quartics. Since $(2, 4, 3, 9)$ is nondegenerate, \mathcal{A} is a finite set. In the next theorem we calculate its cardinality.

Theorem 4.1 *Let Λ be a general net of planar quartics. Then Λ admits 4 polar enneahedra.*

Proof The ideal of $Z \in \mathcal{A}$ is generated by one cubic and 3 quartics. The space Λ_3^\perp is one-dimensional, i.e., Λ is apolar to a unique cubic curve $E \subseteq \mathbb{P}^2$. Since Λ is general, we may (and will) assume that E is smooth. If H denotes the hyperplane divisor on E , then we have an identification $H^0(E, 4H) = R_4/(I_E)_4$. This is a 12-dimensional space, denoted U .

Let $W = \Lambda_4^\perp/(I_E)_4$, which is a 9-dimensional space inside U . Every scheme $Z \subseteq \mathbb{P}^2$ of length 9 which is apolar to Λ is contained in E , and thus defines an effective divisor on E . Then the 3-dimensional space $H^0(E, 4H - Z)$, which is *a priori* inside U , is in fact contained in W . The argument shows that the following diagram is a

fibre square:

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & G(3, W) \\ \downarrow & & \downarrow i_1 \\ E^{(9)} & \xrightarrow{i_2} & G(3, U) \end{array}$$

Here i_1 is the natural inclusion and $i_2(Z) = H^0(E, 4H - Z)$. Since the images of both inclusions have complementary codimensions, it is enough to take the intersection of their classes inside $H^*(G(3, U), \mathbf{Z})$ in order to calculate the degree of \mathcal{A} as a zero-cycle.

Conventions The notation for Schubert calculus follows [12, Section 14.7]. We refer to [1] for some basic cohomological calculations on curves. If X_1, X_2 are varieties, then denote projections by $\pi_i: X_1 \times X_2 \rightarrow X_i$. All cohomology is with \mathbf{Z} -coefficients. If α is a class in $H^*(X_1)$ (resp. $H^*(X_2)$), then its pullback to $H^*(X_1 \times X_2)$ is denoted $\alpha \otimes 1$ (resp. $1 \otimes \alpha$). Cup product is written as juxtaposition.

Firstly, we should find the rank 3 subbundle of $U \otimes \mathcal{O}_{E^{(9)}}$ which defines the inclusion i_2 . Let Δ denote the universal divisor on $E^{(9)} \times E$ (see [1, Chapter IV]), so that $\Delta|_{\{Z\} \times E} = Z \times E$. Define a line bundle $\mathcal{M} = \pi_2^*(\mathcal{O}_E(4H)) \otimes \mathcal{O}(-\Delta)$ on $E^{(9)} \times E$. Applying π_{1*} to the inclusion

$$\mathcal{M} \subseteq \pi_2^*(\mathcal{O}_E(4H)),$$

we have

$$(\mathcal{G} =) \pi_{1*}(\mathcal{M}) \subseteq U \otimes \mathcal{O}_{E^{(9)}}.$$

A moment’s reflection will show that i_2 is induced by the last inclusion.

The image of i_2 has class $\{3, 3, 3\}$. Hence by the Jacobi–Trudi identity, the class of \mathcal{A} in $H^{18}(E^{(9)})$ is given by $c_3(\mathcal{G}^*)^3$, which we now calculate.

The Cohomology Rings of E and $E^{(9)}$ Let $\delta_1, \delta_2 \in H^1(E)$ be a symplectic basis. It will then generate $H^*(E)$. The product $\eta = \delta_1\delta_2 \in H^2(E)$ is the class of a point.

Let \mathcal{L} be a Poincaré line bundle ([1, Chapter IV]) on $E \times \text{Pic}^9(E)$, then $\mathcal{E} = \pi_{2*}(\mathcal{L})$ is a rank 9 bundle on $\text{Pic}^9(E)$. Fix an isomorphism $\text{Pic}^9(E) = E$, then by the calculation of [1, p. 336], $c_1(\mathcal{E}) = -\eta$. Now let $\xi = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) \in H^2(\mathbb{P}\mathcal{E})$. With the identification $\mathbb{P}\mathcal{E} = E^{(9)}$, the ring $H^*(E^{(9)})$ is generated by ξ and (the pullbacks of) δ_1, δ_2 , subject to the relation $\xi^9 = \xi^8\eta$.

The Chern Class of \mathcal{M} and G-R-R Let

$$-\gamma = (\delta_1 \otimes 1)(1 \otimes \delta_2) - (\delta_2 \otimes 1)(1 \otimes \delta_1),$$

a class in $H^{1,1}(E^{(9)} \times E)$. By [1, p. 337-338],

$$c_1(\mathcal{O}(\Delta)) = \xi \otimes 1 + \gamma + 9(1 \otimes \eta),$$

hence

$$c_1(\mathcal{M}) = -\xi \otimes 1 - \gamma + 3(1 \otimes \eta).$$

Now we apply Grothendieck–Riemann–Roch to \mathcal{M} along the projection $E^{(9)} \times E \xrightarrow{\pi_1} E^{(9)}$. Thus

$$\text{ch}(\pi_{1*}\mathcal{M}) \text{td}(E^{(9)}) = \pi_{1*}(\text{ch}(\mathcal{M}) \text{td}(E^{(9)} \times E)).$$

Since $R^i\pi_{1*}\mathcal{M} = 0$ for $i > 0$ and $\text{td}(E) = 1$, this simplifies to

$$\text{ch}(\mathcal{G}) = \pi_{1*}(e^{c_1(\mathcal{M})}).$$

Let n_i denote the i -th Newton class of \mathcal{G} (i.e., the sum of i -th powers of the Chern roots of \mathcal{G}), then $\text{ch}(\mathcal{G}) = \sum_{i \geq 0} n_i/i!$. Now we expand the exponential series, and apply π_{1*} term by term, to get

$$\begin{aligned} n_0 &= 3, & n_1 &= \frac{1}{2}(-6\xi - 2\eta), \\ n_2 &= \frac{1}{3}(9\xi^2 + 6\xi\eta), & n_3 &= \frac{1}{4}(-12\xi^3 - 12\xi^2\eta). \end{aligned}$$

Then

$$c_3(\mathcal{G}) = \frac{1}{6}n_1^3 - \frac{1}{2}n_1n_2 + \frac{1}{3}n_3 = -(\xi^3 + \xi^2\eta).$$

Hence finally

$$c_3(\mathcal{G}^*)^3 = (\xi^3 + \xi^2\eta)^3 = 4\xi^8\eta.$$

Since $\xi^8\eta$ is the class of a point on $E^{(9)}$, we deduce that \mathcal{A} has degree 4.

In order to show that \mathcal{A} is reduced and hence consists of 4 geometric points, we use Kleiman’s transversality result (see [17, Theorem 10.8]). We can reformulate the entire construction in the following way: start with a smooth E and hence U , then specifying a codimension 3 subspace $W \subseteq U$ is tantamount to specifying Λ . Since $G(3, U)$ is a homogeneous space for $\text{GL}(U)$, the intersection is transversal for a general W , so \mathcal{A} is reduced. ■

The next example is that of a pencil of cubic surfaces. We need to show that $(3, 3, 2, 8)$ is nondegenerate, the proof is given in Section 6.

Proposition 4.2 *Let Λ be a general pencil of cubic surfaces. Then Λ admits 3 polar octahedra.*

Proof The calculation is very similar to Theorem 4.1. The ideal of 8 general points in \mathbb{P}^3 is generated by 2 quadrics and 4 cubics. Now Λ_2^\perp is 2-dimensional, hence generates the ideal of a smooth normal elliptic quartic $E \subseteq \mathbb{P}^3$ apolar to Λ , and every $Z \in \mathcal{A}$ is in fact contained in E . Let

$$U = R_3/(I_E)_3, \quad W = \Lambda_3^\perp/(I_E)_3,$$

which are spaces of dimension 12, 10 respectively. Define i_1, i_2 as before, then the following diagram is a fibre square

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & G(4, W) \\ \downarrow & & \downarrow i_1 \\ E^{(8)} & \xrightarrow{i_2} & G(4, U) \end{array}$$

Now i_2 is induced by a rank 4 bundle \mathcal{G} on $E^{(8)}$. The class of \mathcal{A} in $E^{(8)}$ equals

$$c_4(\mathcal{G}^*)^2 = (\xi^4 + \xi^3\eta)^2 = 3\xi^7\eta.$$

The argument for transversality is the same as before. ■

Using similar calculations, we can give alternate proofs of the following results by Schlesinger [25, p. 212]). The original argument used ϑ -functions.

Proposition 4.3 (Schlesinger)

- (1) *Let Λ be a general pencil of planar cubics. Fix a general elliptic curve $E \subseteq \mathbb{P}^2$ apolar to Λ . Then there are 3 polar hexahedra of Λ which are contained in E .*
- (2) *Let Λ be a general pencil of planar quartics. Fix a general elliptic curve $E \subseteq \mathbb{P}^2$ apolar to Λ . Then there are 3 polar octahedra of Λ which are contained in E .*

Proof We will only prove (1), the argument for (2) is identical in essence. Recall that the ideal of 6 general planar points is generated by 4 cubics. Since $(2, 3, 2, 6)$ is nondegenerate¹, $\mathcal{A}(6, \Lambda)$ is 4-dimensional. Consider the incidence correspondence

$$\Phi \subseteq \mathcal{A} \times \mathbb{P}\Lambda_3^\perp, \quad \Phi = \{(Z, E) : Z \subseteq E\}.$$

The projection $\pi_1 : \Phi \rightarrow \mathcal{A}$ is generically a \mathbb{P}^3 -bundle, so $\dim \Phi = 7$. Fix a general elliptic curve E apolar to Λ , and consider the diagram

$$\begin{array}{ccc} & & G(3, \Lambda_3^\perp / (I_E)_3) \\ & & \downarrow i_1 \\ E^{(6)} & \xrightarrow{i_2} & G(3, R_3 / (I_E)_3) \end{array}$$

As usual, i_1 is the inclusion and $i_2(Z) = H^0(E, 3H - Z)$. Then $i_2(Z)$ lies in the image of i_1 , iff Z is apolar to Λ . Calculating as before, the product $[\text{image } i_1] \cdot [\text{image } i_2]$ equals thrice the class of a point. Hence $\pi_2^{-1}(E)$ must be nonempty. This implies that $\pi_2 : \Phi \rightarrow \mathbb{P}\Lambda_3^\perp (\simeq \mathbb{P}^7)$ is dominant. But then it is generically finite, hence for a general E , the fibre $\pi_2^{-1}(E)$ consists of 3 points. ■

It is shown in [2] (using a machine calculation) that $(5, 2, 3, 9)$ is nondegenerate. Now there is a (unique) elliptic sextic curve passing through 9 general points of \mathbb{P}^5 . (The classical reference is [24], also see [5] for a proof using Gale duality.) Hence if Λ is a general net of quadrics in \mathbb{P}^5 and Z a set of 9 general points apolar to Λ , then the elliptic sextic passing through Z is apolar to Λ .

Proposition 4.4 *Let Λ be a general net of quadrics in \mathbb{P}^5 . Fix a general elliptic sextic curve $E \subseteq \mathbb{P}^5$ apolar to Λ . Then there are 4 polar enneahedra of Λ which are contained in E .*

Proof Similar to above. Use the fact that the ideal of 9 general points (resp. an elliptic sextic curve) is generated by 12 (resp. 9) quadrics. ■

¹This is the smallest s possible, because $(2, 3, 2, 5)$ is degenerate by [2].

5 The (2, 3, 3, 6) Case

Now we come to London’s beautiful calculation in [21], where he determines the number of polar hexahedra of a general net of cubic curves. I have rewritten the proof so as to make it more transparent, but all the key ideas are already in the original.

Let Λ be such a net. By Theorem 2.9(i), Λ has a finite number of polar hexahedra. We will count them by setting up a correspondence on a certain elliptic curve.

- 5.1** We begin by motivating the constructions which are to follow. Say $\{F_1, F_2, F_3\}$ is a basis of Λ and $Z = \{L_1, \dots, L_6\}$ one of its polar hexahedra. We have expressions

$$F_j = c_{1j}L_1^3 + \dots + c_{6j}L_6^3, \quad j = 1, 2, 3.$$

Let $\psi \in R_2$ be the form which defines the conic passing through $\{L_2, \dots, L_6\} \subseteq \mathbb{P}S_1$. Since ψ annihilates L_2^3, \dots, L_6^3 , we have $\psi \circ F_j = \text{constant} \times L_1$ for every j , so $\psi \circ \Lambda$ is only a 1-dimensional vector space. It will be seen below (Section 5.2) that all ψ with this property lie on a curve. Similarly if $l_1, l'_1 \in R_1$ annihilate L_1 , then the six derivatives $\{l_1 \circ F_j, l'_1 \circ F_j : j = 1, 2, 3\}$ span only a 5-dimensional space. It will be seen below (Section 5.3) that all 2-dimensional spaces span $\{l_1, l'_1\} \subseteq R_1$ with this property lie on a curve, isomorphic to the previous one.

- 5.2** Now we come to the actual constructions. The symbol $(\leftarrow \rho)$ will appear frequently, it is explained in Remark 5.2. Consider the vector bundle morphism on $\mathbb{P}R_2 (= \mathbb{P}^5)$

$$f_{23}: \mathcal{O}_{\mathbb{P}^5}(-1) \otimes \Lambda \rightarrow S_1$$

coming from the internal product map of Section 2. Define the degeneracy locus $\Psi = \{\text{rank } f_{23} \leq 1\}$. For a general Λ , it is a degree 6 normal elliptic curve in $\mathbb{P}^5 (\leftarrow \rho)$. Note that $\Lambda_2^\perp = 0$ by the generality of Λ , so $\text{rank } f_{23}$ is exactly 1 at each $\psi \in \Psi$.

- 5.3** Now identify $\mathbb{P}S_1$ with the Grassmannian $G(2, R_1)$. The latter is equipped with a rank two tautological bundle $\mathcal{B} \subseteq R_1 \otimes \mathcal{O}_G$. The internal product f_{13} gives a morphism

$$f_{13}: \mathcal{B} \otimes \Lambda \rightarrow S_2$$

The locus $E = \{\text{rank } f_{13} \leq 5\} = \{\det f_{13} = 0\}$ is given by a section of $\mathcal{O}_{\mathbb{P}S_1}(3)$, hence it is a smooth $(\leftarrow \rho)$ degree 3 curve in $\mathbb{P}S_1$. By the generality of Λ , the rank of f_{13} is exactly 5 at every $L \in E (\leftarrow \rho)$.

- 5.4** We have an isomorphism

$$\alpha: E \rightarrow \Psi$$

defined as follows: let $L \in E$, and $U = L^\perp$. By hypothesis, the space $f_{13}(U \otimes \Lambda)$ is 5-dimensional, so it is annihilated by a unique form in $\mathbb{P}R_2$, we declare $\alpha(L)$ to be this form. It is clear that $f_{23}(\alpha(L) \otimes \Lambda)$ is only 1-dimensional (since U annihilates it), so $\alpha(L) \in \Psi$.

If Z is as in Section 5.1 above, then $\alpha(L_1)$ is the conic envelope containing the lines defined by L_2, \dots, L_6 .

- 5.5** Define a correspondence \mathbb{T} on E as follows: $(L, M) \in \mathbb{T}$ iff M lies on the conic defined by $\alpha(L)$. For a fixed L , there are 6 positions of M such that $(L, M) \in \mathbb{T}$. For a fixed M , the elements of Ψ which vanish at M lie on a hyperplane section of Ψ . Via α^{-1} , the points of this hyperplane section correspond to 6 positions of L . This shows that \mathbb{T} has degree $(6, 6)$ and valence zero.
- 5.6** By the general theory of correspondences (see [15, Section 2.5]), there are 12 elements in \mathbb{T} of the form (L, L) , they are called the *united points* of \mathbb{T} . Moreover $\mathbb{T}, \mathbb{T}^{-1}$ have 72 common points, *i.e.*, pairs (L, M) such that $(L, M), (M, L) \in \mathbb{T}$. Hence there are $72 - 12 = 60$ such pairs where (L, M) are distinct.

It is clear that starting from Z , the pairs (L_1, L_2) *etc.* are common to $\mathbb{T}, \mathbb{T}^{-1}$. The next lemma says that the implication is reversible.

Lemma 5.1 Assume $(L, M), (M, L) \in \mathbb{T}$, and $L \neq M$. Let the conics $\alpha(L), \alpha(M)$ intersect in $\{P_1, P_2, P_3, P_4\}$. Then $Z = \{L, M, P_1, \dots, P_4\}$ is a polar hexahedron of Λ .

Proof Recall that the ideal of 6 general points is generated by 4 cubics. Let $l, l' \in R_1$ be generators of L^\perp , and $m, m' \in M^\perp$. Consider the four cubic forms

$$\{l\alpha(L), l'\alpha(L), m\alpha(M), m'\alpha(M)\}.$$

They are linearly independent and each of them vanishes at all points of Z . Hence together they generate $(I_Z)_3$. Moreover, the definition of α implies that each of them annihilates Λ . Hence $I_Z \subseteq \Lambda^\perp$. ■

Now a polar hexahedron of Λ gives $2\binom{6}{2} = 30$ pairs (L_i, L_j) common to $\mathbb{T}, \mathbb{T}^{-1}$. Alternately, starting from a common point we can reconstruct a polar hexahedron as shown above. Hence, following London, we conclude that Λ has $60 \div 30 = 2$ polar hexahedra.

Remark 5.2 At several points in the proof we need to argue that our construction satisfies a certain “good” property, for instance Ψ has codimension 4 as expected and is smooth. This follows from the generality of Λ , as soon as we verify that it holds for a specific Λ . Such points are marked by (\neq) , and I have verified the property in question by a direct computer calculation for a general net in the span of

$$x_0^3, x_1^3, x_2^3, (x_0 + x_1 + x_2)^3, (x_0 - x_1 + x_2)^3, (x_0 - 2x_1 + 3x_2)^3.$$

This was carried out in Macaulay-2. For instance, to verify the last point in Section 5.3, we choose two basis elements with indeterminate entries for an element of $G(2, R_1)$, represent f_{13} by a matrix and check that the ideal defined by all 5×5 minors defines the empty scheme.

6 Nondegeneracy of (3, 3, 2, 8)

To prove this result, we will use the notion of a *grove*, which was introduced in [2]. The general definition is meaningful for any (n, d, r, s) , but we will formulate it only for the case at hand.

Let $\underline{p} = \{p_1, \dots, p_8\}$ (resp. $\underline{Q} = \{Q_1, \dots, Q_8\}$) be points in \mathbb{P}^1 (resp. in \mathbb{P}^3).

Definition 6.1 A *grove* for the data $\underline{p}, \underline{Q}$ is a linear system $\Gamma \subseteq \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$ of projective dimension (say) t , satisfying the following conditions:

- (1) The base locus of Γ contains all the Q_i ;
- (2) $t = 0$ or 1 ;
- (3) either $t = 0$ and the generator of Γ is singular at all Q_i , or $t = 1$ and there is an isomorphism $\gamma: \mathbb{P}^1 \rightarrow \Gamma$ such that for every i , the hypersurface $\gamma(p_i)$ is singular at Q_i .

Now [2, Theorem 2.6] says the following: the quadruple $(3, 3, 2, 8)$ is nondegenerate iff there does not exist a grove for *general* points $\underline{p}, \underline{Q}$ as above. Existence of a grove is an open property of $\underline{p}, \underline{Q}$ (loc. cit.), so it is enough to exhibit some collection of points which does not admit a grove. I concede that the definition of a grove is awkward, in defence one can only say that it is a proof-generated concept in the sense of Lakatos (see [20, Appendix 2]). We begin with a preliminary lemma.

Lemma 6.2 Let E be an elliptic curve and \mathcal{M} a line bundle on E of degree 4. Let Q_1, \dots, Q_8 be distinct points on E . Then it is possible to find points p_1, \dots, p_8 on \mathbb{P}^1 , such that there is no morphism $f: E \rightarrow \mathbb{P}^1$ satisfying the following conditions:

- A. $2 \leq \deg f \leq 4$, and if $\deg f = 4$ then $f^*(\mathcal{O}_{\mathbb{P}^1}(1)) \simeq \mathcal{M}$;
- B. the equality $f(Q_i) = p_i$ holds for at least $4 + \deg f$ values of i .

Proof Since $h^0(\mathcal{M}) = 4$, there are $\infty^4 g_4^1$'s coming from \mathcal{M} . However, modulo automorphisms of \mathbb{P}^1 there are ∞^5 octuples (p_1, \dots, p_8) . Hence for a general octuple, there is no such map of degree 4.

Similarly there are ∞^3 (resp. ∞^1) g_3^1 's (resp. g_2^1 's) on E . Since (B) imposes 4 (resp. 3) conditions in these cases, for a general choice of p_i none of the possibilities can hold. The lemma is proved. ■

Now w, x, y, z be the coordinates in \mathbb{P}^3 . Consider the normal elliptic quartic $E \subseteq \mathbb{P}^9$ defined by the two quadrics

$$G_1 = wx + xy + yz + zw, \quad G_2 = wy + xz.$$

Choose points

$$\begin{aligned} Q_1 &= [1, 0, 0, 0], & Q_3 &= [0, 0, 1, 0], & Q_5 &= [-1, 1, 1, 1], & Q_7 &= [1, 1, -1, 1], \\ Q_2 &= [0, 1, 0, 0], & Q_4 &= [0, 0, 0, 1], & Q_6 &= [1, -1, 1, 1], & Q_8 &= [1, 1, 1, -1], \end{aligned}$$

all lying on E , and the $p_i = [p_{i1}, p_{i2}]$ general in \mathbb{P}^1 .

Assume by way of contradiction that Γ is a grove for the data. If $t = 0$, then the generator of Γ contains at least 16 points of E (counting each Q_i as two), hence it contains E by Bézout's theorem.

Case 1 Assume that Γ contains E as a fixed component (with t possibly 0 or 1). Then Γ is spanned by two cubics of the form

$$C_1 = L_1G_1 + L_2G_2, \quad C_2 = L'_1G_1 + L'_2G_2,$$

where L_1, L'_1 etc. are linear forms and $p_{i1}C_1 + p_{i2}C_2$ is singular at Q_i for $i = 1, \dots, 8$. (The case $C_1 = (\text{constant}) \cdot C_2$ corresponds to $t = 0$.) An elementary linear algebra computation on the Jacobian matrix shows that this is impossible for general p_i .

Case 2 Assume that E is not contained in the base locus of Γ (hence necessarily $t = 1$). Let λ be the linear series obtained by restricting Γ to E and removing the base divisor $\sum Q_i$. Thus λ is a g^1_4 . Let $f: E \rightarrow \mathbb{P}^1$ be the corresponding morphism (of course, only well-defined up to automorphisms of \mathbb{P}^1). Let H denote the hyperplane divisor on E and $\mathcal{M} = \mathcal{O}_E(4H - \sum Q_i)$.

Case 2.1 If λ is base point free (i.e., if Γ has no additional base point on E away from $\sum Q_i$), then $\deg f = 4$ and $f^*(\mathcal{O}_{\mathbb{P}^1}(1)) \simeq \mathcal{M}$. Since the quartic $\gamma(p_i)$ passes doubly through Q_i , we have $f(Q_i) = p_i$ for all i .

Case 2.2 If λ has base points, then $\deg f \leq 3$. The base locus of λ can contain at most $4 - \deg f$ points from the set $\{Q_i\}$, hence $f(Q_i) = p_i$ holds for at least $4 + \deg f$ values of i .

Now the previous lemma implies that either subcase is impossible for general choice of p_i , hence no such grove can exist. We have proved that $(3, 3, 2, 8)$ is nondegenerate. ■

7 Open Problems

Whenever \mathcal{A} is a finite set, we have the obvious enumerative problem of counting its cardinality. Beyond a handful of cases (see [22]) it is entirely open. In particular, I do not know the cardinality of \mathcal{A} for $(2, 4, 4, 10)$ or $(3, 3, 3, 10)$.

It is also of interest to consider the family of positive dimensional schemes (with a fixed Hilbert polynomial) apolar to Λ . For instance, it is known that there are two twisted cubics apolar to a general web of quadrics in \mathbb{P}^3 (see [10, p. 32]).

It is known that a general net of quadrics in \mathbb{P}^5 does not admit a polar octahedron (see [2]), contrary to what one would expect by counting parameters. However it is not known if such a net admits an apolar rational normal quintic curve. A solution to this would help in elucidating the case $(5, 2, 3, 8)$.

Acknowledgements I would like to thank Anthony V. Geramita, Leslie Roberts and Queen's University for financial support while this work was in progress. The pro-

gram Macaulay-2 has been useful, and I am grateful to its authors Dan Grayson and Mike Stillman. I would like to thank the referee for several constructive suggestions.

References

- [1] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of Algebraic Curves, Volume I*. Grundlehren der math. Wissenschaften **267**, Springer-Verlag, New York, 1985.
- [2] E. Carlini and J. Chipalkatti, *On Waring's problem for several algebraic forms*. Comment. Math. Helv. **78**(2003), no. 3, 494–517.
- [3] C. Ciliberto, A. V. Geramita and F. Orecchia, *Remarks on a theorem of Hilbert–Burch*. Boll. Un. Mat. Ital. B (7) **2**(1988), no. 3, 463–483.
- [4] C. Dionisi and C. Fontanari, *Grassmann defectivity à la Terracini*. Le Matematiche, to appear.
- [5] I. Dolgachev, *On certain families of elliptic curves in projective space*. Ann. Mat. Pura Appl. **183**(2004), no. 3, 317–331.
- [6] I. Dolgachev and V. Kanev, *Polar covariants of cubics and quartics*. Adv. in Math. **98**(1993), no. 2, 216–301.
- [7] R. Ehrenborg and G.-C. Rota, *Apolarity and canonical forms for homogeneous polynomials*. European J. Combin. (3) **14**(1993), 157–181.
- [8] D. Eisenbud, *Commutative Algebra, with a View Toward Algebraic Geometry*. Graduate Texts in Math., Springer-Verlag, New York, 1995.
- [9] D. Eisenbud, M. Green and J. Harris, *Cayley–Bacharach theorems and conjectures*. Bull. Amer. Math. Soc. N.S. **33**(1996), no. 3, 295–324.
- [10] G. Ellingsrud and S. A. Strømme, *The number of twisted cubic curves on a quintic threefold*. Math. Scand. **76**(1995), no. 1, 5–34.
- [11] C. Fontanari, *On Waring's problem for many forms and Grassmann defective varieties*. J. Pure Appl. Algebra **74**(2002), no. 3, 243–247.
- [12] W. Fulton, *Intersection Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3 Folge. Springer-Verlag, Berlin, 2nd edition, 1998.
- [13] W. Fulton and J. Harris, *Representation Theory, A First Course*. Graduate Texts in Math., Springer-Verlag, New York, 1991.
- [14] A. V. Geramita, *Inverse Systems of Fat Points*. Queen's Papers in Pure and Applied Math. **X**, Queen's University, 1995.
- [15] P. A. Griffiths and J. Harris, *Principles of Algebraic Geometry*. Wiley Interscience, New York, 1978.
- [16] J. Harris, *Algebraic Geometry, A First Course*. Graduate Texts in Math., Springer-Verlag, New York, 1992.
- [17] R. Hartshorne, *Algebraic Geometry*. Graduate Texts in Math., Springer-Verlag, New York, 1977.
- [18] A. Iarrobino, *Inverse system of a symbolic power II. The Waring problem for forms*. J. Algebra **174**(1995), no. 3, 1091–1110.
- [19] A. Iarrobino and V. Kanev, *Power Sums, Gorenstein Algebras and Determinantal Loci*. Springer Lecture Notes in Math. **1721**, 1999.
- [20] I. Lakatos, *Proofs and Refutations*. Cambridge University Press, 1976.
- [21] F. London, *Über die Polarfiguren der ebenen Curven dritter Ordnung*. Math. Ann. **36**(1890), 535–584.
- [22] K. Ranestad and F.-O. Schreyer, *Varieties of sums of powers*. J. Reine Angew. Math. **525**(2000), 147–181.
- [23] B. Reichstein and Z. Reichstein, *Surfaces parametrizing Waring presentation of smooth plane cubics*. Michigan Math. J. **40**(1993), 95–118.
- [24] T. G. Room, *The Geometry of Determinantal Loci*. Cambridge University Press, Cambridge, 1938.
- [25] O. Schlesinger, *Ueber die Verwerthung der ϑ -Functionen*. Math. Ann. **31**(1888), 183–219.
- [26] A. Terracini, *Sulla rappresentazione delle coppie di forme ternarie mediante somme di potenze di forme lineari*. Ann. Mat. Serie III **XXIV**(1915).

Department of Mathematics
 University of Manitoba
 Winnipeg, MB
 R3T 2N2
 e-mail: chipalka@cc.umanitoba.ca