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MATRIX D.G. NEAR-RINGS

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Matrix near-rings had been defined by Meldrum and Van der Walt in 1986 and although a fair amount of results on the structure of these near-rings have been obtained since then, a satisfactory structure theory has yet to be developed for matrix d.g. near-rings. In this paper we give an alternate definition (in fact the dual definition) for matrix d.g. near-rings and develop a satisfactory structure theory for such d.g. near-rings.

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1. Introduction

In the study of near-rings one would like to have the analogue of matrix rings. A natural choice would be the system $M_n(R)$ of all matrices having entries from a near-ring R together with the normal operations of matrix addition and multiplication. But unfortunately the multiplication is not necessarily associative and thus, in general, $M_n(R)$ is not a near-ring.

Beidleman **[4]** has shown that if R is a near-ring with identity and for some integer $n(>1)$ we have $M_n(R)$ to be a near-ring, then R is a ring. Ligh **[10]** has shown that when $n > 1, M_n(R)$ is a near-ring if and only if R is n-distributive. Thus $M_n(R)$ as defined above fails to be the near-ring (or d.g. near-ring) analogue of matrix rings.

Meldrum and Van der Walt **[14]** defined the matrix near-ring over a near-ring R as the sub near-ring of $Map(R^n, R^n)$ generated by the set $\{f_{ij}^r : R^n \to R^n | r \in R, 1 \leq i, j \leq n\}$ of maps, which in the ring case correspond to the matrices with r in the $(i, j)th$ position and zero elsewhere. A fair amount of results on the structure of these near-rings had been obtained in **[1, 2, 3, 12, 13, 14, 15, 22** and **23]**. However, in our view, a satisfactory structure theory has yet to be developed for matrix d.g. near-rings and we present in this paper an alternate definition (in fact the dual definition) for matrix d.g. near-rings and develop a satisfactory structure theory for such d.g. near-rings.

Meldrum and Van der Walt **[14]** took the view that an $n \times n$ matrix over a ring R may also be considered as an endomorphism of the abelian group R^n (where R^n denotes the direct sum of n copies of $(R, +)$) and their matrices over a near-ring R are maps from R^n to R^n. We start with the characterisation of an $n \times n$ matrix over a ring R as an R-endomorphism of a free R-module of rank n and we characterise an $n \times n$ matrix over a d.g. near-ring R, distributively generated by a semigroup S, as an R-endomorphism of a $\upsilon(R, +)$- free left (R, S)-group Ω on a base with n elements; here $\upsilon(R, +)$ denotes the variety of left (R, S)-groups generated by the left (R, S)-group $(R, +)$. We have shown in **[18]** that the set of all such R-endomorphisms forms a d.g. near-ring and our matrices have been defined in such a manner as to ensure that our matrix d.g. near-ring is near-ring isomorphic to the above endomorphism d.g. near-ring. Further our non-singular matrices correspond to the R-automorphisms of Ω in this isomorphism.

Thus our matrix d.g. near-rings are Neumann d.g. near-rings (named after Hanna Neumann for her work in **[16]**) but not conversely. It may be observed that Hanna Neumann in **[16]** had, in fact, commented on the similarity of her near-rings to ordinary matrix rings.

We define an $m \times n$ matrix over a d.g. near-ring R as a column vector having m rows with an R-word in n variables in each row; an R-word $w(x_1, \ldots, x_n)$ is defined to be zero if $w(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n$ in R and matrix multiplication is by substitution of the variables. Historically, matrices originated from systems of linear equations and matrix multiplication from substitution of the variables. Thus our definition is a very natural generalisation and in the case when R is a ring we get the usual $m \times n$ matrix over R.

In Section 3 we give our definition of matrices over a d.g. near-ring and in Section 4 we obtain, in particular, generalisations of the Wedderburn–Artin Theorem for rings and the Morita criterion for equivalence of the rings R and $M_n(R)$.

In Section 5 we develop the theory of dual (R, S)-groups and prove that the R^n utilised by Meldrum and Van der Walt in their definition of matrix near-rings is the dual left (R, S)-group of our $v(R, +)$-free left (R, S)-group Ω and that our matrix d.g. near-ring is near-ring isomorphic to the matrix d.g. near-ring defined by Meldrum and Van der Walt.

2. Preliminaries and definitions

Throughout this paper we will assume (i) the term near-ring refers to a right near-ring with identity, (ii) R is an abstract d.g. near-ring with identity e, $D(R)$ is the set of distributive elements in R, S is a distributive semigroup generating $(R, +)$ and that 0 and e are in S, (iii) $v(R, +)$ denotes the variety of left (R, S)-groups generated by $(R, +)$, (iv) the basic definitions in **[5]**, **[17]** and **[18]**, (v) n is an arbitrary natural number, (vi) Capital Roman letters signify near-rings and their subsets or matrices and rows of matrices and small Roman letters signify the elements or near-rings, (vii) Capital Greek letters stand for groups or their subsets and small Greek letters for elements of groups or maps.

Definition 2.1. A *right R-group* is an additive group Ω together with a map $(\omega, x) \to \omega x$ of $\Omega \times R \to \Omega$ such that

(i) $(\omega_1 + \omega_2)x = \omega_1 x + \omega_2 x$ for all $\omega_1, \omega_2 \in \Omega$ and $x \in R$;

(ii) $\omega(xy) = (\omega x)y$ for all $\omega \in \Omega$ and $x, y \in R$;

(iii) $\omega e = \omega$ for all $\omega \in \Omega$.

Definition 2.2. An element $\lambda \in \Omega$ is said to be *distributive* if $\lambda(x+y) = \lambda x + \lambda y$ for all $x, y \in R$.

The set $D(\Omega)$ of all distributive elements in Ω is non-empty as $0_\Omega \in D(\Omega)$ and by Proposition 1.2 of **[18]** we have $D(\Omega)D(R) \subseteq D(\Omega)$.

Definition 2.3. A *d.g. right* (R, S)*-group* is a group Ω such that (i) Ω is a right R-group; (ii) there exists a subset Λ of $D(\Omega)$ such that $\Lambda S \subseteq \Lambda$ and Λ generates Ω.

If we wish to specify the distributive subset Λ we shall speak of the d.g. right (R, S)-group (Ω, Λ).

Definition 2.4. A d.g. near-ring R is said to be a *division d.g. near-ring* if

(i) R has no non-trivial right ideals;

(ii) $S^* = S \backslash \{0\}$ forms a multiplicative group for some distributive semigroup S generating $(R, +)$.

Definition 2.5. A d.g. near-ring R is said to be a *regular d.g. near-ring* if there exists a distributive semigroup S generating $(R, +)$ and such that

(i) every right ideal of R is a d.g. right (R, S)-module;

(ii) for each $t \in S$ there exists $s \in S$ such that $tst = t$.

Definition 2.6. The *centre* $C(R)$ of $(R, +)$ is called the *additive centre* of the d.g. near-ring R. Let $Z_S = \{s \in S : st = ts \text{ for all } t \in S\}$ and $Z_S(R)$ be the subgroup of $(R, +)$ generated by Z_S. Z_S and $Z_S(R)$ are called *the centre of* S and *the* S*-centre* of R respectively.

Proposition 2.7. *The* S*-centre of* R *is a d.g. near-ring and* $tz = zt$ *for all* $z \in Z_S(R)$ *and* $t \in S$.

Proof. By Proposition 2.5 of **[17]** we have $Z_S(R)$ is a d.g. near-ring. Since $z \in Z_S(R)$ we have $z = \sum \epsilon_i s_i$ with $\epsilon_i = \pm 1, s_i \in Z_S$ for all i and consequently $tz = t\sum \epsilon_i s_i = \sum \epsilon_i t s_i = \sum \epsilon_i s_i t = (\sum \epsilon_i s_i)t = zt$.

3. Matrices

Definition 3.1. An R*-word* $w(x_1, \ldots, x_n)$ *in the* n *variables* $x_1, \ldots, x_n$ is a formal expression of the form $\sum a_i x'_i$ with $a_i \in R$ and $x'_i \in \{x_1, \ldots, x_n\}$ and $w(x_1, \ldots, x_n)$ is said to be a *reduced word* if the a_i are non-zero and $x'_i \neq x'_{i+1}$ for all i.

Clearly any word has a unique reduced form and we define the sum of two words by juxtaposition and reduction. Further if $w=\sum a_i x_i'$ and $s \in S$ we define $sw=\sum (sa_i)x_i'$. Thus the set Γ_n of all reduced R-words in the n variables $x_i, \ldots, x_n$ forms an S-group and hence a left (R, S)-group.

Now let Ω, Ω' and Ω'' be $v(R,+)$-free left (R,S)-groups on $\Lambda=\{\lambda_1,\ldots,\lambda_m\}, \Lambda'=\{\lambda_1',\ldots,\lambda_m'\}$ and $\Lambda''=\{\lambda_1'',\ldots,\lambda_m''\}$ respectively and let $\phi:\Omega\to\Omega', \psi:\Omega\to\Omega'$ and $\eta:\Omega'\to\Omega''$ be R-homomorphisms. We define the sum $\phi+\psi$ of ϕ and ψ to be the unique R-homomorphism from Ω to Ω' which maps λ onto $\lambda\phi+\lambda\psi$ for all $\lambda\in\Lambda$ and the product $\phi\eta$ as the composition of maps. Thus we have $\lambda(\phi+\psi)=\lambda\phi+\lambda\psi$ and $\lambda(\phi\eta)=(\lambda\phi)\eta$ for all $\lambda\in\Lambda$.

Now the elements of the above groups are expressible, though not uniquely, as R-words on their sets of generators and thus we have

$$\lambda_i\phi=A_i(\lambda_1',\ldots,\lambda_n'),\quad \lambda_i\psi=B_i(\lambda_1',\ldots,\lambda_n'),\quad \lambda_j'\eta=C_j(\lambda_1'',\ldots,\lambda_p''),$$

$$\lambda_i(\phi+\psi)=D_i(\lambda_1',\ldots,\lambda_n'),\quad \lambda_i(\phi\eta)=E_i(\lambda_1'',\ldots,\lambda_p'')$$

where the A_i, B_i, C_j, D_i and E_i are R-words on the respective generators. Now $D_i(\lambda_1',\ldots,\lambda_n')=\lambda_i(\phi+\psi)=A_i(\lambda_1',\ldots,\lambda_n')+B_i(\lambda_1',\ldots,\lambda_n')$ and $E_i(\lambda_1'',\ldots,\lambda_p'')=\lambda_i(\phi\eta)=(\lambda_i\phi)\eta=A_i(\lambda_1',\ldots,\lambda_n')\eta=A_i(\lambda_1'\eta,\ldots,\lambda_n'\eta)=A_i(C_1(\lambda_1'',\ldots,\lambda_p''),\ldots,C_n(\lambda_1'',\ldots,\lambda_p''))$, and as the R-homomorphisms are uniquely determined by these components, we may represent them by these components. For instance, we may represent ϕ by a column matrix having m rows with $A_i(\lambda_1',\ldots,\lambda_n')$ as the element in the ith row. We will use this representation to introduce our matrices over the d.g. near-ring R.

In Γ_n we define $w_1(x_1,\ldots,x_n)=w_2(x_1,\ldots,x_n)$ if $w_1(\underline{r})=w_2(\underline{r})$ for all $\underline{r}\in R^n$. Clearly this is an equivalence relation and the equivalence class of the empty word forms a normal (R,S)-subgroup Δ_n of Γ_n and the other equivalence classes are the cosets of this normal subgroup. Further the difference group $\Gamma_n-\Delta_n$ is a left (R,S)-group.

Definition 3.2. A *$m\times n$ matrix* is a column vector having m rows with an R-word in n variables in each of the rows.

For typographical reasons we shall write them in the transposed form with square brackets; for example

$$A=[A_1(x_1,\ldots,x_n),\ldots,A_m(x_1,\ldots,x_n)]'.$$

Two matrices A and B are said to be equal if they are of the same order and their corresponding rows are equivalent.

Let

$$A_{m\times n}=[A_1(x_1,\ldots,x_n),\ldots,A_m(x_1,\ldots,x_n)]',$$
$$B_{m\times n}=[B_1(x_1,\ldots,x_n),\ldots,B_m(x_1,\ldots,x_n)]',$$
$$C_{n\times p}=[C_1(x_1,\ldots,x_p),\ldots,C_n(x_1,\ldots,x_p)]'$$

and $r \in R$. We define

$$A+B=[A_1(x_1,\ldots,x_n)+B_1(x_1,\ldots,x_n),\ldots,A_m(x_1,\ldots,x_n)+B_m(x_1,\ldots,x_n)]',$$

$$rA=[rA_1(x_1,\ldots,x_n),\ldots,rA_m(x_1,\ldots,x_n)]'$$

and

$$AC=D=[D_1(x_1,\ldots,x_p),\ldots,D_m(x_1,\ldots,x_p)]'$$

where

$$D_i(x_1,\ldots,x_p)=A_i(C_1(x_1,\ldots,x_p),\ldots,C_n(x_1,\ldots,x_p)).$$

Let $I_{n\times n}=[x_1,\ldots,x_n]'$ and $O_{m\times n}=[0,\ldots,0]'$. We then have

Proposition 3.3. *(i)* $I_{m\times m}A_{m\times n}=A_{m\times n}$;

(ii) $A_{m\times n}I_{n\times n}=A_{m\times n}$;

(iii) $O_{m\times n}A_{n\times p}=O_{m\times p}$;

(iv) $A_{m\times n}O_{n\times p}=O_{m\times p}$;

(v) $A_{m\times n}+O_{m\times n}=A_{m\times n}=O_{m\times n}+A_{m\times n}$.

$I_{n\times n}$ will be called the *$n\times n$ identity matrix* and $O_{m\times n}$ the *zero $m\times n$ matrix*.

Definition 3.4. A matrix A is said to be

(i) a *scalar matrix* if it is of the form rI with $r\in R$;

(ii) a *diagonal matrix* if the coefficients of x_j in $A_i(x_1,\ldots,x_n)$ are zero for all $j\neq i$;

(iii) *upper triangular* if the coefficients of x_j in $A_i(x_1,\ldots,x_n)$ are zero for all $j<i$;

(iv) *strictly upper triangular* if the coefficients of x_j in $A_i(x_1,\ldots,x_n)$ are zero for all $j\leq i$;

(v) *lower triangular* if the coefficients of x_j in $A_i(x_1,\ldots,x_n)$ are zero for all $j>i$;

(vi) *strictly lower triangular* if the coefficients of x_j in $A_i(x_1,\ldots,x_n)$ are zero for all $j\geq i$.

If A is an $m\times n$ matrix and T_{ij}, T_{i+rj} and T_{rj} are the elementary row matrices obtained from the $m\times m$ identity matrix I by interchanging the ith and jth rows, adding r times the jth row to the ith row of I and multiplying the jth row of I by r respectively (where r is an element of R), then it can be easily verified that $T_{ij}A$, $T_{i+rj}A$ and $T_{rj}A$ are the matrices obtained from A by performing the corresponding row operations on A.

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4. Matrix d.g. near-ring

We will denote by $M_n(R)$ the set of all $n \times n$ matrices over the d.g. near-ring R and define $M_n(S) = \{A \in M_n(R) : A_i(x_1, \ldots, x_n) = s_i x_i'$ with $s_i \in S$ and $x_i' \in \{x_1, \ldots, x_n\}$ for all $1 \leq i \leq n\}$.

Then it can be easily verified that

(i) $M_n(R)$ forms a right near-ring with identity I;

(ii) $M_n(S)$ forms a distributive semigroup in $M_n(R)$;

(iii) $M_n(S)$ generates $(M_n(R), +)$;

and consequently $M_n(R)$ is a d.g. near-ring.

For the rest of this paper we will assume that Ω is a $v(R, +)$-free left (R, S)-group on $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ and $\overline{R}$ denotes the endomorphism d.g. near-ring of Ω (cf. Proposition 2.2 of [**18**]). Also let $\overline{S} = \{\overline{x} \in \overline{R} : \Lambda\overline{x} \subseteq S\Lambda\}$.

Theorem 4.1. *$(M_n(R), M_n(S))$ is d.g. near-ring isomorphic to $(\overline{R}, \overline{S})$.*

Proof. Let $\phi : M_n(R) \to \overline{R}$ be defined by $\lambda_i \phi(A) = A_i(\lambda_1, \ldots, \lambda_n)$. Since Ω is a $v(R, +)$-free left (R, S)-group, we have $A_i(x_1, \ldots, x_n) = 0$ if and only if $A_i(\lambda_1, \ldots, \lambda_n) = 0$ and consequently ϕ is well defined, as an R-endomorphism of Ω is uniquely determined by its action on Λ. It can be easily verified that ϕ is a near-ring epimorphism. Now suppose $\phi(A) = 0$. Then $A_i(\lambda_1, \ldots, \lambda_n) = 0$ for $i = 1, \ldots, n$ and since Ω is $v(R, +)$-free we have $A_i(\underline{r}) = 0$ for all $\underline{r} \in R^n$. Consequently $A_i(x_1, \ldots, x_n) = 0$ for $i = 1, \ldots, n$ and thus ϕ is a near-ring isomorphism. Clearly $\phi(M_n(S)) = \overline{S}$.

Proposition 4.2. *If R is a ring with identity and $S = R$, then Ω is a free R-module on $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$.*

Proof. Since Ω is $v(R, +)$-free and $(R, +)$ is abelian we have Ω to be abelian and consequently Ω is an R-module. Thus any element of Ω can be represented in the form $\sum_{i=1}^n a_i \lambda_i$ and as Ω is a $v(R, +)$-free left (R, S)-group we have $\sum_{i=1}^n a_i \lambda_i = 0$ if and only if $\sum_{i=1}^n a_i r_i = 0$ for all $r_1, \ldots, r_n$ in R.

Hence $\sum_{i=1}^n a_i \lambda_i = 0$ if and only if $a_i = 0$ for all i and consequently Ω is a free R-module on Λ.

Thus our matrix d.g. near-rings reduce to ordinary matrix rings when R is a ring.

Now let $\phi_0 : R \to \overline{R}$ be defined by $\lambda \phi_0(x) = x\lambda$ for all $\lambda \in \Lambda$ and $x \in R$. Then, as $R \in v(R, +)$, by Proposition 2.5 of [**18**] we have, by identifying R with $\phi_0(R)$,

Proposition 4.3. *(i) R is a sub near-ring of $\overline{R}$;*

(ii) R is a sub near-ring of $M_n(R)$.

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Now let E_{ij} denote the $n \times n$ matrix having x_j as the element in the ith row and zero elsewhere and let $\bar{e}_{\lambda_i\lambda_j}$ be the element of $\overline{R}$ which maps λ_i onto λ_j and λ onto 0 for all $\lambda(\neq \lambda_i)$ in Λ. We will denote $\bar{e}_{\lambda_i\lambda_i}$ by $\bar{e}_{\lambda_i}$.

Theorem 4.4. (*i*) *$\overline{R}$ is a $v(R,+)$-left (R,S)-group on $\{\bar{e}_{\lambda_i\lambda_j} : 1 \le i, j \le n\}$;*

(*ii*) *$M_n(R)$ is a $v(R,+)$-left (R,S)-group on $\{E_{ij} : 1 \le i, j \le n\}$.*

Proof. (i) If $\bar{x} \in \overline{R}, \lambda \in \Lambda$ and $\lambda\bar{x} = \sum a_i\lambda_i'$ with $a_i \in R$ and $\lambda_i' \in \Lambda$, we have $\bar{e}_\lambda\bar{x} = \sum a_i\bar{e}_{\lambda\lambda_i'}$ and consequently $\{\bar{e}_{\lambda_i\lambda_j} : 1 \le i, j \le n\}$ generates $\overline{R}$ as $\bar{x} = \sum_{i=1}^{n} \bar{e}_{\lambda_i}\bar{x}$. By Theorem 3, Corollary 4 of **[18]** we have $v(R,+) = v(\overline{R},+)$ and so $\overline{R}$ is a $v(R,+)$-left (R,S)-group.

(ii) This follows from (i) and Theorem 4.1.

Proposition 4.5. (*i*) *$\{E_{ii} : 1 \le i \le n\}$ is an orthogonal set of idempotents in $M_n(R)$;*

(*ii*) *The set of all diagonal matrices in $M_n(R)$ is a sub near-ring which is near-ring isomorphic to R^n;*

(*iii*) *The set of all upper triangular (lower triangular, strictly upper triangular, strictly lower triangular) matrices in $M_n(R)$ is a sub d.g. near-ring;*

(*iv*) *The set of all strictly upper triangular (lower triangular) matrices in $M_n(R)$ is an ideal in the d.g. near-ring of upper triangular (lower triangular) matrices.*

The proof of this Proposition is straightforward and will be omitted.

Definition 4.6. An $m \times n$ matrix A is said to be

(i) a *kth column matrix* if $A_i(x_1, \ldots, x_n) = a_ix_k$ for all i;

(ii) a *kth row matrix* if $A_i(x_1, \ldots, x_n) = 0$ for all $i \neq k$.

Clearly we have (i) any matrix is a sum of column matrices; (ii) any matrix is a sum of row matrices; and (iii) the matrices E_{ij} are column as well as row matrices.

It can be easily verified that the set of all kth column matrices in $M_n(R)$ is a left $M_n(R)$ submodule and the set of all kth row matrices is a right ideal in $M_n(R)$.

Now by Propositions 5.3 and 5.4 of **[18]** and Theorem 4.1 we have

Theorem 4.7. (*i*) *R is left primitive if and only if $M_n(R)$ is left primitive;*

(*ii*) *R is simple if and only if $M_n(R)$ is simple.*

By Theorem 6 of **[20]** and Theorem 4.1 we have the following generalisation of the Wedderburn–Artin theorem.

Theorem 4.8. *If R is a discrete d.g. near-ring satisfying the descending chain condition for right ideals then the following three conditions are equivalent:*

(*i*) *R is simple and has an irreducible d.g. right (R, S)-module for some S;*

(*ii*) *R is right primitive;*

(*iii*) *R is near-ring isomorphic to $M_n(R_1)$ for some division d.g. near-ring R_1 and some positive integer n.*

By Theorems 6 and 7 of **[21]** and Theorem 4.1 we have

Theorem 4.9. (*i*) *If (R, S) is a division d.g. near-ring then $(M_n(R), M_n(S))$ is a regular d.g. near-ring;*

(*ii*) *If $(R, D(R))$ is a division d.g. near-ring then $(M_n(R), D(M_n(R)))$ is a regular d.g. near-ring.*

Proposition 4.10. *R is near-ring isomorphic to $\bar{e}_\lambda \bar{R} \bar{e}_\lambda$.*

Proof. Define $\phi : R \to \bar{e}_\lambda \bar{R} \bar{e}_\lambda$ by $\lambda\phi(x) = x\lambda$ and $\lambda'\phi(x) = 0$ if $\lambda' \neq \lambda$. It can be easily verified that ϕ is a near-ring homomorphism. Now given $\bar{e}_\lambda \bar{x} \bar{e}_\lambda \in \bar{e}_\lambda \bar{R} \bar{e}_\lambda$ we have $\lambda \bar{e}_\lambda \bar{x} \bar{e}_\lambda = x\lambda$ for some $x \in R$ and $\lambda' \bar{e}_\lambda \bar{x} \bar{e}_\lambda = 0$ for all $\lambda' \neq \lambda$. Thus $\phi(x) = \bar{e}_\lambda \bar{x} \bar{e}_\lambda$ and so ϕ is onto. Now suppose $\phi(x) = 0$. Then $x\lambda = 0$ and as Ω is a $v(R, +)$-free group we have $xy = 0$ for all $y \in R$ and in particular $x = xe = 0$. Thus ϕ is an isomorphism.

Proposition 4.11. *$\bar{R}.\bar{e}_\lambda.\bar{R} = \bar{R}$ where $\bar{R}.\bar{e}_\lambda.\bar{R} = \{\sum \bar{x}_i \bar{e}_\lambda \bar{y}_i : \bar{x}_i, \bar{y}_i \in \bar{R}\}$.*

Proof. Clearly we have $\bar{R}.\bar{e}_\lambda.\bar{R} \subseteq \bar{R}$ and let $\bar{x} \in \bar{R}$. Then $\lambda_i \bar{x} = \sum_{j=1}^{k_i} a_{ij} \lambda'_{ij} = \lambda_i \sum_{j=1}^{k_i} a_{ij} \bar{e}_{\lambda_i \lambda'_{ij}} = \lambda_i \sum_{j=1}^{k_i} a_{ij} \bar{e}_{\lambda_i \lambda} \bar{e}_\lambda \bar{e}_{\lambda \lambda'_{ij}}$ and thus if $\bar{y}_i = \sum_{j=1}^{k_i} a_{ij} \bar{e}_{\lambda_i \lambda} \bar{e}_\lambda \bar{e}_{\lambda \lambda'_{ij}}$ we have $\bar{y}_i \in \bar{R}.\bar{e}_\lambda.\bar{R}$, $\lambda_i(\bar{x} - \bar{y}_i) = 0$ and $\lambda \bar{y}_i = 0$ if $\lambda \neq \lambda_i$. Define $\bar{y} = \sum_{i=1}^{n} \bar{y}_i$. Then $\bar{y} \in \bar{R}.\bar{e}_\lambda.\bar{R}$ and $\lambda_i(\bar{x} - \bar{y}) = 0$ for all i. Hence $\bar{x} = \bar{y} \in \bar{R}.\bar{e}_\lambda.\bar{R}$ and the result follows.

Combining Propositions 4.10 and 4.11 we have

Theorem 4.12. *There exists an idempotent E in $M_n(R)$ such that $EM_n(R)E \cong R$ and $M_n(R).E.M_n(R) = M_n(R)$.*

We note that two rings R and S are Morita equivalent if and only if there exists an idempotent e in S such that

$$eSe \cong R \quad \text{and} \quad S.e.S = S$$

and thus Theorem 4.12 is a generalisation of the result that R and $M_n(R)$ are Morita equivalent when R is a ring.

Theorem 4.13. *$Z_{\bar{S}}(\bar{R}) = \{\bar{x} \in \bar{R} : \bar{x} = r\bar{e} \text{ with } r \in Z_S(R)\}$ where $\bar{e}$ is the identity of $\bar{R}$.*

Proof. Let $\bar{x} \in Z_{\bar{S}}(\bar{R})$. Then $\bar{x}\bar{e}_{\lambda_i\lambda_j} = \bar{e}_{\lambda_i\lambda_j}\bar{x}$ for all i, j and so $\lambda_i\bar{x}\bar{e}_{\lambda_i\lambda_j} = \lambda_j\bar{x}$ for all i, j. Consequently we have $\lambda_i\bar{x} = r\lambda_i$ for all i and thus $\bar{x} = r\bar{e}$ with $r \in R$. Thus if $\bar{t} \in Z_{\bar{S}}$ we have $\bar{t} = t\bar{e}$ with $t \in S$ and, using $\bar{t}(s\bar{e}) = (s\bar{e})\bar{t}$ with $s \in S$, we get $t \in Z_S$. Hence $\bar{x} = \sum \epsilon_i\bar{t}_i = \sum \epsilon_i(t_i\bar{e}) = \sum(\epsilon_i t_i)\bar{e} = r\bar{e}$ with $r \in Z_S(R)$ and $\epsilon_i = \pm 1$. Conversely let $r \in Z_S(R)$. Then $r = \sum \epsilon_i t_i$ with $t_i \in Z_S$ and so $(t_i\bar{e})\bar{t} = \bar{t}(t_i\bar{e})$ for all $\bar{t} \in \bar{S}$. Thus $t_i\bar{e} \in Z_{\bar{S}}$ and consequently $r\bar{e} \in Z_{\bar{S}}(\bar{R})$.

Corollary 4.14. $Z_{\bar{S}}(\bar{R}) \cong Z_S(R)$.

This corollary is a generalisation of the result that the centre of a matrix ring is ring isomorphic to the centre of the base ring.

5. Dual (R, S)-groups

Let $\Omega^* = Hom_R(\Omega, R)$ be the set of all left R-homomorphisms from Ω into R. If $x \in R$ and $\alpha^*, \beta^* \in \Omega^*$, denote by $\alpha^* + \beta^*$ and $\alpha^* x$ the unique R-homomorphisms from Ω to R defined respectively by $\lambda(\alpha^* + \beta^*) = \lambda\alpha^* + \lambda\beta^*$ and $\lambda(\alpha^* x) = (\lambda\alpha^*)x$ for all $\lambda \in \Lambda$. For each $\lambda \in \Lambda$ let λ^* denote the element of Ω^* defined by $\lambda\lambda^* = e$ and $\lambda'\lambda^* = 0$ for all $\lambda'(\neq \lambda)$ in Λ. Also let $\Lambda_S^* = \{\alpha^* \in \Omega^* : \lambda\alpha^* \in S \text{ for all } \lambda \in \Lambda\}$ and $\Lambda^* = \{\lambda^* : \lambda \in \Lambda\}$.

Proposition 5.1. (Ω^*, Λ_S^*) *is a d.g. right* (R, S)*-group.*

Proof. Let $x, y \in R$ and $s \in S$. We have $\lambda((\alpha^* + \beta^*)x) = (\lambda\alpha^* + \lambda\beta^*)x = (\lambda\alpha^*)x + (\lambda\beta^*)x = \lambda(\alpha^* x) + \lambda(\beta^* x)$, $\lambda(\alpha^*(xy)) = (\lambda\alpha^*)(xy) = ((\lambda\alpha^*)x)y = (\lambda(\alpha^* x))y = \lambda((\alpha^* x)y)$ and $\lambda(\alpha^* e) = (\lambda\alpha^*)e = \lambda\alpha^*$ for all $\lambda \in \Lambda$ and thus Ω^* is a right R-group. Also if $\alpha^* \in \Lambda_S^*$ we have $\lambda(\alpha^*(x + y)) = (\lambda\alpha^*)(x + y) = (\lambda\alpha^*)x + (\lambda\alpha^*)y = \lambda(\alpha^* x) + \lambda(\alpha^* y)$ and $\lambda(\alpha^* s) = (\lambda\alpha^*)s \in S$ for all $\lambda \in \Lambda$.

Thus Λ_S^* is a set of distributive elements in Ω^* such that $\Lambda_S^* S \subseteq \Lambda_S^*$. Now given $\alpha^* \in \Omega^*$ let $\lambda_i\alpha^* = \sum_{j=1}^{k_i} \epsilon_{ij}s_{ij}$ with $\epsilon_{ij} = \pm 1$ and $s_{ij} \in S$. Let γ_{ij}^* be the element of Ω^* defined by $\lambda_i\gamma_{ij}^* = s_{ij}$ and $\lambda\gamma_{ij}^* = 0$ for all $\lambda(\neq \lambda_i)$ in Λ. Then $\gamma_{ij}^* \in \Lambda_S^*$ and $\alpha^* = \sum_{i=1}^{n} \sum_{j=1}^{k_i} \epsilon_{ij}\gamma_{ij}^*$. Thus Λ_S^* generates Ω^* and the result follows.

Proposition 5.2. $\omega(\omega^* x) = (\omega\omega^*)x$ *for all* $\omega \in \Omega$, $\omega^* \in \Omega^*$ *and* $x \in R$.

Proof. If $\omega = \sum \pm s_i\lambda_i'$ with $s_i \in S$ and $\lambda_i \in \Lambda$ we have $\omega(\omega^* x) = (\sum \pm s_i\lambda_i')(\omega^* x) = \sum \pm (s_i\lambda_i')(\omega^* x) = \sum \pm s_i(\lambda_i'(\omega^* x)) = \sum \pm s_i(\lambda_i'\omega^*)x = (\sum \pm s_i(\lambda_i'\omega^*))x = ((\sum \pm s_i\lambda_i')\omega^*)x = (\omega\omega^*)x$.

Now by Proposition 3.2 of [18] and Proposition 4.3 we have

Proposition 5.3. (*i*) Ω^* *is a left* $(\bar{R}, \bar{S})$*-group with* $\bar{x}\alpha^*$ *being defined by* $\lambda(\bar{x}\alpha^*) = (\lambda\bar{x})\alpha^*$ *for all* $\lambda \in \Lambda$;

(*ii*) Ω^* *is a left* (R, S)*-group.*

By the Corollary 3 of Theorem 3 of [**18**] we have

Proposition 5.4. $\Omega^* \in v(R, +)$.

Now given $x \in R, \bar{x} \in \bar{R}$ and $\alpha^* \in \Omega^*$, we have $\lambda((\bar{x}\alpha^*)x) = (\lambda(\bar{x}\alpha^*))x = ((\lambda\bar{x})\alpha^*)x = (\lambda\bar{x})(\alpha^* x) = \lambda(\bar{x}(\alpha^* x))$ for all $\lambda \in \Lambda$ and consequently we have

Proposition 5.5. $(\bar{x}\alpha^*)x = \bar{x}(\alpha^* x)$ *for all* $x \in R, \bar{x} \in \bar{R}$ *and* $\alpha^* \in \Omega^*$.

Theorem 5.6. *(i)* (Ω^*, Λ_S^*) *is right R-isomorphic to* (R^n, S^n)*;*

(ii) An element of Ω^* *has a unique representation in the form* $\sum_{i=1}^n \lambda_i^* x_i$ *with* $x_i \in R$*;*

(iii) Λ^* *generates* Ω^* *as a right R-group and* $\Lambda^* \subseteq \Lambda_S^*$.

Proof. (i) Define $\phi : \Omega^* \to R^n$ by $(\phi(\alpha^*))_i = \lambda_i \alpha^*$ for $i = 1, \ldots, n$. Then from the proofs of Theorems 2 and 3 of [**18**] we have ϕ is a left R-isomorphism. Now $(\phi(\alpha^* x))_i = \lambda_i(\alpha^* x) = (\lambda_i \alpha^*)x = (\phi(\alpha^*))_i x$ for $i = 1, \ldots, n$ and so ϕ is a right R-homomorphism. Also clearly we have $\phi(\Lambda_S^*) = S^n$.

(ii) Since Ω^* is right R-isomorphic to R^n and $(\phi(\lambda_j^*))_i = e$ if $j = i$ and is zero if $j \neq i$, any element of Ω^* has a unique representation in the form $\sum_{i=1}^n \lambda_i^* x_i$.

(iii) This follows from (ii) and the definition of Λ_S^*.

Definition 5.7. The d.g. right (R, S)-group Ω^* is called the *dual* of the left (R, S)-group Ω and Λ^* the *dual basis* to Λ.

Now let $hom_R^r(\Omega^*, R)$ be the set of all right R-homomorphisms from Ω^* to R and let $\Omega^{**} = Hom_R(\Omega^*, R)$ where $Hom_R(\Omega^*, R)$ is the subgroup of $Map(\Omega^*, R)$ generated by $hom_R^r(\Omega^*, R)$.

Given $x \in R$ and $\omega^{**} \in \Omega^{**}$ define $x\omega^{**}$ by $(x\omega^{**})\omega^* = x(\omega^{**}\omega^*)$ for all $\omega^* \in \Omega^*$.

Proposition 5.8. *(i)* $S.hom_R^r(\Omega^*, R) \subseteq hom_R^r(\Omega^*, R)$*;*

(ii) Ω^{**} *is a left* (R, S)*-group.*

Proof. (i) Let $s \in S$ and $\omega^{**} \in hom_R^r(\Omega^*, R)$. Then $(s\omega^{**})(\omega_1^* + \omega_2^*) = s(\omega^{**}(\omega_1^* + \omega_2^*)) = s(\omega^{**}\omega_1^* + \omega^{**}\omega_2^*) = s(\omega^{**}\omega_1^*) + s(\omega^{**}\omega_2^*) = (s\omega^{**})\omega_1^* + (s\omega^{**})\omega_2^*$ for all $\omega_1^*, \omega_2^* \in \Omega^*$, and $(s\omega^{**})(\omega^* x) = s(\omega^{**}(\omega^* x)) = s.((\omega^{**}\omega^*)x) = (s.(\omega^{**}\omega^*))x = ((s\omega^{**})\omega^*)x$ for all $\omega^* \in \Omega^*$ and $x \in R$. Thus $s\omega^{**} \in hom_R^r(\Omega^*, R)$.

(ii) By (i) we have $x\omega^{**} \in \Omega^{**}$ for all $x \in R$ and $\omega^{**} \in \Omega^{**}$. Clearly $e\omega^{**} = \omega^{**}$. Now given $s \in S$ and $\omega_1^{**}, \omega_2^{**} \in \Omega^{**}$ we have $(s(\omega_1^{**} + \omega_2^{**}))(\omega^*) = s((\omega_1^{**} + \omega_2^{**})\omega^*) = s(\omega_1^{**}\omega^* + \omega_2^{**}\omega^*) = s(\omega_1^{**}\omega^*) + s(\omega_2^{**}\omega^*) = (s\omega_1^{**})\omega^* + (s\omega_2^{**})\omega^* = (s\omega_1^{**} + s\omega_2^{**})\omega^*$ and $((st)\omega_1^{**})\omega^* = (st)(\omega_1^{**}\omega^*) = s(t(\omega_1^{**}\omega^*)) = s((t\omega_1^{**})\omega^*) = (s(t\omega_1^{**}))\omega^*$ for all $\omega^* \in \Omega^*$. Hence Ω^{**} is a left S-group and consequently is a left (R, S)-group.

Now as, by Theorem 5.6, every element of Ω^* has a unique representation in the form $\sum_{i=1}^{n}\lambda_i^* x_i$, we will for each $i \in \{1,\ldots,n\}$ define a map λ_i^{**} from Ω^* to R by $\lambda_i^{**}(\sum_{j=1}^{n}\lambda_j^* x_j) = x_i$ and let $\Lambda^{**} = \{\lambda_1^{**},\ldots,\lambda_n^{**}\}$.

Proposition 5.9. *(i)* $\Lambda^{**} \subseteq hom_R^r(\Omega^*, R)$;

(ii) Λ^{**} *generates* Ω^{**} *as a left* (R,S)*-group.*

Proof. (i) $\lambda_i^{**}(\sum_{j=1}^{n}\lambda_j^* x_j + \sum_{j=1}^{n}\lambda_j^* y_j) = \lambda_i^{**}(\sum_{j=1}^{n}\lambda_j^*(x_j + y_j)) = x_i + y_i = \lambda_i^{**}(\sum_{j=1}^{n}\lambda_j^* x_j)+ \lambda_i^{**}(\sum_{j=1}^{n}\lambda_j^* y_i)$ and $\lambda_i^{**}((\sum_{j=1}^{n}\lambda_j^* x_j)x) = \lambda_i^{**}(\sum_{j=1}^{n}\lambda_j^*(x_j x)) = x_i x = (\lambda_i^{**}(\sum_{j=1}^{n}\lambda_j^* x_j))x$. Hence λ_i^{**} is a right R-homomorphism.

(ii) If $\omega^{**} \in hom_R^r(\Omega^*, R)$ and $\omega^* \in \Omega^*$ then $\omega^{**}(\omega^*) = \omega^{**}(\sum_{j=1}^{n}\lambda_j^* x_j) = \sum_{j=1}^{n}(\omega^{**}\lambda_j^*)x_j = \sum_{j=1}^{n}(\omega^{**}\lambda_j^*)(\lambda_j^{**}(\sum_{i=1}^{n}\lambda_i^* x_i)) = (\sum_{j=1}^{n}(\omega^{**}\lambda_j^*)\lambda_j^{**})(\omega^*)$ for all $\omega^* \in \Omega^*$ and consequently $\omega^{**} = \sum_{j=1}^{n}(\omega^{**}\lambda_j^*)\lambda_j^{**}$. But $hom_R^r(\Omega^*, R)$ generates Ω^{**} and thus Λ^{**} generates Ω^{**} as a left (R,S)-group.

Theorem 5.10. *(i)* $\Omega^{**} \in v(R,+)$;

(ii) Ω^{**} *is left* R*-isomorphic to* Ω.

Proof. (i) Let $w(x_1,\ldots,x_m)$ be an R-word in the m variables $x_1,\ldots,x_m$ such that $w(r_1,\ldots,r_m) = 0$ for all $r_1,\ldots,r_m$ in R. Then given $\omega_1^{**},\ldots,\omega_m^{**} \in \Omega^{**}$ we have $w(\omega_1^{**},\ldots,\omega_m^{**})(\omega^*) = w(\omega_1^{**}\omega^*,\ldots,\omega_m^{**}\omega^*) = 0$ for all $\omega^* \in \Omega^*$ and so $w(\omega_1^{**},\ldots,\omega_m^{**}) = 0$. Thus $\Omega^{**} \in v(R,+)$.

(ii) Since Ω is a $v(R,+)$-free left (R,S)-group on Λ and $\Omega^{**} \in v(R,+)$ let $\phi : \Omega \to \Omega^{**}$ be the left R-homomorphism defined by $\phi(\lambda_i) = \lambda_i^{**}$ for $i = 1,\ldots,n$. Now ϕ is surjective as Λ^{**} generates Ω^{**}. Suppose $\phi(\omega) = 0$ and $\omega = w(\lambda_1,\ldots,\lambda_n)$ where w is an R-word. Then $\phi(w) = w(\lambda_1^{**},\ldots,\lambda_n^{**})$ and so $w(\lambda_1^{**}\omega^*,\ldots,\lambda_n^{**}\omega^*) = w(\lambda_1^{**},\ldots,\lambda_n^{**})(\omega^*) = 0$ for all $\omega^* \in \Omega^*$.

Now given $r_1,\ldots,r_n$ in R, choosing $\omega^* = \sum_{j=1}^{n}\lambda_j^* r_j$ we have $w(r_1,\ldots,r_n) = w(\lambda_1^{**}\omega^*,\ldots,\lambda_n^{**}\omega^*) = 0$ and consequently we have $\omega = 0$. Thus Ω is left R-isomorphic to Ω^{**}.

Definition 5.11. The left (R,S)-group Ω^{**} is called the *dual* of the d.g. right (R,S)-group Ω^* and the *double dual* of the left (R,S)-group Ω and Λ^{**} is called the *dual basis* to Λ^* and the *double dual basis* to Λ.

Definition 5.12. (i) For $\omega \in \Omega$ and $\omega^* \in \Omega^*$ we denote by $\omega^*\omega$ the R-endomorphism of Ω defined by $\lambda(\omega^*\omega) = (\lambda\omega^*)\omega$ for all $\lambda \in \Lambda$.

(ii) $\Omega.\Omega^*$ and $\Omega \circ \Omega^*$ denote respectively the subgroup and normal subgroup of $(R,+)$ generated by the set $\{\omega\omega^* : \omega \in \Omega, \omega^* \in \Omega^*\}$ and $\Omega^*.\Omega$ and $\Omega^* \circ \Omega$ denote respectively the subgroup and normal subgroup of $(\overline{R},+)$ generated by the set $\{\omega^*\omega : \omega^* \in \Omega^*, \omega \in \Omega\}$.

Proposition 5.13. *(i)* $\omega^*(x\omega) = (\omega^*x)\omega$ *for all* $x \in R, \omega \in \Omega$ *and* $\omega^* \in \Omega^*$*;*

(ii) $(\omega_1^* + \omega_2^*)\omega = \omega_1^*\omega + \omega_2^*\omega$ *for all* $\omega \in \Omega$ *and* $\omega_1^*, \omega_2^* \in \Omega^*$*;*

(iii) $\alpha^*(\omega_1 + \omega_2) = \alpha^*\omega_1 + \alpha^*\omega_2$ *for all* $\alpha^* \in \Lambda_S^*$ *and* $\omega_1, \omega_2 \in \Omega$*;*

(iv) $(\omega\omega^*)\omega_1 = \omega(\omega^*\omega_1)$ *for all* $\omega, \omega_1 \in \Omega$ *and* $\omega^* \in \Omega^*$.

Proof. (i), (ii) and (iii). For all $\lambda \in \Lambda$ we have (i) $\lambda((\omega^*x)\omega) = (\lambda(\omega^*x))\omega = ((\lambda\omega^*)x)\omega = (\lambda\omega^*)(x\omega) = \lambda(\omega^*(x\omega))$;

(ii) $\lambda((\omega_1^* + \omega_2^*)\omega) = (\lambda(\omega_1^* + \omega_2^*))\omega = (\lambda\omega_1^* + \lambda\omega_2^*)\omega = (\lambda\omega_1^*)\omega + (\lambda\omega_2^*)\omega = \lambda(\omega_1^*\omega + \omega_2^*\omega)$ and

(iii) $\lambda(\alpha^*(\omega_1 + \omega_2)) = (\lambda\alpha^*)(\omega_1 + \omega_2) = (\lambda\alpha^*)\omega_1 + (\lambda\alpha^*)\omega_2 = \lambda(\alpha^*\omega_1) + \lambda(\alpha^*\omega_2) = \lambda(\alpha^*\omega_1) + \lambda(\alpha^*\omega_2) = \lambda(\alpha^*\omega_1 + \alpha^*\omega_2)$ and the results follow.

(iv) Suppose $\omega = \sum \pm s_i\lambda_i'$ with $s_i \in S$ and $\lambda_i' \in \Lambda$. Then $(\omega\omega^*)\omega_1 = ((\sum \pm s_i\lambda_i')\omega^*)\omega_1 = (\sum \pm s_i(\lambda_i'\omega^*))\omega_1 = \sum \pm s_i(\lambda_i'\omega^*)\omega_1 = \sum \pm s_i(\lambda_i'(\omega^*\omega_1)) = (\sum \pm s_i\lambda_i')(\omega^*\omega_1) = \omega(\omega^*\omega_1)$.

Proposition 5.14. *If* $\bar{x} \in \bar{R}, \omega \in \Omega$ *and* $\omega^* \in \Omega^*$ *we have*

(i) $\bar{x}(\omega^*\omega) = (\bar{x}\omega^*)\omega$;

(ii) $(\omega^*\omega)\bar{x} = \omega^*(\omega\bar{x})$;

(iii) $\omega(\bar{x}\omega^*) = (\omega\bar{x})\omega^*$.

Proof. (i) For $\lambda \in \Lambda$ let $\lambda\bar{x} = \sum a_i\lambda_i'$ with $a_i \in R$ and $\lambda_i' \in \Lambda$. Then $\lambda(\bar{x}(\omega^*\omega)) = (\lambda\bar{x})(\omega^*\omega) = (\sum a_i\lambda_i')(\omega^*\omega) = \sum a_i(\lambda_i'(\omega^*\omega)) = \sum a_i((\lambda_i'\omega^*)\omega) = \sum(a_i(\lambda_i'\omega^*))\omega = (\sum a_i(\lambda_i'\omega^*))\omega$ and $\lambda((\bar{x}\omega^*)\omega) = (\lambda(\bar{x}\omega^*))\omega = ((\lambda\bar{x})\omega^*)\omega = ((\sum a_i\lambda_i')\omega^*)\omega = (\sum a_i(\lambda_i'\omega^*))\omega$ and so (i) is proved.

(ii) $\lambda((\omega^*\omega)\bar{x}) = (\lambda(\omega^*\omega))\bar{x} = ((\lambda\omega^*)\omega)\bar{x} = (\lambda\omega^*)(\omega\bar{x}) = \lambda(\omega^*(\omega\bar{x}))$ as $\bar{x}$ is an R-homomorphism of Ω and the result follows.

(iii) By definition of $\bar{x}\omega^*$ we have $\lambda(\bar{x}\omega^*) = (\lambda\bar{x})\omega^*$ for all $\lambda \in \Lambda$ and consequently if $\omega = \sum a_i\lambda_i'$ we have $\omega(\bar{x}\omega^*) = (\sum a_i\lambda_i')(\bar{x}\omega^*) = \sum a_i(\lambda_i'(\bar{x}\omega^*)) = \sum a_i((\lambda_i'\bar{x})\omega^*) = (\sum a_i(\lambda_i'\bar{x}))\omega^* = ((\sum a_i\lambda_i')\bar{x})\omega^* = (\omega\bar{x})\omega^*$.

Theorem 5.15. *(i)* $\Omega.\Omega^* = R$*;*

(ii) $\Omega^*.\Omega = \bar{R}$.

Proof. (i) Trivial.

(ii) From the definition we have $\Omega^*.\Omega \subseteq \bar{R}$. Now let $\bar{x} \in \bar{R}, \lambda \in \Lambda$ and $\lambda\bar{x} = \sum a_i\lambda_i'$ with $a_i \in R$ and $\lambda_i' \in \Lambda$. Define ω_i^* in Ω^* by $\lambda\omega_i^* = a_i$ and $\lambda'\omega_i^* = 0$ if $\lambda' \neq \lambda$. Then $\bar{e}_\lambda\bar{x} = \sum \omega_i^*\lambda_i' \in \Omega^*.\Omega$ for all $\lambda \in \Lambda$ and by the Corollary of Theorem 2 of [**18**] we have $\bar{x} \in \Omega^*.\Omega$.

Let Ω_1, Ω_2 and Ω_3 be $v(R,+)$-free left (R,S)-groups, $\bar{x}, \bar{y} \in Hom_R(\Omega_1, \Omega_2)$ and $\bar{z} \in Hom_R(\Omega_2, \Omega_3)$.

Definition 5.16. The *dual map* $\bar{x}^* : \Omega_2^* \to \Omega_1^*$ of $\bar{x}$ is defined by $\omega_1(\bar{x}^*(\omega_2^*)) = (\omega_1\bar{x})\omega_2^*$ for all $\omega_1 \in \Omega_1$ and $\omega_2^* \in \Omega_2^*$.

It is easily verified that $(\bar{x}+\bar{y})^* = \bar{x}^* + \bar{y}^*$ and $(\bar{x}\bar{z})^* = \bar{x}^*\bar{z}^*$.

Let $\overline{R}^*$ denote the matrix d.g. near-ring over R defined by Meldrum and Van der Walt. Since by Theorem 5.6 we have $\Omega^* \cong \overline{R}^n$ as d.g. right (R,S)-groups, we may consider the elements of $\overline{R}^*$ as elements of $Map(\Omega^*, \Omega^*)$, f_{ij}^r as the element of $Map(\Omega^*, \Omega^*)$ such that $f_{ij}^r(\sum_{k=1}^n \lambda_k^* r_k) = \lambda_i^* r r_j$ and I^n as $\Omega^*.I$.

Theorem 5.17. *$\overline{R}$ is near-ring isomorphic to $\overline{R}^*$.*

Proof. Define $\psi : \overline{R} \to Map(\Omega^*, \Omega^*)$ by $\psi(\bar{x}) = \bar{x}^*$. Then ψ is clearly a near-ring monomorphism. Further $(r\bar{e}_{\lambda_i\lambda_j})^* = f_{ij}^r$ for all $r \in R$. Now by definition we have $\{f_{ij}^r : r \in R, 1 \le i,j \le n\}$ generates $\overline{R}^*$ and by Theorem 4.4 we have $\overline{R}$ is generated by the set $\{r\bar{e}_{\lambda_i\lambda_j} : r \in R, 1 \le i,j \le n\}$. Consequently we have $\psi(\overline{R}) = \overline{R}^*$ and the result follows.

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