COFINITENESS AND FINITENESS OF GENERALIZED
LOCAL COHOMOLOGY MODULES

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Abstract

Let $I$ be an ideal of a commutative Noetherian local ring $R$, and $M$ and $N$ two finitely generated modules. Let $t$ be a positive integer. We mainly prove that (i) if $H^i_I(M, N)$ is Artinian for all $i < t$, then $H^i_I(M, N)$ is $I$-cofinite for all $i < t$ and $\text{Hom}(R/I, H^t_I(M, N))$ is finitely generated; (ii) if $d = \text{pd}(M) < \infty$ and $\dim N = n < \infty$, then $H^{d+n}_I(M, N)$ is $I$-cofinite. We also prove that if $M$ is a nonzero cyclic $R$-module, then $H^i_I(N)$ is finitely generated for all $i < t$ if and only if $H^i_I(M, N)$ is finitely generated for all $i < t$.


Keywords and phrases: local cohomology, generalized local cohomology, cofinite, Artinian.

1. Introduction

Let $R$ be a commutative Noetherian ring and $I$ a proper ideal of $R$. In 1969, Grothendieck proposed the following conjecture.

Conjecture 1.1. Let $N$ be a finitely generated $R$-module and let $I$ be an ideal of $R$. Then $\text{Hom}(R/I, H^i_I(N))$ is finitely generated for all $i \geq 0$.

Hartshorne provided a counter-example to this conjecture in [9]. He defined an $R$-module $L$ to be $I$-cofinite if $\text{Supp}_R(L) \subseteq V(I)$ and $\text{Ext}_R^i(R/I, L)$ is a finitely generated $R$-module for any $i \geq 0$, where $V(I)$ denotes the set of prime ideals of $R$ containing $I$, and he asked the following question.

Question 1.2. Let $N$ be a finitely generated $R$-module and let $I$ be an ideal of $R$. Then is $H^i_I(N)$ $I$-cofinite for all $i$?

In general, the answer is also no, even if $R$ is a regular local ring. See [6] for a counter-example to this question.

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The generalized local cohomology module

\[ H^i_I(M, N) = \lim_{\to n} \Ext^i_R(M/I^nM, N) \]

for all \( R \)-modules \( M \) and \( N \) was introduced by Herzog in [10]. Clearly, it is a generalization of the usual local cohomology module. The study of generalized local cohomology modules was continued by many authors (see, for example, [2, 16]). In [17], Yassemi asked whether Question 1.2 holds for generalized local cohomology. And the cofiniteness of generalized local cohomology modules is studied by Divaani-Aazar and Sazeedeh [8] and Khashyarmanesh and Yassi [11].

At the same time, Conjecture 1.1 inspires us to think about the following question.

**Question 1.3.** Let \( M \) and \( N \) be two finitely generated \( R \)-modules. When is \( \Hom(R/I, H^i_I(M, N)) \) finitely generated?

Asadollahi, Khashyarmanesh and Salarian [1] proved that if \( H^i_I(M, N) \) is finitely generated for all \( i < t \), then \( \Hom(R/I, H^i_I(M, N)) \) is finitely generated. As an analogue of this result, we show that if \( H^i_I(M, N) \) is Artinian for all \( i < t \), then \( H^i_I(M, N) \) is \( I \)-cofinite for all \( i < t \) and \( \Hom(R/I, H^i_I(M, N)) \) is finitely generated. We also prove that if \( d = \pd(M) < \infty \) and \( \dim N = n < \infty \), then \( H^{d+n}_I(M, N) \) is \( I \)-cofinite, which is a generalization of [6, Theorem 3].

Throughout this paper, \((R, m)\) is a commutative Noetherian local ring (with nonzero identity), \( M \) and \( N \) are finitely generated \( R \)-modules and \( I \) is a proper ideal of \( R \). We refer the reader to [3] or [4] for any unexplained terminology.

**2. Results**

We begin this section with some lemmas.

**Lemma 2.1.** Let \( M \) be a finitely generated \( R \)-module. If \( H^i_I(M) \) is Artinian for all \( i < t \), then \( H^i_I(M) \) is \( I \)-cofinite for all \( i < t \) and \( \Hom(R/I, H^i_I(M)) \) is finitely generated.

**Proof.** This can be deduced from [7, Theorem 2.1] and [13, Proposition 4.3]. \( \square \)

**Lemma 2.2.** Let \( M \) be a finitely generated \( R \)-module. If \( L \) is Artinian and \( I \)-cofinite, then \( \Ext^i_R(M, L) \) is \( I \)-cofinite for all \( i \).

**Proof.** Since \( L \) is Artinian, \( \Ext^i_R(M, L) \) is Artinian for all \( i \). By [13, Proposition 4.3], it suffices to prove that \( \Hom_R(R/I, \Ext^i_R(M, L)) \) is finitely generated. In the following, we show that \( \Hom_R(R/I, \Ext^i_R(M, L)) \) is of finite length. Since

\[
\Hom_R(R/I, \Ext^i_R(M, L)) \cong \Hom_R(R/I, \Ext^i_R(M, L)) \otimes \hat{R} \\
\cong \Hom_{\hat{R}}(\hat{R}/I\hat{R}, \Ext^i_{\hat{R}}(\hat{M}, L)),
\]

we may assume that \( R \) is \( m \)-adic complete.
Set $E = E(R/m)$, an injective envelope of $R/m$. By [15, Theorem 11.57],

$$\text{Hom}_R(\text{Hom}_R(R/I, \text{Ext}_R^i(M, L)), E) \cong R/I \otimes \text{Tor}_i^R(M, \text{Hom}_R(L, E)).$$

By Matlis duality, $R/I \otimes \text{Tor}_i^R(M, \text{Hom}_R(L, E))$ is finitely generated, so it is enough to show that it is Artinian. Since $L$ is $I$-cofinite and Artinian, $\text{Hom}_R(R/I, L)$ is of finite length, and then $\text{Hom}_R(\text{Hom}_R(R/I, L), E) \cong R/I \otimes \text{Hom}_R(L, E)$ is of finite length. In particular,

$$\text{Supp}_R[R/I \otimes \text{Hom}_R(L, E)] = V(I) \cap \text{Supp}_R[\text{Hom}_R(L, E)] = \{m\}.$$

Therefore

$$\text{Supp}_R[R/I \otimes \text{Tor}_i^R(M, \text{Hom}_R(L, E))] \subseteq V(I) \cap \text{Supp}_R[\text{Hom}_R(L, E)] = \{m\}.$$

This completes the proof. \hfill \Box

The following lemma plays a key role in the proof of our first main result.

**Lemma 2.3.** Let $M$ be a finitely generated $R$-module and $s$ be a nonnegative integer. Let $L$ be an $R$-module such that $H^i_I(L)$ is Artinian and $I$-cofinite for all $i < s$. If $H^i_I(M, L)$ is Artinian for all $i < s$, then $H^i_I(M, L)$ is $I$-cofinite for all $i < s$.

**Proof.** The proof is by induction on $s$. When $s = 1$, by the hypothesis, $H^0_I(L)$ is Artinian and $I$-cofinite. Then $\text{Hom}(R/I, H^0_I(M, L)) \cong \text{Hom}(M, \text{Hom}(R/I, H^0_I(L)))$, which is of finite length. By [13, Proposition 4.3], the result holds.

Suppose that $s > 1$, and the result holds for the case $s - 1$. The short exact sequence

$$0 \longrightarrow H^0_I(L) \longrightarrow L \longrightarrow L/H^0_I(L) \longrightarrow 0$$

yields the long exact sequence

$$\cdots \longrightarrow H^i_I(M, H^0_I(L)) \longrightarrow H^i_I(M, L) \longrightarrow H^i_I(M, L/H^0_I(L)) \longrightarrow \cdots.$$

Since $H^0_I(L)$ is $I$-torsion, $H^i_I(M, H^0_I(L)) \cong \text{Ext}_R^i(M, H^0_I(L))$. Then, by Lemma 2.2, $H^i_I(M, H^0_I(L))$ is $I$-cofinite and Artinian for all $i$. By [13, Corollary 4.4], it is enough for us to prove that $H^i_I(M, L/H^0_I(L))$ is $I$-cofinite for all $i < s$. So we can assume that $\Gamma_I(L) = 0$. Taking an injective hull $E$ of $L$, then we have the short exact sequence

$$0 \longrightarrow E \longrightarrow L \longrightarrow L/E \longrightarrow 0.$$ 

Consequently, from the long exact sequences of the above short exact sequence, $H^{i+1}_I(M, L) \cong H^{i+1}_I(M, E/L)$ and $H^{i+1}_I(L) \cong H^{i+1}_I(E/L)$ for all $i$. Thus, $H^i_I(E/L)$ is Artinian and $I$-cofinite, and $H^i_I(M, E/L)$ is Artinian for all $i < s - 1$. Now by the induction hypothesis, the result is proved. \hfill \Box

The following lemma has already been proved. However, we cannot find the original proof, so we give our own.

**Lemma 2.4.** Assume that $0 \longrightarrow L_1 \longrightarrow L_2 \longrightarrow L_3 \longrightarrow 0$ is an exact sequence of finitely generated $R$-modules. Then we have the long exact sequence

$$\cdots \longrightarrow H^i_I(L_3, N) \longrightarrow H^i_I(L_2, N) \longrightarrow H^i_I(L_1, N) \longrightarrow H^{i+1}_I(L_3, N) \longrightarrow \cdots.$$
Let $M$ and $N$ be two finitely generated $R$-modules. If, for some nonnegative integer $t$, $H^i(M, N)$ is Artinian for all $i < t$, then $H^i(M, N)$ is $I$-cofinite for all $i < t$ and $\text{Hom}(R/I, H^i(M, N))$ is finitely generated.

**Proof.** Let $0 \rightarrow N \rightarrow E^*$ be a minimal injective resolution of $N$. Note that $\Gamma_I(E)$ is injective if $E$ is injective. Then $\text{Hom}(-, \Gamma_I(E))$ is an exact functor. So $0 \rightarrow \text{Hom}(L_3, \Gamma_I(E^*)) \rightarrow \text{Hom}(L_2, \Gamma_I(E^*)) \rightarrow \text{Hom}(L_1, \Gamma_I(E^*)) \rightarrow 0$ is an exact sequence of $R$-complexes. By a well-known theorem of homology theory, we have a long exact sequence

$$\cdots \rightarrow H^i(\text{Hom}(L_3, \Gamma_I(E^*))) \rightarrow H^i(\text{Hom}(L_2, \Gamma_I(E^*))) \rightarrow H^i(\text{Hom}(L_1, \Gamma_I(E^*))) \rightarrow H^{i+1}(\text{Hom}(L_3, \Gamma_I(E^*))) \rightarrow \cdots.$$ 

Suppose that $M$ is a finitely generated $R$-module. Then

$$\Gamma_I(\text{Hom}(M, E^*)) \cong \text{Hom}(M, \Gamma_I(E^*)),$$

and so

$$H^i_I(M, N) = \lim_{\substack{n \to \infty}} \text{Ext}^1_R(M/I^n M, N) \cong H^i(\Gamma_I(\text{Hom}(M, E^*)))$$

$$\cong H^i(\text{Hom}(M, \Gamma_I(E^*))).$$

Hence, we obtain the long exact sequence that we wished to prove. \qed

For any submodule $K$ of a finitely generated $R$-module $L$, we use $K : L \langle m \rangle$ to denote the submodule $\{x \in L | m^n x \subseteq K \text{ for some } n > 0\}$. A sequence $x_1, \ldots, x_n$ of elements in $m$ is said to be an $m$-filter regular sequence on a module $N$ if

$$(x_1, \ldots, x_{i-1})N :_N x_i \subseteq (x_1, \ldots, x_{i-1})N :_N \langle m \rangle$$

for all $i = 1, \ldots, n$. The f-depth of an ideal $I$ on a module $N$ is defined as the length of any maximal $m$-filter regular sequence on $N$ in $I$; we denote it by $f$-depth($I, N$).

As the analogue of a result in the local cohomology modules case, the authors of [5] proved that $f$-depth($I + \text{Ann} M, N$) is equal to $\text{Min}\{r | H^r_I(M, N) \text{ is not Artinian}\}$. On the other hand, from the definition of generalized local cohomology modules, it is well known that, for all $i$,

$$H^i_{I + \text{Ann} M} N(M, N) \cong H^i_{I + \text{Ann} M} M(M, N) \cong H^i_I(M, N).$$

Now we are in the position to present our first main result.

**Theorem 2.5.** Let $M$ and $N$ be two finitely generated $R$-modules. If, for some nonnegative integer $t$, $H^i_I(M, N)$ is Artinian for all $i < t$, then $H^i_I(M, N)$ is $I$-cofinite for all $i < t$ and $\text{Hom}(R/I, H^i_I(M, N))$ is finitely generated.

**Proof.** Since $H^i_I(M, N)$ is Artinian for all $i < t$, $H^i_{I + \text{Ann} M} N(M, N) \cong H^i_{I + \text{Ann} M} M(M, N) \cong H^i_I(M, N)$ is Artinian for all $i < t$ by [5, Theorem 2.2]. Then by Lemma 2.1, $H^i_{I + \text{Ann} M} N(M, N)$ is $(I + \text{Ann} M)$-cofinite for all $i < t$. Therefore, for any $i < t$, $H^i_{I + \text{Ann} M} N(M, N)$ is $(I + \text{Ann} M)$-cofinite by Lemma 2.3. In particular, $\text{Hom}(R/(I + \text{Ann} M), H^i_{I + \text{Ann} M} N(M, N))$ is...
Let $I$ be an ideal of $R$, and let $M, N$ be two finitely generated modules. By Lemma 2.6, we can assume that $\text{Ann } M \subseteq I$.

Let $0 \rightarrow K \rightarrow R^n \rightarrow M \rightarrow 0$ be an exact sequence of finitely generated $R$-modules. By Lemma 2.4, we have the exact sequence

$$\cdots \rightarrow H^i_{I}(K, N) \xrightarrow{\alpha} H^i_{I}(M, N) \rightarrow H^i_{I}(R^n, N) \rightarrow \cdots.$$  

It is clear that $H^i_{I}(R^n, N) \cong \bigoplus_{i=1}^{\infty} H^i_{I}(N)$ for any $i$. Then $H^i_{I}(R^n, N)$ and $H^i_{I}(K, N)$ are Artinian for all $i < t$ by [5, Theorem 2.2]. By virtue of the former part of the result, we know that $H^i_{I}(K, N)$ is $I$-cofinite for all $i < t$. Let $L$ denote the image of $\alpha$ in the above long exact sequence. By [13, Corollary 4.4], $L$ is $I$-cofinite. From the exact sequence

$$0 \rightarrow L \xrightarrow{\alpha} H^i_{I}(M, N) \rightarrow H^i_{I}(R^n, N) \rightarrow \cdots,$$

we have the exact sequence

$$0 \rightarrow \text{Hom}(R/I, L) \rightarrow \text{Hom}(R/I, H^i_{I}(M, N)) \rightarrow \text{Hom}(R/I, H^i_{I}(R^n, N)).$$

By Lemma 2.1, the right term of the above exact sequence is finitely generated. Then the result follows from the above exact sequence. 

The following lemma is a generalization of [14, Lemma 3.4].

**Lemma 2.6** ([12, Theorem 3.2]). Let $M$ be a finitely generated $R$-module such that $d = \text{pd}(M) < \infty$. Let $N$ be a finitely generated $R$-module and assume that $n$ is an integer, and $x_1, x_2, \ldots, x_n$ is an I-filter regular sequence on $N$. Then

$$H^{i+n}_{I}(M, N) \cong H^i_{I}(M, H^n_{(x_1, x_2, \ldots, x_n)}(N))$$

for all $i \geq d$. 

**Proposition 2.7.** Let $I$ be an ideal of $R$, and let $M, N$ be two finitely generated $R$-modules such that $d = \text{pd}(M) < \infty$ and $\dim N = n < \infty$. Then $H^{d+n}_{I}(M, N) \cong \text{Ext}^d_R(M, H^n_{I}(N))$. In particular, $H^{d+n}_{I}(M, N)$ is Artinian.
Let $N$ be a finitely generated $R$-module and let $t$ be a positive integer. The 'only if' part has been proved in [14, Lemma 3.4],

$$H^n_{(x_1,x_2,\ldots,x_n)}(N) \cong H^0_I(H^n_{(x_1,x_2,\ldots,x_n)}(N)) \cong H^n_I(N).$$

Therefore, by Lemma 2.6,

$$H^{d+n}_I(M,N) \cong H^d_I(M,H^n_{(x_1,x_2,\ldots,x_n)}(N)) \cong H^d_I(M,H^n_I(N)) \cong \text{Ext}^d_R(M,H^n_I(N)).$$

This completes the proof. \hfill \qed

The following theorem is our second main result, which generalizes [6, Theorem 3].

**Theorem 2.8.** Let $I$ be an ideal of $R$, and let $M$, $N$ be two finitely generated $R$-modules such that $d = \text{pd}(M) < \infty$ and $\dim N = n < \infty$. Then $H^{d+n}_I(M,N)$ is $I$-cofinite.

**Proof.** By [6, Theorem 3], we know that $H^n_I(N)$ is $I$-cofinite. Then by Lemma 2.2 and Proposition 2.7, the result follows. \hfill \qed

In the last part of this note, we discuss the finiteness of $H^n_I(M,N)$.

**Lemma 2.9.** Let $N$ be a finitely generated $R$-module and $M$ a nonzero cyclic $R$-module. Let $t$ be a positive integer. If $H^i_I(N)$ is finitely generated for all $i < t$, then $H^i_I(N)$ is finitely generated if and only if $\text{Hom}(M,H^i_I(N))$ is finitely generated.

**Proof.** The 'only if' part is clear. Now suppose that $\text{Hom}(M,H^i_I(N))$ is finitely generated. Note that $\text{Hom}(M,H^i_I(N))$ is $I$-torsion, then there exists an integer $n$ such that $I^n\text{Hom}(M,H^i_I(N)) = 0$. Assume that $M$ is generated by an element $m$. For any $x \in H^i_I(N)$, we can find an element $f \in \text{Hom}(M,H^i_I(N))$ such that $f(m) = x$. Since $I^n f = 0$, $I^n x = 0$, and so $I^n H^i_I(N) = 0$. Since $H^i_I(N)$ is finitely generated for all $i < t$, by [3, Proposition 9.1.2], there exists an integer $r$, $I^r H^i_I(N) = 0$ for all $i < t$. Thus, $I^r H^i_I(N) = 0$ for all $i < t + 1$. Again by [3, Proposition 9.1.2], $H^i_I(N)$ is finitely generated for all $i < t + 1$. In particular, $H^i_I(N)$ is finitely generated. \hfill \qed

**Proposition 2.10.** Let $N$ be a finitely generated $R$-module and let $t$ be a positive integer. If $M$ is a nonzero cyclic $R$-module, then $H^i_I(N)$ is finitely generated for all $i < t$ if and only if $H^i_I(M,N)$ is finitely generated for all $i < t$.

**Proof.** The 'only if' part has been proved in [11, Theorem 1.1(iv)]. Now we suppose that $H^i_I(M,N)$ is finitely generated for all $i < t$. By induction on $t$, we can assume that $H^i_I(N)$ is finitely generated for all $i < t - 1$. Then by [11, Theorem 1.1(iii)], it follows that $\text{Hom}(M,H^{i-1}_I(N))$ is finitely generated from the fact that $H^{i-1}_I(M,N)$ is finitely generated. Then $H^{i-1}_I(N)$ is finitely generated by Lemma 2.9. \hfill \qed
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References


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