RANDOM LAGUERRE TESSELLATIONS

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Abstract

A systematic study of random Laguerre tessellations, weighted generalisations of the well-known Voronoi tessellations, is presented. We prove that every normal tessellation with convex cells in dimension three and higher is a Laguerre tessellation. Tessellations generated by stationary marked Poisson processes are then studied in detail. For these tessellations, we obtain integral formulae for geometric characteristics and densities of the typical k-faces. We present a formula for the linear contact distribution function and prove various limit results for convergence of Laguerre to Poisson–Voronoi tessellations. The obtained integral formulae are subsequently evaluated numerically for the planar case, demonstrating their applicability for practical purposes.

Keywords: Laguerre tessellation; power tessellation; weighted Voronoi tessellation; k-face density; Poisson process; random tessellation; stochastic geometry

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1. Introduction

Random tessellations are widely used to model natural cellular structures ranging from organic tissues to telecommunication networks to the origins of the universe. Perhaps the most popular model is the Voronoi tessellation in \( \mathbb{R}^d \) which is defined by an at most countable set of distinct ‘generator’ points or nuclei \( \varphi = \{x_1, x_2, \ldots \} \subset \mathbb{R}^d \) as follows. With each \( x_i \in \varphi \), there is associated a cell \( C(x_i, \varphi) \) consisting of the points of \( \mathbb{R}^d \) which are closer to \( x_i \) than to any other \( x_j \in \varphi \). Being the intersection of half-spaces, i.e.

\[
C(x_i, \varphi) = \bigcap_{x_j \in \varphi} \{ y \in \mathbb{R}^d : \| y - x_i \| \leq \| y - x_j \| \},
\]

all the cells are nonempty polytopes and it is also customary to call \( x_i \) the centre or centroid of the cell \( C_i = C(x_i, \varphi) \), meaning that there is a one-to-one correspondence between the cells and \( \varphi \). In the case when the nuclei set is a random point process \( \Phi \), the tessellation \( \{C(x, \Phi) : x \in \Phi\} \) becomes a random object, too, and it is usually described by means of the random closed set of the cells’ boundaries. In particular, when \( \Phi \) is a homogeneous Poisson process, we speak of the Poisson–Voronoi tessellation. A multitude of results is available for both nonrandom and random Voronoi tessellations. A comprehensive account of these can be found in the monographs [7] and [20].

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Although Voronoi tessellations proved extremely useful in modelling many natural phenomena, in some situations they may still be too restrictive. In particular, the cell geometry depends entirely on the mutual position and the distances between the nuclei; in this respect all the nuclei bear the same ‘weight’. But we can also envisage practical scenarios when different nuclei have different ‘power’, so that the more powerful has a bigger cell. This idea leads to the replacement of the Euclidean norm in (1.1) by a power distance. Namely, if each nucleus $x_i$ has an associated weight $w_i$ then the new cells are defined to be

$$C_i = \bigcap_{(x_j, w_j) \in \phi} \{ y \in \mathbb{R}^d : \text{pow}(y, (x_i, w_i)) \leq \text{pow}(y, (x_j, w_j)) \},$$

where $\text{pow}(y, (x, w)) = \|y - x\|^2 - w$. If $w$ is positive then it is indeed the power of the point $y$ with respect to a sphere $s(x, r)$ centred in $x$ with radius $r = \sqrt{w}$, hence the name. In this paper we mainly consider the case of positive weights and label the nuclei $x_i$ with these radii $r_i$ rather than with $w_i$.

The power distance has been considered by several authors [6], [8], [9]. The first analyses of the corresponding deterministic tessellations seem to be [1], [3], and [11]. Often these tessellations are called power tessellations; however, the synonym Laguerre tessellations has also been established and will be used in this paper.

It is clear that when all weights are the same, the Laguerre tessellation is the Voronoi tessellation. When they are not, much similarity to the Voronoi case still remains: the cells are all polytopes and, under some mild regularity assumptions of general position type imposed on the nuclei, the Laguerre tessellation is also normal, i.e. each $k$-dimensional face lies in the intersection of exactly $d - k + 1$ cells (vertices are zero-dimensional faces). But there are also differences, the most striking, perhaps, are that a nucleus may not be contained in its cell and that a Laguerre cell may be empty. Hence, for Laguerre tessellations, the notions of nuclei and cell centroids are different; see Figure 1.

In this paper the authors present the first systematic study of random Laguerre tessellations. The next section contains notation and formal definitions of the objects we will be dealing with: tessellations, their general properties, and their moment characteristics. In Section 3 we establish important topological properties of Laguerre tessellations and state that every normal...
tessellation in dimension three and higher is a Laguerre tessellation. The proof of this new result is given in Appendix A. Section 4 concentrates on Poisson–Laguerre tessellations. We first show a necessary and sufficient condition for the existence of the tessellation in terms of the moments of the radius distribution. Then we present general expressions for the intensities and the mean $k$-content of the $k$-faces of the tessellation. The proofs of these results are rather technical and therefore deferred to Appendix B. In Section 5 some limit results studying the convergence of Laguerre tessellations to Voronoi tessellations are presented. Finally, the obtained general expressions are illustrated for the planar case in Section 6. The formulae are specified, numerically evaluated for some examples, and further results are obtained. In particular, it is possible to explicitly evaluate the probability of a cell being empty. The paper concludes with appendices containing the proofs.

2. Preliminaries

2.1. Basic notation

Throughout this paper, we work in $d$-dimensional Euclidean space $\mathbb{R}^d$ equipped with the Euclidean norm $\| \cdot \|$ and the corresponding scalar product $(\cdot, \cdot)$. For $x \in \mathbb{R}^d$ and $r \geq 0$, let $b(x, r)$ denote the closed $d$-dimensional ball of radius $r$ centred in $x$ and let $s(x, r) = \partial b(x, r)$ denote the sphere given by its boundary. The $d$-dimensional unit sphere is denoted by $S^{d-1}$. Let $B^d$ for the Borel sets in $\mathbb{R}^d$, $\lambda_d$ for the $d$-dimensional Lebesgue measure, and $\sigma$ for the surface measure on $S^{d-1}$. Let $|B| = \lambda_d(b(0, 1))$ denote the volume, and let $\sigma_d = \sigma(S^{d-1})$ denote the surface area of the unit ball in $\mathbb{R}^d$, i.e.

$$
\omega_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} \quad \text{and} \quad \sigma_d = 2\frac{\pi^{d/2}}{\Gamma(d/2)}.
$$

Let $\mathcal{L}^d_k$ and $\mathcal{B}^d_k$ denote the set of $k$-dimensional linear and affine subspaces, respectively, of $\mathbb{R}^d$. Write $SO_d$ for the group of rotations about the origin in $\mathbb{R}^d$ and write $\nu$ for its unique rotational invariant probability measure. The translation $x + B$ of a set $B \subset \mathbb{R}^d$ by a point $x \in \mathbb{R}^d$ is defined via $x + B = \{x + b: b \in B\}$. The rotation $\vartheta B$ of $B$ by $\vartheta \in SO_d$ is defined analogously. For $k \in \{0, \ldots, d\}$, denote the $k$-dimensional Hausdorff measure by $\mathcal{H}^k$. Finally, we denote the indicator function of a set $B \subset \mathbb{R}^d$ by $1_B$, i.e. $1_B(x) = 1$ if $x \in B$ and $1_B(x) = 0$ otherwise.

2.2. Tessellations of $\mathbb{R}^d$

A tessellation of $\mathbb{R}^d$ is a countable set $T = \{C_i: i \in \mathbb{N}\}$ of sets $C_i \subset \mathbb{R}^d$ (the cells of the tessellation) such that

(i) $\text{int}(C_i) \cap \text{int}(C_j) = \emptyset$, $i \neq j$;

(ii) $\bigcup_{i \in \mathbb{N}} C_i = \mathbb{R}^d$;

(iii) $T$ is locally finite, i.e. $\#\{i \in \mathbb{N}: C_i \cap B \neq \emptyset\} < \infty$ for all bounded $B \subset \mathbb{R}^d$; and

(iv) each cell of the tessellation is a compact set with interior points.

If, in addition, all the cells are convex, as it will always be in this paper, then [24, Lemma 6.1.1] implies that the cells are bounded $d$-dimensional polytopes.

The faces of a convex polytope $P$ are the intersections of $P$ with its supporting hyperplanes [22, Section 2.4]. We call a face of $P$ of dimension $s$, $s \in \{0, \ldots, d - 1\}$, an $s$-face of $P$. For convenience, the polytope $P$ itself is considered as a $d$-face. Write $\mathcal{F}_s(P)$ for the set of $s$-faces of a polytope $P$ and $\mathcal{F}_s(T) = \bigcup_{i \in \mathbb{N}} \mathcal{F}_s(C_i)$ for the set of $s$-faces of all cells of the tessellation $T$. 

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Furthermore, let $F(y)$ be the intersection of all cells of the tessellation containing the point $y$. Then $F(y)$ is a finite intersection of $d$-polytopes and, since it is nonempty, $F(y)$ is an $s$-dimensional polytope for some $s \in \{0, \ldots, d\}$. Therefore, we may introduce

$$\delta_s(T) = \{ F(y) : \dim F(y) = s, y \in \mathbb{R}^d \}, \quad s = 0, \ldots, d,$$

the set of $s$-faces of the tessellation $T$. Then an $s$-face $H \in \mathcal{F}_s(C)$ of a cell $C$ of $T$ is the union of all those $s$-faces $F \in \delta_s(T)$ of the tessellation contained in $H$.

A tessellation $T$ is called face-to-face if the faces of the cells and the faces of the tessellation coincide, i.e. if $\delta_s(T) = \mathcal{F}_s(T)$ for $s = 0, \ldots, d$. For $s = 0$ and $s = d$, this is always true.

A tessellation $T$ is called normal if it is face-to-face and every $s$-face of $T$ is contained in the boundary of exactly $d - s + 1$ cells for $s = 0, \ldots, d - 1$.

Write $\mathbb{T}$ for the set of all tessellations in $\mathbb{R}^d$ and equip it with a suitable $\sigma$-field $\mathcal{T}$ as described in [16, p. 46]. A random tessellation in $\mathbb{R}^d$ is then a random element $X$ on a probability space $(\Omega, \mathcal{A}, P)$ with range $(\mathbb{T}, \mathcal{T})$. It is called normal or face-to-face if its realisations are almost surely normal or face-to-face, respectively.

The structure of a random tessellation is usually described by means of geometric and topological characteristics of its $k$-faces. For stationary tessellations, the easiest such characteristics are the intensities of the $k$-faces, i.e. the mean number of $k$-faces per unit volume. In order to formalise this, we first define the notion of the centroid of a $k$-face.

Denote by $\mathcal{P}_k$ the set of $k$-dimensional polytopes in $\mathbb{R}^d$, and let $c_k : \mathcal{P}_k \times \mathbb{T} \rightarrow \mathbb{R}^d$ be a measurable centroid function, i.e.

$$c_k(F + x, T + x) = c_k(F, T) + x, \quad x \in \mathbb{R}^d, F \in \mathcal{P}_k, T \in \mathbb{T}, \quad (2.1)$$

such that $c_k(F) \neq c_k(F')$ for different $k$-faces $F, F' \in \delta_k(T)$. We call the point $c_k(F, T)$ the centroid of the $k$-face $F \in \delta_k(T)$. For a random tessellation $X$, we can now introduce the point process $N_k$ of centroids of the $k$-faces of $X$. By (2.1), $N_k$ is almost surely a simple and stationary point process whose intensity $\gamma_k$ is then given by

$$\gamma_k = \mathbb{E} \left[ \sum_{F \in \delta_k(X)} 1_{[0,1]^d}(c_k(F, X)) \right], \quad k = 0, \ldots, d.$$ 

The values of $\gamma_k$ do not depend on the choice of the centroid function $c_k$ [16, p. 47].

Provided that the intensities above are finite, the Palm distribution $P^0_k$ of $N_k$ can be defined. Under $P^0_k$, there is a $k$-face $C_k(0)$ with centroid in the origin. Its distribution is called the distribution of the typical $k$-face of the tessellation $X$.

Further random measures induced by a random tessellation are the measures

$$M_k(B) = \sum_{F \in \delta_k(X)} \mathcal{H}^k(F \cap B), \quad k = 0, \ldots, d, \quad B \in \mathcal{B}^d.$$ 

Their intensities

$$\mu_k = \mathbb{E} \left[ \sum_{F \in \delta_k(X)} \mathcal{H}^k(F \cap [0, 1]^d) \right], \quad k = 0, \ldots, d,$$

can be interpreted as the mean total $k$-content of the $k$-faces of the tessellation per unit volume. Also, the Palm probability measure $Q^0_k$ of $M_k$ is of special interest. With respect to this measure, the origin is almost surely contained in a $k$-face $F_k(0)$ of $X$. 

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The Palm measures $P_k^0$ and $Q_k^0$ are closely related. In particular, their intensities satisfy

$$\gamma_k = \mu_k \mathbb{E}_{M_k}^{(0)}[\mathcal{H}^d(F_k(0))^{-1}], \quad (2.2)$$

where $\mathbb{E}_{M_k}^{(0)}$ denotes the expectation with respect to $Q_k^0$.

As shown in [13], the mean values of the cell characteristics of a planar face-to-face tessellation are completely determined by the values of $\mu_0$ and $\mu_1$ (usually denoted by $L_A$). For a spatial tessellation, the required parameters are $\mu_0$, $\mu_1$ ($L_V$), $\mu_2$ ($S_V$), and the cell intensity, $\gamma_3$.

3. Laguerre tessellations

For $y, x \in \mathbb{R}^d$ and $r \geq 0$, define the power of $y$ with respect to the sphere $s(x, r)$ as

$$\text{pow}(y, s(x, r)) = \|y - x\|^2 - r^2.$$

Let $\varphi \subset \mathbb{R}^d \times \mathbb{R}_+^d$ be an at most countable set such that $\min_{(x, r) \in \varphi} \text{pow}(y, s(x, r))$ exists for each $y \in \mathbb{R}^d$. Then the Laguerre cell of $(x, r) \in \varphi$ is defined as

$$C((x, r), \varphi) = \{y \in \mathbb{R}^d : \min_{s(x, r) \in \varphi} \text{pow}(y, s(x, r)) \leq \min_{s(x', r') \in \varphi} \text{pow}(y, s(x', r'))\}, \quad (x', r') \in \varphi.$$

The point $x$ is called the nucleus of the cell $C((x, r), \varphi)$, and the Laguerre diagram $L(\varphi)$ is the set of the nonempty Laguerre cells of $\varphi$. If the radii of all spheres in $\varphi$ are equal then $L(\varphi)$ is the Voronoi tessellation of the set $\{x : (x, r) \in \varphi\}$.

We will often identify a pair $(x, r) \in \mathbb{R}^d \times \mathbb{R}_+$ with the sphere $s(x, r)$ and use both forms of notation synonymously. Also, the abbreviation $s_i = s(x_i, r_i)$ will be used.

Note that a Laguerre cell does not necessarily contain its nucleus and that a nucleus does not necessarily generate a cell. A necessary condition for a cell to be empty is that the generating sphere is completely contained in the union of the remaining spheres. However, this is not a sufficient condition, as Figure 1 shows.

Given two spheres $s_1 = s(x_1, r_1)$ and $s_2 = s(x_2, r_2)$ in $\mathbb{R}^d$, the points $z \in \mathbb{R}^d$ satisfying $\text{pow}(z, s_1) = \text{pow}(z, s_2)$ form a hyperplane $Ra(s_1, s_2)$ given by

$$Ra(s_1, s_2) = \{z \in \mathbb{R}^d : 2\langle z, x_1 - x_2 \rangle = \|x_1\|^2 - \|x_2\|^2 + r_2^2 - r_1^2\},$$

which is perpendicular to the line joining $x_1$ and $x_2$ and called the radical axis of $s_1$ and $s_2$. If two spheres intersect then their radical axis passes through their intersections. If two spheres have equal radii, their radical axis is the perpendicular bisector of the line joining their centres.

Every $s$-face $F \in \delta_s(L(\varphi))$ can be written as

$$F = F(s_0, \ldots, s_k, \varphi) = \bigcap_{i=0}^k C(s_i, \varphi), \quad s_0, \ldots, s_k \in \varphi, \quad (3.1)$$

with a suitable number of cells involved. Then $F(s_0, \ldots, s_k, \varphi)$ is included in the affine subspace $\{y \in \mathbb{R}^d : \min_{s(x, r) \in \varphi} \text{pow}(y, s(x, r)) \leq \min_{s(x', r') \in \varphi} \text{pow}(y, s(x', r'))\}$.

For $\varphi \subset \mathbb{R}^d \times \mathbb{R}_+$, introduce the following regularity conditions:

(R1) for every $y \in \mathbb{R}^d$ and every $t \in \mathbb{R}$, only finitely many elements $(x, r) \in \varphi$ satisfy $\|y - x\|^2 - r^2 \leq t$; and

(R2) the convex hull of $\{x : (x, r) \in \varphi\}$ is the whole space $\mathbb{R}^d$. 

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If the set of radii is bounded, condition (R1) implies the local finiteness of the set of points \( \{x : (x, r) \in \varphi\} \).

Furthermore, we say that the points of \( \varphi \) are in general position if

1. \((GP1)\) no \(k + 1\) nuclei are contained in a \((k - 1)\)-dimensional affine subspace of \( \mathbb{R}^d \) for \( k = 2, \ldots, d \); and
2. \((GP2)\) no \(d + 2\) points have equal power with respect to some point in \( \mathbb{R}^d \).

In the case of equal radii this is exactly the property addressed as general quadratic position in [18, p. 5].

**Theorem 3.1.** If the set \( \varphi \subset \mathbb{R}^d \times \mathbb{R}_+ \) satisfies the regularity conditions \( (R1) \) and \( (R2) \) then the set of the Laguerre cells \( C((x, r), \varphi), (x, r) \in \varphi \), with nonvanishing interior is a face-to-face tessellation of \( \mathbb{R}^d \). If, in addition, the points of \( \varphi \) are in general position then all the cells of \( L(\varphi) \) have dimension \( d \) and the Laguerre tessellation \( L(\varphi) \) is normal.

For the proof, we refer the reader to [12] and [21].

Aurenhammer [2] gave a complete characterisation of the set of Laguerre diagrams generated by finite sets of spheres. A very pleasing result is that each finite normal cell complex can be realised as a Laguerre diagram. However, diagrams with finitely many cells necessarily contain unbounded cells; hence, they do not belong to \( T \). Theorem 3.2, below, generalises Aurenhammer’s results to the case of infinitely many spheres (or cells), which then also applies to tessellations in the sense defined above. The proof can be found in Appendix A.

**Theorem 3.2.** Every normal tessellation of \( \mathbb{R}^d \) with convex cells for \( d \geq 3 \) is a Laguerre tessellation.

Note that the above statement cannot be strengthened to include \( d = 2 \), a counterexample is given in [2].

### 4. Poisson–Laguerre tessellations

From now on we will assume that the set \( \varphi \) is a realisation of a stationary, independently marked Poisson process \( \Phi \) on \( \mathbb{R}^d \times \mathbb{R}_+ \) with intensity \( \lambda > 0 \) and mark distribution \( \rho \). The proofs of the statements in this section are mainly given in Appendix B.

**Theorem 4.1.** Suppose that \( R \) is a positive random variable with distribution \( \rho \). Then the Laguerre tessellation of \( \Phi \) exists, i.e. \( \min_{(x, r) \in \Phi} \text{pow}(y, (x, r)) > -\infty \) almost surely for all \( y \in \mathbb{R}^d \), if and only if \( \text{E}[(R \lor 1)^d] < \infty \), where \( a \lor b = \max(a, b) \).

From now on we will assume that the mark distribution \( \rho \) of \( \Phi \) satisfies the condition \( \text{E}[(R \lor 1)^d] < \infty \).

**Theorem 4.2.** The Laguerre tessellation of \( \Phi \) is a normal random tessellation.

Now we pass to the properties related to the Palm probability measure \( Q^0_k \) defined in Section 2. Its complete description in terms of integrals has been obtained in [12], the special case of the Voronoi tessellation has been studied in [5]. Here, we only state the formulae for the mean number \( \gamma_k \) and the mean \( k \)-content \( \mu_k \) of \( k \)-faces per unit volume.
For a natural number \( m \) and \( x_0, \ldots, x_m \in \mathbb{R}^m \), let \( \Delta_m(x_0, \ldots, x_m) \) be the \( m \)-dimensional volume of the convex hull of \( x_0, \ldots, x_m \) in \( \mathbb{R}^m \). For \( w_0, \ldots, w_m \geq 0 \), define

\[
V_{m,k}(w_0, \ldots, w_m) = \int_{\mathbb{R}^m} \cdots \int_{\mathbb{R}^m} \Delta_{m+1}^{k+1}(w_0u_0, \ldots, w_mu_m) \sigma(du_0) \cdots \sigma(du_m). 
\]

In the remainder of this section,

\[
p(t) = \exp\left(-\lambda \omega(t) \int_0^\infty (|t + r^2|^+)^{d/2} \rho(dr)\right),
\]

where \( r^+ = \max(t, 0) \), is the probability that the power from the origin to each point of \( \Phi \) exceeds \( t \); see (B.1) in Appendix B.

**Theorem 4.3.** Let \( \Phi \) be a stationary marked Poisson process with intensity \( \lambda > 0 \) and mark distribution \( \rho \) satisfying \( \mathbb{E}[(R \vee 1)^d] < \infty \). The intensities \( \mu_k, 0 < k < d \), are given by the formula

\[
\mu_k = \frac{\lambda^{m+1}/4(m+1)!}{c_{m+k}}, 
\]

where \( m = d - k \) and \( c_{m+k} = \sigma_{d-m+1} \cdots \sigma_1/\sigma_m \). For \( k = d \), we have \( \mu_d = 1 \), for \( k = 0 \),

\[
\mu_0 = \frac{\lambda^{d+1}/2(d+1)!}{2(d+1)!} 
\]

The formulae for \( \mu_k \) cannot be evaluated further because of the lack of an explicit formula for \( V_{m,k}(w_0, \ldots, w_m) \). However, a formula by Miles \cite{14} shows that

\[
V_{m,k}(1, \ldots, 1) = 2^{m+1} \pi^{m(m+1)/2} \Gamma((1/2)(m+1)(d+1) - m) \prod_{i=1}^m \Gamma((1/2)(k+1+i))/\Gamma((i/2). \]

For a degenerate radius distribution \( \rho \), i.e. the case of a Poisson–Voronoi tessellation, applying (4.2) to (4.1) leads to the well-known values

\[
\mu_k^V = \frac{\lambda^{m+d} \pi^{m/2} \Gamma((dm + k + 1)/2) \Gamma((d+1)/2)^{m+k/d} \Gamma(m + k/d)}{d(m+1)! \Gamma((dm + k + 1)/2) \Gamma((d+1)/2)^m \Gamma((k+1)/2)}. \]

While this formula is explicit, the computation of \( \mu_k \) for other radius distributions \( \rho \) usually requires numerical integration. Some examples will be considered in Section 6.
Theorem 4.4. For $0 < k < d$, the intensity $\gamma_k$ of the $k$-faces is given by

\[
\gamma_k = \frac{\lambda^{m+1}}{4(m+1)!} cdm\sigma_k \times \int_0^\infty \cdots \int_0^\infty \int_{r^2_i = 0}^\infty \prod_{i=0}^{m-2}(t + r^2_i)^{(m-2)/2} V_{m,k}((t + r^2_0)^{1/2}, \ldots, (t + r^2_m)^{1/2}) \times \int_0^\infty (t + s)^{(k-2)/2} E[A_{L^\perp}(t, s, \Phi^{t+s})^{-1}] ds \, dt \, d\rho(dr_0) \cdots d\rho(dr_m),
\]

where $L \in \mathbb{R}^d_m$ is a fixed subspace of $\mathbb{R}^d$, $\Phi^l = \Phi \cap \{(x, r) : \text{pow}(0, (x, r)) > t\}$,

\[
A_{L^\perp}(t, s, r) = \int_0^\infty l^{-1} \int_{\mathbb{R}^{d-1} \cap L^\perp} 1[\tau(l, t, s, v, u) \leq \text{pow}(lv, (x, r)), (x, r) \in \eta] \sigma(L^\perp)(dv) \, dl
\]

with $\tau(l, t, s, u, v) = l^2 + t + s - 2ls^{1/2}(u, v)$ for fixed $u \in \mathbb{S}^{d-1} \cap L^\perp$, and $\sigma(L^\perp)$ is the surface measure on the $(d-m)$-dimensional sphere $\mathbb{S}^{d-1} \cap L^\perp$.

The formula for the cell intensity $\gamma_d$ reads

\[
\gamma_d = \frac{\lambda \sigma_d}{2} \int_0^\infty \int_{r^2_o = 0}^\infty (t + r^2_0)^{(d-2)/2} p(t) E[A(t, r_0, u, \Phi^t)^{-1}] dt \, d\rho(dr_0),
\]

where $u \in \mathbb{S}^{d-1}$ is fixed and

\[
A(t, r_0, u, \eta) = \int_0^\infty l^{-1} \int_{\mathbb{S}^{d-1}} 1[\xi(l, t, r_0, v, u) \leq \text{pow}(lv, (x, r)), (x, r) \in \eta] \sigma(dv) \, dl
\]

with $\xi(l, t, r_0, u, v) = l^2 + t - 2((l + r^2_0)^{1/2} + (l + r^2_1)^{1/2})(u, v)$.

4.1. Typical faces

In Section 2 we defined two Palm measures $P_0^0$ and $Q_0^0$. Under $P_0^0$, there is a $k$-face $C_k(0)$ with centroid in the origin, allowing us to call this random element a typical $k$-face. In contrast, with respect to $Q_0^0$, the origin has been ‘uniformly’ chosen on the $k$-dimensional boundary of the tessellation which gives rise to a $k$-face $F_k(0)$ containing the origin. Then $F_k(0)$ is, in a sense, a typical edge weighted with its $k$-content. In particular, $F_d(0)$ is simply the cell containing the origin. In the following, we will give the formulae for the mean $k$-content of the $k$-faces $F_k(0)$ and $C_k(0)$.

Denote by $\kappa(l, r_1, r_2)$ the volume of the union of two balls with radii $r_1$ and $r_2$ and centres separated by distance $l$. There is an explicit expression for this union, which is rather cumbersome; see, e.g. [12, Proposition 3.3.4]. Introduce, for $l \geq 0$ and $t_1, t_2 \in \mathbb{R}$,

\[
\xi(l, t_1, t_2) = \exp\left(-\lambda \int_0^\infty \kappa(l, (I_1 + r^2_1)^{1/2}, (I_2 + r^2_2)^{1/2}) \rho(dr)\right).
\]
Theorem 4.5. The mean $k$-content of the $k$-dimensional face $F_k(0)$ for $0 < k < d$ is given by

$$
\mu_k E^0_{\mathcal{M}_d}[\mathcal{H}^k(F_k(0))] \equiv \frac{\lambda^{m+1}}{4(m+1)!} c_{d,m} \sigma^k 
$$

$$
\times \int_0^\infty \cdots \int_0^\infty \int_{\min r_i}^\infty \prod_{i=0}^{m} \frac{(t+r_i)^{(m-2)/2} V_{m,k}((t+r_i)^{1/2}, \ldots, (t+r_m)^{1/2})}{1} \times \int_\mathbb{S}^{d-1} \int_0^{\infty} \int_{l^d-\cap L^\perp} \xi(l, s + t, \tau(l, t, s, u, v)) \sigma(dv) \, ds \, dt \rho(dr_0) \ldots \rho(dr_m),
$$

where $u \in \mathbb{S}^{d-1} \cap L^\perp$ is a fixed vector and the function $\xi$ is defined in (4.3). For the mean volume of $F_d(0)$, we have

$$
\mathbb{E}^0_{\mathcal{H}_d}[\mathcal{H}^d(F_d(0))] \equiv \frac{\lambda \sigma_d}{2} \int_0^{\infty} \int_{-r_0^2}^{\infty} (t + r_0^2)^{(d-2)/2} \int_0^{\infty} \int_{l^d-\cap L^\perp} \xi(l, t, \rho(l, t, r_0, u, v)) \sigma(dv) \, ds \, dt \rho(dr_0).
$$

A formula for the mean $k$-content of the typical $k$-face $C_k(0)$ is obtained from the previous results via the relation

$$
\gamma_k \mathbb{E}^0_{\mathcal{N}_d}[\mathcal{H}^k(C_k(0))] = \mu_k.
$$

Further formulae for distributions related to the typical $k$-faces $F_k(0)$ and $C_k(0)$, in particular the joint distribution of their generators, are given in [12, Section 3.3].

4.2. Contact distributions

Recall that, for a random closed set $X$ and a convex compact set $B$ in $\mathbb{R}^d$ containing the origin, the contact distribution function $H_B$ is defined via

$$
H_B(r) = P(X \cap rB \neq \emptyset | 0 /\in X), \quad r \geq 0.
$$

The random closed set of interest here is the union of cell boundaries of the tessellation. Since the origin is almost surely contained in the cell $F_d(0)$, we have $H_B(r) = 1 - P(rB \subset F_d(0))$ for every choice of $B$. Important special cases are the spherical contact distribution function $H_s$, where $B = b(0, 1)$ is the unit ball centred in the origin, and the linear contact distribution function $H_l(v)$, where $B = l(v)$ is a line segment of unit length in direction $v \in \mathbb{S}^{d-1}$.

Contact and chord length distributions of the Poisson–Voronoi tessellation have been studied in [19], while the Voronoi tessellation with respect to more general point processes has been investigated in [10]. For Poisson–Laguerre tessellations, we have the following result.

Theorem 4.6. The linear contact distribution function $H_l(v)$ for $v \in \mathbb{S}^{d-1}$ is given by

$$
1 - H_l(v)(r) = \frac{\lambda}{2} \int_0^{\infty} \int_{-r_0^2}^{\infty} (t + r_0^2)^{(d-2)/2} \times \int_{\mathbb{S}^{d-1}} \xi(r, t, \rho(r, t, r_0, u, v)) \sigma(du) \, dr_0, \quad r \geq 0,
$$

where $\xi(l, t_1, t_2)$ is defined in (4.3).
Since the Poisson–Laguerre tessellation is isotropic, the values of $H_l(v)(r)$ do not depend on the direction $v$ of the line segment.

An expression for the spherical contact distribution function can also be obtained and is given in [12, Corollary 3.3.16].

### 5. Limit results

Since a Voronoi tessellation can be interpreted as a Laguerre tessellation with respect to a degenerate distribution of radii, it is natural to consider Poisson–Voronoi tessellations as limits of Poisson–Laguerre tessellations when changing the parameters of the mark distribution. In this section we present some limit results which we prove in Appendix B.

Consider a stationary marked Poisson process $\Phi_1$ on $\mathbb{R}^d \times \mathbb{R}_+$ with intensity $\lambda$ and mark distribution $\rho$ satisfying $E[(R \vee 1)^d] < \infty$. Write $\Phi_v = \{(x, vr) : (x, v) \in \Phi\}$, $v > 0$, for a mark-scaled version of the point process $\Phi_1$.

Theorem 5.1. Almost surely, as $v \downarrow 0$, the boundary $F_{d-1}(L(\Phi_v))$ of the Laguerre tessellation $L(\Phi_v)$ converges in Wijsman topology to the boundary $F_{d-1}(L^V (\hat{\Phi}))$ of the Voronoi tessellation $L^V (\hat{\Phi})$ constructed with respect to the Poisson process $\hat{\Phi} = \{x \in \mathbb{R}^d : (x, v) \in \Phi\}$ with intensity $\lambda$.

The next limiting regime is when there is an atom at the maximum value of the radius distribution, implying that the corresponding largest radius cells will eventually dominate the others.

Theorem 5.2. Assume that the mark distribution $\rho$ is supported by a bounded segment $[0, s]$ and is such that $\rho(\{s\}) = p > 0$. Denote by $\Phi_s = \{x \in \mathbb{R}^d : (x, s) \in \Phi\}$ the subset of points carrying the weight $s$ (which is a Poisson process with intensity $p\lambda$ in $\mathbb{R}^d$). Then almost surely for any bounded set $W \subset \mathbb{R}^d$ there exists $v_0 = v_0(\Phi, W) > 0$ such that the boundary of the Laguerre tessellation $L(\Phi_v)$ inside $W$ coincides with the boundary of the Voronoi tessellation $L^V (\hat{\Phi})$ restricted to $W$ for all $v > v_0$.

This theorem also implies the almost sure Wijsman convergence of the boundaries $F_{d-1}(L(\Phi_v))$ to their Voronoi tessellation counterparts. But the result formulated above is stronger, of ‘a finite coupling time’ type.

Convergence in the above schemes is also discussed in [12] in terms of the Palm distributions $P^{\Phi}_k$ corresponding to the $k$-faces of the tessellation.

### 6. The planar case

For applications, the cases in which $d = 2$ and $d = 3$ are of special interest. In this section we discuss the planar case in detail: more explicit formulae for $\mu_0$ and $\mu_1$ are obtained and evaluated for some examples. Furthermore, we derive a formula for the probability $p_0$ that the typical point of $\Phi$ generates a nonempty cell.

The main problem when working with the expressions in Theorem 4.3 is the lack of explicit general formulae for $\Delta^{k+1}(w_0, \ldots, w_m)$ and $V_{m,k}(w_0, \ldots, w_m)$. However, in some special cases it is possible to overcome this problem. In the two-dimensional case we have to
Consider \( \Delta_2 \) and \( \Delta_1^2 \). Unfortunately, \( \Delta_2(w_0u_0, w_1u_1, w_2u_2) \) remains intractable. But we have
\[
\Delta_1^2(w_0u_0, w_1u_1) = w_0^2 + w_1^2 - 2(u_0, u_1)w_0w_1,
\]
and, therefore, \( V_{1,1}(w_0, w_1) = 4(w_0^2 + w_1^2) \).

For the intensities \( \mu_0 \) and \( \mu_1 \), we obtain
\[
\mu_0 = \frac{\lambda^3}{12} \iint_{B_+^2} \int_{\min r_i^2}^{\infty} \exp\left( -\lambda \pi \int_0^{\infty} [t + r^2]^+ \rho(dr) \right)
\]
\[
\times V_{2,0}(t + r_0^2)^{1/2}, (t + r_1^2)^{1/2}, (t + r_2^2)^{1/2}) dt \rho(\text{dr}_0) \rho(\text{dr}_1) \rho(\text{dr}_2)
\]
\[
\mu_1 = \frac{\lambda^2 \pi}{12} \iint_{B_+^2} \int_{\min r_i^2}^{\infty} \frac{2t + r_0^2 + r_1^2}{(t + r_0^2)^{1/2}(t + r_1^2)^{1/2}}
\]
\[
\times \int_{r_2^2}^{\infty} \exp\left( -\lambda \pi \int_0^{\infty} [t + s + r^2]^+ \rho(dr) \right)
\]
\[
\times s^{-1/2} ds \rho(\text{dr}_0) \rho(\text{dr}_1) \rho(\text{dr}_2).
\]

These two formulae provide all the parameters which are required for computing the mean values of the cell characteristics using the mean-value relations for normal tessellations [13]. In particular, we can derive a formula for the probability \( p_0 \) that the typical point of \( \Phi \) generates a nonempty cell: since the intensity of cells is given by \( \gamma_2 = \mu_0 \lambda \) and \( \mu_0 = \gamma_0 = 2\gamma_2 \), we have
\[
p_0 = \frac{\lambda^2}{24} \iint_{B_+^2} \int_{\min r_i^2}^{\infty} \exp\left( -\lambda \pi \int_0^{\infty} [t + r^2]^+ \rho(dr) \right)
\]
\[
\times V_{2,0}(t + r_0^2)^{1/2}, (t + r_1^2)^{1/2}, (t + r_2^2)^{1/2})
\]
\[
\times d\rho(\text{dr}_0) \rho(\text{dr}_1) \rho(\text{dr}_2).
\]

As an example, we consider the Poisson–Laguerre tessellation for the case where \( \rho \) is a two-atom distribution, taking the value \( a \) with probability \( q \) and the value \( b > a \) with probability \( 1 - q \). The parameters are chosen as \( \lambda = 100, a = 0.01, b = 0.01, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30 \), and \( q = 0.5 \). This means that we start with a Poisson–Voronoi tessellation of intensity \( \lambda = 100 \) and gradually increase the value of the larger radius.

The formulae for \( \mu_0(= \gamma_0) \) and \( \mu_1(= L_\lambda) \) are evaluated using the numerical integration functions of MATHEMATICA®. From these, the mean values of characteristics of the typical nonempty cell are computed using the mean-value relations. The results are summarised in Table 1. For comparison, the values for Poisson–Voronoi tessellations with intensities \( \lambda = 100 \) and \( \lambda = 50 \) are included.

When investigating the intensities \( \mu_k \) of the measures \( M_k \), we may ask not only for the total value of \( \mu_k \) but also for the contribution of each class of \( k \)-faces to this value. Hence, we write \( \mu_0(r_0, r_1, r_2) \) for the intensity of vertices whose neighbours carry the weights \( r_0, r_1, r_2 \) and \( \mu_1(r_0, r_1) \) for the total length of edges whose neighbours carry the weights \( r_0 \) and \( r_1 \). Clearly,
\[
\mu_0 = \mu_0(a, a, a) + \mu_0(a, a, b) + \mu_0(a, b, a) + \mu_0(b, b, b),
\]
\[
\mu_1 = \mu_1(a, a) + \mu_1(a, b) + \mu_1(b, b).
\]

The results of the numerical evaluation for the example discussed above are presented in Table 2.
Table 1: Mean values of cell characteristics for a two-dimensional Laguerre tessellation generated by a stationary Poisson process of intensity $\lambda = 100$ with marks independently drawn from a two-atom distribution taking the values 0.01 and $b$ with probability 0.5 each. The columns $PV_{100}$ and $PV_{50}$ contain the values for Poisson–Voronoi tessellations with intensities $\lambda = 100$ and $\lambda = 50$, respectively. Given are the intensities of the $k$-faces $\gamma_k$, the mean total edge length per unit volume $L_A$, the mean length $l_1$ of the typical edge, and the mean area $a_2$ and the perimeter $u_2$ of the typical cell.

<table>
<thead>
<tr>
<th>$b$</th>
<th>$PV_{100}$</th>
<th>0.05</th>
<th>0.10</th>
<th>0.15</th>
<th>0.20</th>
<th>0.25</th>
<th>0.30</th>
<th>$PV_{50}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_0$</td>
<td>200,000</td>
<td>192,406</td>
<td>148,398</td>
<td>110,968</td>
<td>101,050</td>
<td>100,043</td>
<td>100,001</td>
<td>100,000</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>300,000</td>
<td>288,609</td>
<td>222,597</td>
<td>166,452</td>
<td>151,574</td>
<td>150,065</td>
<td>150,001</td>
<td>150,000</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>100,000</td>
<td>96,203</td>
<td>74,199</td>
<td>55,484</td>
<td>50,525</td>
<td>50,022</td>
<td>50,001</td>
<td>50,000</td>
</tr>
<tr>
<td>$L_A$</td>
<td>20,000</td>
<td>19,203</td>
<td>16,283</td>
<td>14,529</td>
<td>14,173</td>
<td>14,142</td>
<td>14,142</td>
<td>14,142</td>
</tr>
<tr>
<td>$l_1$</td>
<td>0.067</td>
<td>0.066</td>
<td>0.073</td>
<td>0.087</td>
<td>0.094</td>
<td>0.094</td>
<td>0.094</td>
<td>0.094</td>
</tr>
<tr>
<td>$a_2$</td>
<td>0.010</td>
<td>0.010</td>
<td>0.014</td>
<td>0.018</td>
<td>0.020</td>
<td>0.020</td>
<td>0.020</td>
<td>0.020</td>
</tr>
<tr>
<td>$u_2$</td>
<td>0.400</td>
<td>0.399</td>
<td>0.439</td>
<td>0.524</td>
<td>0.561</td>
<td>0.566</td>
<td>0.566</td>
<td>0.566</td>
</tr>
</tbody>
</table>

Table 2: Contributions of different cell types to $\mu_0$ and $\mu_1$.

<table>
<thead>
<tr>
<th>$b$</th>
<th>$PV_{100}$</th>
<th>0.05</th>
<th>0.10</th>
<th>0.15</th>
<th>0.20</th>
<th>0.25</th>
<th>0.30</th>
<th>$PV_{50}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_0(b, b, b)$</td>
<td>25,000</td>
<td>35,627</td>
<td>67,743</td>
<td>92,562</td>
<td>99,263</td>
<td>99,969</td>
<td>99,999</td>
<td>100,000</td>
</tr>
<tr>
<td>$\mu_0(a, b, b)$</td>
<td>75,000</td>
<td>77,291</td>
<td>48,678</td>
<td>12,652</td>
<td>1.331</td>
<td>0.058</td>
<td>0.001</td>
<td>0.000</td>
</tr>
<tr>
<td>$\mu_0(a, a, b)$</td>
<td>75,000</td>
<td>62,341</td>
<td>26,699</td>
<td>5.013</td>
<td>0.408</td>
<td>0.015</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$\mu_0(a, a, a)$</td>
<td>25,000</td>
<td>17,148</td>
<td>5.279</td>
<td>0.741</td>
<td>0.047</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$\mu_1(b, b)$</td>
<td>5,000</td>
<td>6,935</td>
<td>11,239</td>
<td>13,615</td>
<td>14,100</td>
<td>14,141</td>
<td>14,142</td>
<td>14,142</td>
</tr>
<tr>
<td>$\mu_1(a, b)$</td>
<td>10,000</td>
<td>8,839</td>
<td>3,987</td>
<td>0.766</td>
<td>0.063</td>
<td>0.002</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>$\mu_1(a, a)$</td>
<td>5,000</td>
<td>3,430</td>
<td>1,056</td>
<td>0.148</td>
<td>0.010</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Convergence to a Poisson–Voronoi tessellation of intensity $\lambda = 50$ with increasing $b$ in line with Theorem 5.2 is clearly visible in both tables. It turns out that already for $b = 0.3$ nearly all of the cells generated by points with the smaller weight have disappeared.

A number of further numeric results can be found in [12]. These include numerical evaluations of the formulae presented here for the case of a uniform distribution of radii. Distributions of cell characteristics, namely the probability density functions of the area, perimeter, and number of edges of a typical Poisson–Laguerre cell in $\mathbb{R}^2$ are studied by simulation. In addition, the spatial case, $d = 3$, is discussed in detail.

Appendix A. Proof of Theorem 3.2

In this section we consider the more general situation where the weights of the points do not necessarily have to be positive. Hence, we will work with weights $w_i$ rather than with the radii $r_i$. Nevertheless, we will use the abbreviation $s_i$ for the pairs $(x_i, w_i)$.

**Definition A.1.** Consider a tessellation $T = \{C_i : i \in \mathbb{N}\}$ of $\mathbb{R}^d$. An orthogonal dual of $T$ is a point set $D(T) = \{x_i : i \in \mathbb{N}\}$ in $\mathbb{R}^d$ with the following properties.

1. **Duality.** Each cell $C_i$ of $T$ is associated with exactly one point $x_i \in D(T)$ such that $x_i \neq x_j$ for $i \neq j$.
Theorem A.1. A tessellation $T$ of $\mathbb{R}^d$ is the Laguerre tessellation of some set $\psi$ if and only if an orthogonal dual $D(T)$ of $T$ exists.

Proof. If $T$ is a Laguerre tessellation of $\mathbb{R}^d$, the set of nuclei yields the required orthogonal dual. Conversely, consider a tessellation $T = \{C_i : i \in \mathbb{N}\}$ of $\mathbb{R}^d$ and assume the existence of an orthogonal dual $D(T) = \{x_i : i \in \mathbb{N}\}$. Sort the points $x_i$ in increasing distance from the origin, ordering points with equal distance lexicographically. We will now iteratively assign weights $w_i$ to the points $x_i$ such that $T$ is the Laguerre tessellation of the set $\psi = \{(x_i, w_i) : i \in \mathbb{N}\}$. The weight $w_i$ can be chosen arbitrarily. Consider three cells $C_i, C_j,$ and $C_m$, $i < j < m$, such that $F = C_i \cap C_j \cap C_m \neq \emptyset$, and assume that $w_i$ has already been constructed. Denote the closed half-space bounded by $Ra(s_j, s_i)$ by

$$H(s_j, s_i) = \{z \in \mathbb{R}^d : 2(z, x_i - x_j) \geq \|x_i\|^2 - \|x_j\|^2 + w_j - w_i\},$$

and write $s_j \sim s_j$ if $C_i \subset H(s_j, s_i)$ and $C_j \subset H(s_j, s_i)$. A result in [2, Fact 1] shows that, for any weight $w_i$ of $s_j$ and for any cell $C_j$ distinct from $C_i$, there exists a $w_j$ such that $s_j \sim s_i$. Hence, we find weights $w_j$ and $w_m$ such that $s_j \sim s_i$ and $s_m \sim s_i$. Then $F \subset Ra(s_j, s_i) \cap Ra(s_j, s_m)$. Since the radical axes of $s_i, s_j$, and $s_m$ are not parallel, they intersect in a common $(d - 2)$-dimensional subspace of $\mathbb{R}^d$. Therefore, we also have $F \subset Ra(s_j, s_m)$. Obviously, $F \subset C_j \cap C_m$. Since both $C_j \cap C_m$ and $Ra(s_j, s_m)$ are orthogonal to the line joining $x_j$ and $x_m$ and contain $F$, we conclude that $C_j \cap C_m \subset Ra(s_j, s_m)$. Furthermore, the orientation of $D(T)$ yields $C_j \subset H(s_j, s_m)$. This implies transitivity of the relation $\sim$ for $F \neq \emptyset$, which allows the construction of a set $\psi = \{s_i : i \in \mathbb{N}\}$ such that $T = L(\psi)$.

In order to give the proof of Theorem 3.2, we first fix some notation. Let $T = \{C_i : i \in \mathbb{N}\}$ be a normal tessellation of $\mathbb{R}^d$. For $i \geq d$, define $Q_i = \bigcup_{j=1}^i C_j$, and let $v$ be a vertex in the boundary of $Q_i$. Denote by $e_1, \ldots, e_\ell$ the edges in $\partial Q_i$ having $v$ as a vertex. We say that $Q_i$ is concave at $v$ if the convex hull of $\{e_1, \ldots, e_\ell\}$ is not contained in $Q_i$. Since $T$ is normal, concavity of $Q_i$ implies the existence of a unique cell $C$ of $T$ containing all faces $F$ in $\partial Q_i$ with $v \in F$. This cell is called proper with respect to $Q_i$.

Proof of Theorem 3.2. Let $T$ be a normal tessellation of $\mathbb{R}^d$ with $d \geq 3$. We will introduce a certain ordering $C_1, C_2, \ldots$ of the cells $C_i$ of $T$ which can then be used for an iterative construction of an orthogonal dual $D(T) = \{x_1, x_2, \ldots\}$.

Choose $C_1, \ldots, C_d$ such that they share a common edge. A set of points $x_1, \ldots, x_d$ that satisfies duality, orthogonality, and orientation can easily be found. For $i > d$, choose $C_i$ proper with respect to $Q_{i-1}$ such that $Q_i$ is simply connected. For $j < i$, denote by $F_{j,i}$ the face $C_j \cap C_i$ (if it exists) and denote by $L_{j,i}$ the line orthogonal to $F_{j,i}$ through $x_j$. Define $F_i = Q_{i-1} \cap C_i$. If we show that $x_i = \bigcap_{j \in F_i} L_{j,i}$ is a singleton, this implies orthogonality of the set $X_i = \{x_1, \ldots, x_i\}$. Duality and orientation of $X_i$ follow from the convexity of $C_i$.

Call a vertex in $F_i \setminus \partial Q_i$ an inner vertex of $F_i$. By [2, Claim 1], for any inner vertex $v$ of $F_i$, the intersection $\bigcap_{v \in F_i} L_{j,i}$ is a single point $z$. If the set of inner vertices of $F_i \setminus \partial Q_i$ contains exactly one element, we set $x_i = z$. Otherwise, let $v$ and $v'$ be two inner vertices joined by the edge $e$. Then there exist unique points $z = \bigcap_{v \in F_i} L_{j,i}$ and $z' = \bigcap_{v' \in F_i} L_{l,i}$. Since, by
Theorem A.1 shows that thus, for each independently marked Poisson process such a Boolean model, the condition \( E \) \( y \) \( \Phi_1 \). Hence, \( \exp \) implies condition (R1). Regularity condition (R2) is a consequence of the stationarity of \( \Phi_2 \) \( 2 \) \( \{ (x, r) \in \Phi \} \). The proof that (GP1) and (GP2) are fulfilled is standard and similar to the one that the points of a Poisson point process are almost surely in general quadratic position; see, e.g. [12] and [17, Proposition 4.1.2].

Appendix B. Proofs from Section 4

Proof of Theorem 4.1. For suitable functions \( v \), the generating functional \( G \) of an independently marked Poisson process \( \Phi \) is given by the formula

\[
G(v) = E \left[ \prod_{(x, r) \in \Phi} v(x, r) \right] = \exp \left( -\lambda \int (1 - v(x, r))\lambda_d(dx)\rho(dr) \right).
\]

Let \( y \in \mathbb{R}^d \) be an arbitrary point, and denote by \( p(t) \) the probability that the power from \( y \) to each point of \( \Phi \) exceeds \( t \). Then

\[
p(t) = E \left[ \prod_{(x, r) \in \Phi} 1[\text{pow}(y, (x, r)) > t] \right] = \exp \left( -\lambda \int_0^\infty \int_{\mathbb{R}^d} 1[\text{pow}(y, (x, r)) \leq t]\lambda_d(dx)\rho(dr) \right)
\]

\[
= \exp \left( -\lambda \int_0^\infty \int_{\mathbb{R}^d} \int_{\|y-x\|}^\infty \lambda_d(dx)\rho(dr) \right)
\]

\[
= \exp \left( -\lambda \phi_d \int_0^\infty \|t^+\|_2^d/2\rho(dr) \right),
\]

where \( t^+ = \max(t, 0) \). If \( E[(R \lor 1)^d] < \infty \) then

\[
P \left( \inf_{(x, r) \in \Phi} \text{pow}(y, (x, r)) = -\infty \right) = \lim_{t \to -\infty} P \left( \inf_{(x, r) \in \Phi} \text{pow}(y, (x, r)) < t \right)
\]

\[
= \lim_{t \to -\infty} (1 - p(t)) = 0.
\]

Thus, for each \( y \in \mathbb{R}^d \), we have at least one point \( (x, r) \in \Phi \) minimising \( \text{pow}(y, (x, r)) \). Hence, \( y \in C((x, r), \Phi) \). On the other hand, \( E[(R \lor 1)^d] = \infty \) implies that \( p(t) = 0 \) for each \( t \in \mathbb{R} \) and, therefore, \( \inf_{(x, r) \in \Phi} \text{pow}(y, (x, r)) = -\infty \) with probability 1.

Proof of Theorem 4.2. We will check the applicability of Theorem 3.1. The property \( \text{pow}(y, (x, r)) \leq t \) for \( (x, r) \in \Phi \) is equivalent to \( y \in b(x, \sqrt{|t + r^2|}) \) for \( (x, r) \in \Phi \). In other words, \( y \) is contained in the Boolean model of balls \( b(x, \sqrt{|t + r^2|}) \) with \( (x, r) \in \Phi \). For such a Boolean model, the condition \( E[R^2] < \infty \) implies that, for each bounded set \( B \subset \mathbb{R}^d \), the set \( \{ (x, r) \in \Phi : b(x, \sqrt{|t + r^2|}) \cap B \neq \emptyset \} \) is almost surely finite. With \( B = \{ y \} \), this implies condition (R1). Regularity condition (R2) is a consequence of the stationarity of \( \Phi \) [24, Satz 1.3.5]. The proof that (GP1) and (GP2) are fulfilled is standard and similar to the one that the points of a Poisson point process are almost surely in general quadratic position; see, e.g. [12] and [17, Proposition 4.1.2].
The proof of Theorem 4.3 is based on the next statement generalising a transformation formula given in [24, p. 314].

**Proposition B.1.** Let \( L \in \mathcal{L}_d^k \), and let \( f : L^{k+1} \to \mathbb{R}_+ \) be a measurable function. Furthermore, consider strictly monotonic increasing, differentiable functions \( r_i : [a, \infty) \to \mathbb{R}_+, \ i = 0, \ldots, k \), with an arbitrary real number \( a \). Then we have

\[
\int_{L} \cdots \int_{L} f(x_0, \ldots, x_k) \lambda_d(dx_0) \cdots \lambda_d(dx_k) = k! \int_{L} \int_{a}^{\infty} \prod_{i=0}^{k} (r_i(t))^{k-1} \dot{r}_i(t)) \times \Delta_k \left( \frac{1}{r_0(t)} u_0, \ldots, \frac{1}{r_k(t)} u_k \right) \times \sigma_L(du_0) \cdots \sigma_L(du_k) \, dt \mathcal{H}^k(dy),
\]

where \( \dot{r}_i \) is the derivative of the function \( r_i, \ i = 0, \ldots, k \).

**Proof.** We will only prove this proposition for \( L = \mathbb{R}^k \subset \mathbb{R}^d \). The statement then follows by introducing an appropriate coordinate system on \( L \). Define a mapping

\[
T : \mathbb{R}^k \times [a, \infty) \times S^{k-1} \times \cdots \times S^{k-1} \to (\mathbb{R}^k)^{k+1},
\]

\[
(y, t, u_0, \ldots, u_k) \mapsto \begin{pmatrix} y + r_0(t) u_0 \\ \vdots \\ y + r_k(t) u_k \end{pmatrix}.
\]

It is easy to see that \( T \) is injective. We have to show that the determinant of the Jacobian of \( T \) is given by

\[
D = D(y, t, u_0, \ldots, u_k) = k! \prod_{i=0}^{k} (r_i(t))^{k-1} \dot{r}_i(t)) \Delta_k \left( \frac{1}{r_0(t)} u_0, \ldots, \frac{1}{r_k(t)} u_k \right).
\]

Assume that the unit vectors \( u_i \) are given in spherical coordinates, and let \( \dot{u}_i \) denote the derivative of \( u_i \) with respect to these coordinates. Writing \( E_k \) for the \( k \)-dimensional unit matrix we obtain

\[
D = \begin{vmatrix}
E_k & \dot{r}_0(t) u_0 & r_0(t) \dot{u}_0 & 0 & \cdots & 0 \\
E_k & \dot{r}_1(t) u_1 & r_1(t) \dot{u}_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
E_k & \dot{r}_k(t) u_k & r_k(t) \dot{u}_k & 0 & \cdots & r_k(t) \dot{u}_k
\end{vmatrix}
\]
For $\widetilde{D} = (\prod_{i=0}^{k} 1/r_{i}(t))^{k-1} D$, this leads to

$$\widetilde{D}^{2} = \begin{vmatrix}
E_{k} & E_{k} & \ldots & E_{k} \\
\dot{r}_{0}(t)u_{0} & \dot{r}_{1}(t)u_{1} & \ldots & \dot{r}_{k}(t)u_{k} \\
\dot{u}_{0} & 0 & \ldots & 0 \\
0 & \dot{u}_{1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \dot{u}_{k} \\
\end{vmatrix}
$$

and

$$\begin{align*}
(k + 1)E_{k} & \sum \dot{r}_{i}(t)u_{i} \quad \dot{u}_{0} \quad \dot{u}_{1} \quad \ldots \quad \dot{u}_{k} \\
\sum \dot{r}_{i}(t)u_{i}^{2} & \sum \dot{r}_{i}(t)u_{i} \\
\dot{u}_{0} & 0 \quad E_{k-1} \quad 0 \quad \ldots \quad 0 \\
\dot{u}_{1} & 0 \quad 0 \quad E_{k-1} \quad \ldots \quad 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\dot{u}_{k} & 0 \quad 0 \quad 0 \quad \ldots \quad E_{k-1} \\
\end{align*}
$$

Now we will use the formula $|A| = |B||A_{22}|$, where

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad B = A_{11} - A_{12}A_{22}^{-1}A_{21},$$

if the matrix $A$ is symmetric and $A_{22}^{-1}$ exists. In our case we have $A_{22} = E_{(k+1)(k-1)}$. Furthermore, note that $E_{k} - \dot{u}_{i}\dot{u}_{i}^{T} = u_{i}u_{i}^{T}$, as the matrix $u_{i}\dot{u}_{i}$ is orthogonal. Using this relation, we obtain

$$\widetilde{D}^{2} = \begin{vmatrix}
(k + 1)E_{k} - \sum \dot{u}_{i}\dot{u}_{i}^{T} & \sum \dot{r}_{i}(t)u_{i} \\
\sum \dot{r}_{i}(t)u_{i} & \sum \dot{r}_{i}(t)u_{i}^{2} \\
\end{vmatrix}
$$

$$= \begin{vmatrix}
\sum u_{i}u_{i}^{T} & \sum \dot{r}_{i}(t)u_{i} \\
\sum \dot{r}_{i}(t)u_{i} & \sum \dot{r}_{i}(t)u_{i}^{2} \\
\end{vmatrix}
$$

$$= \begin{vmatrix}
u_{0} & \ldots & u_{k} \\
\dot{r}_{0}(t) & \ldots & \dot{r}_{k}(t) \\
\end{vmatrix}
$$

$$\begin{vmatrix}
\dot{u}_{0} \\
\dot{u}_{1} \\
\vdots \\
\dot{u}_{k} \\
\end{vmatrix}
$$

$$= (k!)^{2} \prod_{i=0}^{k} \dot{r}_{i}(t)^{2} \Delta_{k}^{2} \left( \frac{1}{\dot{r}_{0}(t)} \frac{1}{\dot{r}_{k}(t)} \right),$$

which completes the proof.

**Proof of Theorem 4.3.** We will concentrate on the case in which $1 \leq k \leq d - 1$. The proofs for the cases in which $k = 0$ and $k = d$ work similarly. Write $m = d - k$. Using the definition of $\mu_{k}$, the Slivnyak–Mecke formula (see, e.g. [4, pp. 186–187]), and (3.1), we obtain

$$\mu_{k} = \mathbb{E} \left[ \sum_{F \in \mathcal{H}_{k}(X)} \mathcal{H}^{k}(F \cap [0, 1)^{d}) \right]
$$

$$= \frac{1}{(m + 1)!} \mathbb{E} \left[ \sum_{y \notin \Phi} \int_{[0, 1)^{d}} \mathbb{1}_{y \in [0, 1)^{d}} \cap F(s_{0}, \ldots, s_{m}, \Phi) \mathcal{H}^{k}(dy) \right]$$

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Using the affine Blaschke–Petkantschin formula (see, e.g. [23, Satz 6.1.5]), we obtain

\[ \nu_m(\Theta L) \]

where \( \sum^\Phi \) denotes summation over pairwise distinct points of \( \Phi \). Denote the affine hull of \( F((x_0, r_0), \ldots, (x_m, r_m), \Phi) \) by \( G((x_0, r_0), \ldots, (x_m, r_m)) \). Then,

\[
\mu_k = \frac{\lambda^{m+1}}{(m+1)!} c_{dm}(m!) \\lambda_m(\Phi) \cdot \int_0^\infty \cdots \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} E[\mathbf{1}[y \in [0, 1]^d \cap F((x_0, r_0), \ldots, (x_m, r_m), \Phi)]] \times \mathcal{H}_d(dy) \lambda_d(dx_0) \cdots \lambda_d(dx_m) \rho(dr_0) \cdots \rho(dr_m),
\]

It is well known that there exist unique invariant measures \( \nu_m \) on \( L_m^d \) and \( \eta_m \) on \( G^d_m \) such that \( \nu_m(L_m^d) = 1 \) and that, for any measurable function \( f : G_m^d \rightarrow \mathbb{R}_+ \),

\[
\int_{G_m^d} f(E) \eta_m(dE) = \int_{L_m^d} \int_{L_m^d} f(L + y) \mathcal{H}^{d-m}(dy) \nu_m(dL) = \int_{L_m^d} \int_{L_m^d} f(\theta(y + L_0)) \mathcal{H}^{d-m}(dy) \nu(d\theta), \tag{B.2}
\]

where \( L_0 \in L_m^d \) is a fixed linear subspace of \( \mathbb{R}^d \) (see, e.g. [23, Satz 1.3.3, Satz 1.3.4, and p. 29]).

For an arbitrary subspace \( L \in L_m^d \) and \((x_0, r_0), \ldots, (x_m, r_m) \in \partial L \times \mathbb{R}_+ \), we find a unique point \( z = z((x_0, r_0), \ldots, (x_m, r_m)) \in \partial L \) such that \( G((x_0, r_0), \ldots, (x_m, r_m)) = z + \partial L^\perp \).

Furthermore, \( z((x + x_0, r_0), \ldots, (x + x_m, r_m)) = z((x_0, r_0), \ldots, (x_m, r_m)) \) for any \( x \in \partial L^\perp \).

Now fix a subspace \( L \in L_m^d \) and apply (B.2), which yields

\[
\mu_k = \frac{\lambda^{m+1}}{(m+1)!} c_{dm}(m!) \times \int_0^\infty \cdots \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p(\rho(y, (x + x_0, r_0))) \times \mathbf{1}[y \in [0, 1]^d \cap x + z((x_0, r_0), \ldots, (x_m, r_m)) + \partial L^\perp] \times \mathcal{H}_d(dy) \Delta_m^k(x_0, \ldots, x_m) \mathcal{H}^{d-m}(dx_0) \cdots \mathcal{H}^{d-m}(dx_m) \mathcal{H}_d(dx) \nu(d\theta) \rho(dr_0) \cdots \rho(dr_m).
\]
Now the change of coordinates introduced in Proposition B.1 with \( z = z(s_0, \ldots, s_m) \), \( t = \text{pow}(z, s_0) \), and \( r_i(t) = (t + r_i^2)^{1/2} \) yields
\[
\mu_k = \frac{\lambda^{m+1} c_{dm}(m\mu_k)^{k+1}}{2(m+1)!} \int_0^\infty \cdots \int_0^\infty \int_{-\min t_i}^{t_i} \prod_{i=0}^m (t + r_i^2)^{(m-2)/2}
\times \int_{S_0} \int_{L^\perp} \int_L \int_S p(t + \|x + z - y\|^2)
\times I[y \in [0, 1]^d \cap \gamma L^\perp \mathcal{H}^k(dy) \mathcal{H}^m(dz) \mathcal{H}^k(dx)
\times \int_{S^{d-1} \cap \gamma L} \cdots \int_{S^{d-1} \cap \gamma L} \Delta_m^{k+1}((t + r_0^2)^{1/2}u_0, \ldots, (t + r_m^2)^{1/2}u_m)
\times \sigma_L(du_0) \cdots \sigma_L(du_m) \nu(d\rho) d\rho \cdots d\rho(\rho(\rho_m)).
\]

Substitute \((u_0, \ldots, u_m, z, x)\) by \((\vartheta u_0, \ldots, \vartheta u_m, \vartheta z, \vartheta x)\) and use the fact that \(\Delta_m(\cdot)\) and \(\mathcal{H}^k(\cdot)\) are invariant under rotations. This yields
\[
\mu_k = \frac{\lambda^{m+1} c_{dm}(m\mu_k)^{k+1}}{2(m+1)!} \int_0^\infty \cdots \int_0^\infty \int_{-\min t_i}^{t_i} \prod_{i=0}^m (t + r_i^2)^{(m-2)/2}
\times \int_{S_0} \int_{L^\perp} \int_L \int_S p(t + \|\vartheta(x + z) - y\|^2)
\times I[y \in [0, 1]^d \cap \gamma (x + z + L^\perp)] \mathcal{H}^k(dy) \mathcal{H}^m(dz) \mathcal{H}^k(dx) \nu(d\vartheta)
\times \int_{S^{d-1} \cap L} \cdots \int_{S^{d-1} \cap L} \Delta_m^{k+1}((t + r_0^2)^{1/2}u_0, \ldots, (t + r_m^2)^{1/2}u_m)
\times \sigma_L(du_0) \cdots \sigma_L(du_m) \nu(d\rho) d\rho \cdots d\rho(\rho(\rho_m)).
\]

By the change of variables \(y_0 = y - \vartheta(x + z) \in \vartheta L^\perp\) we get
\[
\mu_k = \frac{\lambda^{m+1} c_{dm}(m\mu_k)^{k+1}}{2(m+1)!} \int_0^\infty \cdots \int_0^\infty \int_{-\min t_i}^{t_i} \prod_{i=0}^m (t + r_i^2)^{(m-2)/2}
\times \int_{S_0} \int_{L^\perp} \int_L \int_S p(t + \|y_0\|^2) I[y_0 + \vartheta(x + z) \in [0, 1]^d \cap \vartheta (x + z + L^\perp)]
\times \mathcal{H}^k(dy_0) \mathcal{H}^m(dz) \mathcal{H}^k(dx) \nu(d\vartheta)
\times \int_{S^{d-1} \cap L} \cdots \int_{S^{d-1} \cap L} \Delta_m^{k+1}((t + r_0^2)^{1/2}u_0, \ldots, (t + r_m^2)^{1/2}u_m)
\times \sigma_L(du_0) \cdots \sigma_L(du_m) \nu(d\rho) d\rho \cdots d\rho(\rho(\rho_m)).
\]

For fixed \(y_0 \in \vartheta L^\perp\), we have
\[
\int_{L^\perp} \int_L I[y_0 + \vartheta(x + z) \in [0, 1]^d \cap \vartheta (x + z + L^\perp)] \mathcal{H}^m(dz) \mathcal{H}^k(dx)
= \int_{L^\perp} \int_L I[y_0 + x + z \in [0, 1]^d] \mathcal{H}^m(dz) \mathcal{H}^k(dx)
= 1,
\]
Theorem 4.3 yields

\[ \mu_k = \frac{\lambda^{m+1}}{(m+1)!} c_{dm}(m!)^{k+1} \]

\[ \times \int_0^\infty \cdots \int_0^\infty \int_{-\min_i r_i}^{\infty} p(t + r_i^2/(m-2)/2) \int_0^\infty \sigma_L(d\mu_0) \sigma_L(d\mu_m) d\rho(d\nu) \cdots \rho(d\nu_m), \]

which, introducing spherical coordinates \( y_0 = s^{1/2} u \) in \( L^2 \), reads

\[ \mu_k = \frac{\lambda^{m+1}}{4(m+1)!} c_{dm}(m!)^{k+1} \sigma_L \]

\[ \times \int_0^\infty \cdots \int_0^\infty \int_{-\min_i r_i}^{\infty} p(t + s^{(k-2)/2}) ds \]

\[ \times \int_{S^{d-1} \cap L} \int_{S^{d-1} \cap L} \Delta_m^{k+1} ((t + r_0^2/2 + \ldots + r_m^2/2) u_0, \ldots, (t + r_m^2/2) u_m) \]

\[ \times \sigma_L(d\mu_0) \cdots \sigma_L(d\mu_m) d\rho(d\nu) \cdots \rho(d\nu_m), \]

Proofs of Theorem 4.4 and Theorem 4.5. Again, we restrict attention to the case in which \( 0 < k < d \). With respect to the Palm probability measure of \( \mathcal{M}_k \), the origin is almost surely contained in a unique \( k \)-face \( F_k(0) = F(S_k,0, \ldots, S_k,m, \Phi) \) generated by the spheres \( S_k,0, \ldots, S_k,m \). Write \( G_k(0) \) for the affine hull of \( F_k(0) \) and choose \( v \in G_k(0) \cap S^{d-1} \) and \( l \geq 0 \). Then \( l v \in F_k(0) \) holds if and only if \( \text{pow}(l v, S_k,0) \leq \text{pow}(l v, (x,r)) \) for all \( (x,r) \in \Phi \). Hence,

\[ \mathcal{H}^k (F_k(0)) = \int_0^\infty \cdots \int_{S^{d-1} \cap G_k(0)} 1 \{ \text{pow}(l v, S_k,0) \leq \text{pow}(l v, (x,r)), (x,r) \in \Phi \} \]

\[ \times \sigma_{G_k(0)}(dv) dl. \]

Let \( h: \mathbb{R}_+ \to \mathbb{R}_+ \) be a measurable function. Then a calculation similar to the proof of Theorem 4.3 yields

\[ \mu_k \mathbb{E}_k [h(\mathcal{H}^k (F_k(0)))] \]

\[ = \frac{\lambda^{m+1}}{4(m+1)!} c_{dm}(m!)^{k+1} \int_0^\infty \cdots \int_0^\infty \int_{-\min_i r_i}^{\infty} p(t + s^{(k-2)/2}) ds \]

\[ \times \int_{S^{d-1} \cap L} \int_{S^{d-1} \cap L} \mathbb{E}[h(\tilde{A}_{G}^{\Phi})(t,s,\theta u, \Phi^{+\tau})] \sigma_L(du) ds \]

\[ \times \int_{S^{d-1} \cap L} \int_{S^{d-1} \cap L} \Delta_m^{k+1} ((t + r_0^2/2 + \ldots + r_m^2/2) u_0, \ldots, (t + r_m^2/2) u_m) \]

\[ \times \sigma_L(d\mu_0) \cdots \sigma_L(d\mu_m) d\rho(d\nu) \cdots \rho(d\nu_m), \]
where $\Phi' = \Phi \cap \{(x, r) : \text{pow}(0, (x, r)) > t\}$ and
\[
\tilde{A}_L(t, s, u, \eta) = \int_0^\infty \int_{\mathbb{S}^{d-1} \cap L} \mathbb{I}[\tau(l, t, s, u, v) \leq \text{pow}(l(v, (x, r)), (x, r) \in \eta)\sigma_L(dv)] dl
\]
with $\tau(l, t, s, u, v) = l^2 + t + s - 2l^{1/2}(u, v)$. Now, by a short calculation using the invariance under rotations of $\Phi^{s+i}$ and $\sigma_{L\perp}$, we obtain
\[
\mu_k E_{Ml}[h(\hat{E}^k(F_k(0)))]
\]
\[
= \frac{\lambda_{m+1}}{4(m+1)!} c_{dm}(m!)^{k+1} \sigma_k
\]
\[
\times \int_0^\infty \cdots \int_0^\infty \int_{\min_i r_i}^\infty \left( \prod_{i=0}^m (t + r_i^2)^{(m-2)/2} \right) \]
\[
\times \int_0^\infty p(t+s)s^{(k-2)/2} E[h(\tilde{A}_L(t, s, u, \Phi^{s+i}))] ds
\]
\[
\times \int_{\mathbb{S}^{d-1} \cap L} \int_{\mathbb{S}^{d-1} \cap L} \Delta_{m+1}^{k+1} ((t + r_0^2)^{1/2} u_0, \ldots, (t + r_m^2)^{1/2} u_m)
\]
\[
\times \sigma_L(du_0) \cdots \sigma_L(du_m) d\rho(du_0) \cdots d\rho(du_m).
\]
For the proof of Theorem 4.5, choose $h(x) = x$ and check that
\[
p(s + t) P(\tau(l, t, s, u, v) \leq \text{pow}(l(v, (x, r)), (x, r) \in \Phi^{s+i}) = \xi(l, s + t, \tau(l, t, s, u, v)).
\]
The formula for $\gamma_k$ is obtained using (2.2) and $h(x) = x^{-1}$.

**Proof of Theorem 5.1.** The theorem easily follows by observing that in every realisation of $\Phi$ the power $\text{pow}(y, (x, vr))$ at any point $y$ from any nucleus $x$ of $\Phi_e$ tends to $\|y - x\|^2$, the distance function generating the Voronoi tessellation $L^V(\Phi)$.

**Proof of Theorem 5.2.** Let $\phi = \{(x_i, r_i)\}$ be a configuration of $\Phi$, let $\phi_e = \{(x, vr) : (x, r) \in \phi\}$, and let $\tilde{\phi} = \{x : (x, s) \in \phi\}$. Given $W$, $\phi$, and $y \in W$, consider
\[
D = D(v, \phi)
\]
\[
= \text{pow}(y, (x, vr)) - \text{pow}(y, (\tilde{x}(y), vs))
\]
\[
= \|y - x\|^2 - \|y - \tilde{x}(y)\|^2 + v^2(s^2 - r^2),
\]
(B.3)
where $\tilde{x}(y)$ is (possibly one of) the point(s) of $\tilde{\phi}$ closest to $y$. When $x = \tilde{x} \in \tilde{\phi}$ in (B.3), then $D = \|y - \tilde{x}\|^2 - \|y - \tilde{x}(y)\|^2 \geq 0$. When $(x, r) \in \phi, r < s$, and $\|y - x\| \geq \|y - \tilde{x}(y)\|$, $D \geq \|y - x\|^2 - \|y - \tilde{x}(y)\|^2 \geq 0$.

Finally, let $m$ be the maximal value of the weights of the points $(x, r) \in \phi$ such that $r < s$ and $\|y - x\| < \|y - \tilde{x}(y)\|$. By local finiteness, there is only a finite number of such points, implying that $m < s$. Then
\[
D \geq -\|y - \tilde{x}(y)\|^2 + v^2(s^2 - m^2),
\]
which is positive for all $v^2 > \|y - \tilde{x}(y)\|^2/(s^2 - m^2)$. Thus, we have shown that, for such large $v$, the minimum power at $y$ is provided by the nucleus $(\tilde{x}(y), vs)$, so that $y$ belongs to the
Laguerre cell $C((\tilde{x}(y), \tilde{\phi}), \phi_v)$ and also implying that $y$ belongs to the Voronoi cell $C^V(\tilde{x}, \tilde{\phi})$ constructed with respect to nuclei set $\tilde{\phi}$. Thus, for all $v > v_0$, all the points in $W$ belong to the Voronoi cells centred at their closest $\tilde{\phi}$-points. As $v_0$, we may take the square root of the ratio of the maximal diameter $d$ of all the cells $C^V(\tilde{x}, \phi_v)$ intersecting $W$ to $\zeta^2$ minus the squared maximum radius among all non-$\tilde{\phi}$-points in the $d$-neighbourhood of $W$.

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