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Non-Selfadjoint Perturbations of Selfadjoint Operators in Two Dimensions IIIa. One Branching Point

Michael Hitrik and Johannes Sjöstrand

Abstract. This is the third in a series of works devoted to spectral asymptotics for non-selfadjoint perturbations of selfadjoint h-pseudodifferential operators in dimension 2, having a periodic classical flow. Assuming that the strength ϵ of the perturbation is in the range $h^2 \ll \epsilon \ll h^{1/2}$ (and may sometimes reach even smaller values), we get an asymptotic description of the eigenvalues in rectangles $[-1/C, 1/C] + i\epsilon[F_0 - 1/C, F_0 + 1/C]$, $C \gg 1$, when ϵF_0 is a saddle point value of the flow average of the leading perturbation.

1 Introduction

This work is the third in a series devoted to non-selfadjoint perturbations of self-adjoint semiclassical pseudodifferential operators in two dimensions, whose classical bicharacteristic flow is periodic on each energy surface. The general background is an observation by A. Melin and the second author [18] that there are quite general classes of analytic non-selfadjoint operators in dimension 2 for which (in the semi-classical limit when $h \to 0$) the eigenvalues in some h-independent region of the complex plane are determined by a Bohr–Sommerfeld quantization condition, very much as for selfadjoint operators in dimension 1. Recall here that for *selfadjoint operators* in dimension 2 or higher, this can happen (to the authors' knowledge) only in the completely integrable case.

The previous works in this series are [14, 15], and more recently, in collaboration with S. Vũ Ngọc, the authors have begun a study of the case when the classical flow of the unperturbed operator is no longer periodic but rather possesses invariant Lagrangian tori with a Diophantine property; see [16] for the first work in this direction.

In this work, we continue with the perturbed periodic case. After switching on a perturbation of size ϵ, the spectrum will be confined to a band of width $\mathcal{O}(\epsilon)$, and the more precise distribution of eigenvalues is very much governed by the flow average of the imaginary part of the leading symbol of the perturbation. In the previous works, we studied the eigenvalues associated with non-critical values of this flow average or with non-degenerate maxima or minima in a suitable sense (after restriction to the 2-dimensional manifold of trajectories in an energy surface). In this paper we

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study the remaining generic case, namely that of eigenvalues associated with a non-degenerate saddle point.

We will work under the general assumptions of [14, 15], which we now recall. Let M denote $\mathbf{R}^2$ or a compact real-analytic manifold of dimension 2. We shall let $M^{\mathbf{C}}$ stand for a complexification of M (obtained by covering M by finitely many real and analytic coordinate patches and then passing to corresponding complex ones in the natural way) so that $M^{\mathbf{C}} = \mathbf{C}^2$ in the case when $M = \mathbf{R}^2$.

When $M = \mathbf{R}^2$, let $P_\epsilon = P(x, hD_x, \epsilon; h)$ be the Weyl quantization on $\mathbf{R}^2$ of a symbol $P(x, \xi, \epsilon; h)$ depending smoothly on $\epsilon \in \operatorname{neigh}(0, \mathbf{R})$ with values in the space of holomorphic functions of (x, ξ) in a tubular neighborhood of $\mathbf{R}^4$ in $\mathbf{C}^4$, with

$$|P(x, \xi, \epsilon; h)| \le \mathcal{O}(1) m(\operatorname{Re}(x, \xi)) \tag{1.1}$$

there. Here m is assumed to be an order function on $\mathbf{R}^4$, in the sense that $m > 0$ and for some $C_0, N_0 > 0$,

$$m(X) \le C_0 \langle X - Y \rangle^{N_0} m(Y), \quad X, Y \in \mathbf{R}^4, \ \langle X - Y \rangle := (1 + |X - Y|^2)^{\frac{1}{2}}. \tag{1.2}$$

We also assume that

$$m \ge 1, \tag{1.3}$$

and

$$P(x, \xi, \epsilon; h) \sim \sum_{j=0}^{\infty} p_{j,\epsilon}(x, \xi) h^j, \quad h \to 0, \tag{1.4}$$

in the space of such functions. We make the ellipticity assumption

$$|p_{0,\epsilon}(x, \xi)| \ge \frac{1}{C} m(\operatorname{Re}(x, \xi)), \quad |(x, \xi)| \ge C, \tag{1.5}$$

for some $C > 0$.

When M is a compact manifold, we let $P_\epsilon = \sum_{|\alpha| \le m} a_{\alpha,\epsilon}(x; h)(hD_x)^\alpha$, be a differential operator on M, such that for every choice of local coordinates, centered at some point of M, $a_{\alpha,\epsilon}(x; h)$ is a smooth function of ϵ with values in the space of bounded holomorphic functions in a complex neighborhood of $x = 0$. We further assume that

$$a_{\alpha,\epsilon}(x; h) \sim \sum_{j=0}^{\infty} a_{\alpha,\epsilon,j}(x) h^j, \quad h \to 0,$$

in the space of such functions. The semi-classical principal symbol in this case is given by $p_{0,\epsilon}(x, \xi) = \sum a_{\alpha,\epsilon,0}(x)\xi^\alpha$, and we make the ellipticity assumption

$$|p_{0,\epsilon}(x, \xi)| \ge \tfrac{1}{C} \langle \xi \rangle^m, \ (x, \xi) \in T^*M, \ |\xi| \ge C, \tag{1.6}$$

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for some large $C > 0$. (Here we assume that M has been equipped with some Riemannian metric, so that $|\xi|$ and $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$ are well defined.)

Sometimes, we write p_ϵ for $p_{0,\epsilon}$ and simply p for $p_{0,0}$. Assume

(1.7) $$P_{\epsilon=0} \text{ is formally selfadjoint.}$$

In the case when M is compact, we let the underlying Hilbert space be $L^2(M, \mu(dx))$ for some positive real-analytic density $\mu(dx)$ on M.

Under these assumptions, P_ϵ will have discrete spectrum in some fixed neighborhood of $0 \in \mathbf{C}$, when $h > 0, \epsilon \geq 0$ are sufficiently small, and the spectrum in this region will be contained in a band $|\operatorname{Im} z| \leq \mathcal{O}(\epsilon)$.

Assume for simplicity that (with $p = p_{\epsilon=0}$)

(1.8) $$p^{-1}(0) \cap T^*M \text{ is connected.}$$

Let $H_p = p'_\xi \cdot \frac{\partial}{\partial x} - p'_x \cdot \frac{\partial}{\partial \xi}$ be the Hamilton field of p. In this work, we will always assume that for $E \in \operatorname{neigh}(0, \mathbf{R})$,

(1.9) the H_p-flow is periodic on $p^{-1}(E) \cap T^*M$ with period $T(E) > 0$ depending analytically on E.

Let $q = \frac{1}{i}(\frac{\partial}{\partial\epsilon})_{\epsilon=0} p_\epsilon$, so that

(1.10) $$p_\epsilon = p + i\epsilon q + \mathcal{O}(\epsilon^2 m),$$

in the case when $M = \mathbf{R}^2$, and $p_\epsilon = p + i\epsilon q + \mathcal{O}(\epsilon^2\langle\xi\rangle^m)$ in the compact case. Let

$$\langle q\rangle = \frac{1}{T(E)} \int_{-T(E)/2}^{T(E)/2} q \circ \exp tH_p \, dt \text{ on } p^{-1}(E) \cap T^*M.$$

Notice that $p, \langle q\rangle$ are in involution $0 = H_p\langle q\rangle =: \{p, \langle q\rangle\}$. In [14], we saw how to reduce ourselves to the case when $p_\epsilon = p + i\epsilon\langle q\rangle + \mathcal{O}(\epsilon^2)$, near $p^{-1}(0) \cap T^*M$. An easy consequence of this, also remarked upon in [14], is that the spectrum of P_ϵ in $\{z \in \mathbf{C}\,;\, |\operatorname{Re} z| < \delta\}$ is confined to $]-\delta, \delta[+ i\epsilon]\langle \operatorname{Re} q\rangle_{\min,0} - o(1), \langle \operatorname{Re} q\rangle_{\max,0} + o(1)[$, when $\delta, \epsilon, h \to 0$, where $\langle \operatorname{Re} q\rangle_{\min,0} = \min_{p^{-1}(0)\cap T^*M} \langle \operatorname{Re} q\rangle$ and similarly for $\langle \operatorname{Re} q\rangle_{\max,0}$. We shall mainly think about the case when $\langle q\rangle$ is real-valued but will work under the more general assumption that

(1.11) $$\operatorname{Im}\langle q\rangle \text{ is an analytic function of } p \text{ and } \operatorname{Re}\langle q\rangle,$$

in a region of T^*M, where $|p| \leq 1/|\mathcal{O}(1)|$.

Let $\Lambda_{0,F_0} = \{\rho \in T^*M\,;\, p(\rho) = 0,\ \operatorname{Re}\langle q\rangle(\rho) = F_0\}$. Assume

(1.12) $T(0)$ is the minimal period for the H_p-flow at every point of Λ_{0,F_0} and Λ_{0,F_0} is connected.

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The connectedness assumption is for convenience only and can easily be removed. Then $\Sigma_0 := p^{-1}(0)/\exp(\mathbf{R}H_p)$ is a symplectic 2-dimensional manifold near the image $\widetilde{\Lambda}_{0,F_0}$ of Λ_{0,F_0}. We consider $\mathrm{Re}\langle q\rangle$ as an analytic function on $\mathrm{neigh}(\widetilde{\Lambda}_{0,F_0}, \Sigma_0)$. Assume that

(1.13) this function has F_0 as critical value and the corresponding critical point is unique, non-degenerate and of signature 0.

Then $\widetilde{\Lambda}_{0,F_0}$ is an ∞-shaped curve, and $\langle q\rangle$ is an analytic function in a neighborhood of that curve (which is the level-curve of $\mathrm{Re}\langle q\rangle$ corresponding to F_0).

In the following, we may assume that $F_0 = 0$ for simplicity. In Section 2 we shall construct an ϵ-dependent canonical transformation κ_ϵ which is an ϵ-perturbation of a real canonical transformation κ_0, with

$$\kappa_\epsilon, \kappa_0\colon\ \mathrm{neigh}(\{\tau = 0\}, (T^*S^1)^{\mathbf{C}}_{t,\tau}) \times \mathrm{neigh}(K_{0,0}, \mathbf{C}^2_{x,\xi}) \to \mathrm{neigh}(\Lambda_{0,0}, T^*M^{\mathbf{C}}),$$

such that $p \circ \kappa_0 = g(\tau)$, $\langle q\rangle \circ \kappa_0 = \langle q\rangle(\tau, x, \xi)$ and

$$p_\epsilon \circ \kappa_\epsilon = g(\tau) + i\epsilon\langle q\rangle(\tau, x, \xi) + \mathcal{O}(\epsilon^2), \tag{1.14}$$

where the $\mathcal{O}(\epsilon^2)$ is also independent of t. Here $K_{0,0} \subset \mathbf{R}^2$ is an ∞-shaped curve with the self-crossing at $(0,0)$ and $(0,0)$ is the saddle point for the function $(x,\xi) \mapsto \mathrm{Re}\langle q\rangle(0,x,\xi)$ with $\langle q\rangle(0,x,\xi) = F_0\ (= 0)$.

In the present work it seems quite essential to assume that

(1.15) the subprincipal symbol of $P_{\epsilon=0}$ vanishes.

(In [14] this assumption was an optional one that permitted to get improved results.)

After further reductions for the lower order symbols, described in Section 2, we get a microlocal reduction of P_ϵ near $\Sigma_{0,0}$ to an operator $\widehat{P}_\epsilon(hD_t, x, hD_x; h)$ with symbol

$$\begin{aligned}\widehat{P}_\epsilon(\tau, x, \xi; h) &= g(\tau) + i\epsilon\Big(\langle q\rangle(\tau, x, \xi) + \mathcal{O}(\epsilon) + \frac{h^2}{i\epsilon} p_2(\tau, x, \xi) + \frac{h}{i}\widetilde{p}_1 + \cdots\Big)\\ &= g(\tau) + i\epsilon Q\Big(\tau, x, \xi, \epsilon, \frac{h^2}{\epsilon}; h\Big).\end{aligned} \tag{1.16}$$

The operator $\widehat{P}_\epsilon$ is only microlocally defined near

$$\{(t, \tau, x, \xi) \in T^*S^1 \times T^*\mathbf{R}\,;\ \tau = 0,\ (x,\xi) \in K_{0,0}\},$$

but that allows us to define asymptotically its eigenvalues in a rectangle $]-\frac{1}{C}, \frac{1}{C}[+ i\epsilon]\frac{1}{C}, \frac{1}{C}[$, and they are of the form

$$g\Big(hk - \frac{S_0}{2\pi} - \frac{k_0 h}{4}\Big) + i\epsilon w_{j,k},\ k \in \mathbf{Z}, \tag{1.17}$$

where $w_{j,k}$ are the eigenvalues near 0 of $Q(hk - \frac{S_0}{2\pi} - \frac{k_0}{4}h, x, hD_x, \epsilon, \frac{h^2}{\epsilon}; h)$ in the microlocal space $L^2_{\theta'}(\mathbf{R})$ defined with Floquet conditions along the two loops of $K_{0,0}$ as in [14]. Here $\theta' = (\theta_1, \theta_2) \in \mathbf{R}^2$ with $\theta_j = \frac{S_j}{2\pi} + \frac{k_j h}{4}$, $k_j \in \mathbf{Z}$, and (S_0, S_1, S_2) appear as action differences when quantizing κ_0, while k_0, k_1, k_2 are Maslov indices.

For $\tau \in \operatorname{neigh}(0, \mathbf{R})$, let $R(\tau)$ be the real analytic curve formed by the values of $\langle q\rangle(\tau, \cdot)$. That $R(\tau)$ is a curve follows from (1.11) and we see that $R(\tau)$ is of the form $\operatorname{Im} w = r(\tau, \operatorname{Re} w)$, where r is analytic in a neighborhood of 0. Also, let $\rho_c^0(\tau) \in \operatorname{neigh}((0,0), \mathbf{R}^2)$ be the critical point of $\rho \mapsto \langle q\rangle(\tau, \rho)$ (with $\rho_c^0(0) = (0,0)$), and let $\rho_c(\tau) \in \operatorname{neigh}((0,0), \mathbf{C}^2)$ be the critical point of the principal symbol

$$(x, \xi) \mapsto Q^0\Big(\tau, x, \xi, \epsilon, \frac{h^2}{\epsilon}\Big) = \langle q\rangle + \mathcal{O}(\epsilon) + \frac{h^2}{i\epsilon} p_2(\tau, x, \xi)$$

of $Q(\tau, x, \xi, \epsilon, h^2/\epsilon; h)$ appearing in (1.16). Clearly, $\rho_c(\tau) = \rho_c^0(\tau) + \mathcal{O}(\epsilon + \frac{h^2}{\epsilon})$. Put $w_c(\tau) = \langle q\rangle(\tau, \rho_c^0(\tau))$ and introduce the exceptional boxes

(1.18)

$$\mathcal{B}(\tau) = \left\{ w\,;\, |\operatorname{Re}(w - w_c^0(\tau)| \le C_0\Big(\epsilon + \frac{h^2}{\epsilon}\Big),\ |\operatorname{Im} w - r(\tau, \operatorname{Re} w)| \le \frac{C_0\big(\epsilon + \frac{h^2}{\epsilon}\big)}{|\ln\big(\epsilon + \frac{h^2}{\epsilon}\big)|} \right\},$$

for some fixed sufficiently large $C_0 > 0$.

Theorem 1.1 *We make the assumptions above and especially* (1.7), (1.9), (1.11), (1.12), (1.13), (1.15), *and put* $\tau_k = hk - \frac{S_0}{2\pi} - \frac{k_0 h}{4}$, $k \in \mathbf{Z}$. *Assume furthermore that* $h^2 \ll \epsilon \ll h^{1/2}$. *For* $C > 0$ *sufficiently large, the eigenvalues of* P_ϵ *in* $]-\frac{1}{C}, \frac{1}{C}[+ i\epsilon]F_0 - \frac{1}{C}, F_0 + \frac{1}{C_0}[$ *are of the form* (1.17), *where the following can be said about the* $w_{j,k}$*:*

- *The number of* $w_{j,k}$ *in* $\mathcal{B}(\tau_k)$ *is* $\mathcal{O}\big(\frac{\epsilon}{h} + \frac{h}{\epsilon}\big)\,|\ln\big(\epsilon + \frac{h^2}{\epsilon}\big)|$.
- *If* $w_{j,k} \notin \mathcal{B}(\tau_k)$, *then* $|\operatorname{Re}(w_{j,k} - w_c^0(\tau_k))| > C_0(\epsilon + \frac{h^2}{\epsilon})$, *with* C_0 *as in* (1.18).
- *There is a bijection* b_k *between the set of these* $w_{j,k}$ *outside* $\mathcal{B}(\tau_k)$ *and the union of three sets of points away from* $\mathcal{B}(\tau_k)$*:* $E_{\text{ext}} = E_e$, $E_{\text{leftint}} = E_{li}$, $E_{\text{rightint}} = E_{ri}$ *such that*

$$b_k(w) - w = \mathcal{O}\Big(e^{\frac{-|\operatorname{Re}(w - w_c^0(\tau_k))|}{Ch}} \frac{h}{|\ln|\operatorname{Re}(w - w_c^0(\tau_k))||} + h^\infty \Big).$$

- *Here* E_e *is a subset of* $\{\operatorname{Re}(w - w_c^0(\tau_k)) < -C_0(\epsilon + \frac{h^2}{\epsilon})\}$ *and* E_{li}, E_{ri} *are subsets of* $\{\operatorname{Re}(w - w_c^0(\tau_k)) > C_0(\epsilon + \frac{h^2}{\epsilon})\}$ *(or vice versa, but we only stick to the first option for simplicity) that can be described by Bohr–Sommerfeld conditions*

$$b_\Theta\Big(w, \epsilon, \frac{h^2}{\epsilon}, \tau_k; h\Big) = 2\pi\Big(j + \frac{1}{2}\Big)h,\ j \in \mathbf{Z},\ \Theta = e,\ li,\ ri, \tag{1.19}$$

where

$$b_\Theta\Big(w, \epsilon, \frac{h^2}{\epsilon}, \tau_k; h\Big) \sim \sum_{\nu=0}^{\infty} b_\Theta^\nu(w, \epsilon, \frac{h^2}{\epsilon}, \tau_k; h) h^\nu,$$

in the space of bounded functions of $w, \epsilon, h^2/\epsilon, \tau$, *that are smooth near* $(0,0,0)$ *in* $(\epsilon, h^2/\epsilon, \tau)$ *and holomorphic in* w *for*

$$|\operatorname{Im}(w - w_c(\tau))| \le \tfrac{1}{C}|\operatorname{Re}(w - w_c(\tau))|,\ \pm\operatorname{Re}(w - w_c(\tau_k)) \ge C_0\Big(\epsilon + \frac{h^2}{\epsilon}\Big),$$

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with a "$-$" when $\Theta = e$ and "+" for $\Theta = li,\ ri$.

- *Further, b^1_Θ is holomorphic in a full neighborhood of $w = w_c(\tau)$,*

$$b^\nu_\Theta = \mathcal{O}(|w - w_c(\tau)|^{1-\nu}), \quad \nu \geq 2,$$

and

$$b^0_e(w, \epsilon, \frac{h^2}{\epsilon}, \tau_k) - 2\mu \ln(-\mu), \quad b_{li} - \mu \ln \mu, \quad b_{ri} - \mu \ln \mu$$

are holomorphic in a neighborhood of $w = w_c(\tau)$. Here μ is a renormalized spectral parameter defined by $w = K_{\epsilon, h^2/\epsilon}(\tau_k, \mu; h)$, with K given in Propositions 6.2, 6.1. *Finally b^0_Θ can be described as actions along suitable cycles in the complexified cotangent space, see Section* 10.

Inside the exceptional boxes the eigenvalues $w_{j,k}$ (for each fixed k) continue to accumulate to roughly at most five curves where three of the curves are the extensions of the curves carrying the E_e, E_{li}, E_{ri} (defined by replacing $2\pi h j$ (1.19) by a continuous real parameter) and one of the new curves, which exists under certain conditions, can be related to barrier top resonances in dimension 1. There are at most two and at least one point (if we exclude degenerate cases) where three of the curves terminate and form a "Y". Away from those points we may have crossings of two of the curves (like, for instance, the ones carrying E_{li} and E_{ri}). Away from the Y points and with some margin, the distribution of eigenvalues can be described by Bohr–Sommerfeld rules as in the theorem, and near the Y points as well as elsewhere, we can get quite detailed estimates for the distribution of eigenvalues. Indeed, the eigenvalues can be identified with zeros of quite explicitly given holomorphic functions which in most regions appear as the sum of four exponential functions, and for such functions it is possible to study the distribution of zeros quite in detail. (See Davies [7] for inspiring results in this direction and Hager [11] for quite elaborated results obtained in parallel with the present work.) The appearance of Y-shaped eigenvalue distributions for non-selfadjoint operators in one dimension seems to be quite well known and we refer to Shkalikov [24], Redparth [22] and further references given there, as well as to the recent works by L. Nedelec [19] and E. Servat and A. Tovbis [23]. The Y-shaped eigenvalue distribution is also readily observed numerically; see Figure 1–3 illustrating the main result of this work.

Unfortunately it turned out to be exceedingly difficult to give a concise and precise description of what happens inside the exceptional boxes in the form of a theorem in less than several pages, so instead we refer the reader to Sections 8–10 where this description can be found.

In Section 13 we apply our results to the study of barrier top resonances for potentials of the form $-x^2 + \mathcal{O}(x^4)$, $\mathbf{R}^2 \ni x \to 0$. In Section 12, we make a remark that permits improving the domain of validity in the direction of small resonances. This gives an improvement also in the applications to barrier tops in [14, 15] and allows us in the present work to treat resonances E with $h^{1-\delta} \ll |E| \ll h^{1/3}$ for every fixed $\delta > 0$, while a direct application of Theorem 1.1 would only give the range $h^{2/3} \ll |E| \ll h^{1/3}$. (In this special situation one can say that the lower bound $\epsilon \gg h^2$, can be replaced by $\epsilon \gg h^{N_0}$ for every fixed $N_0 > 0$.)

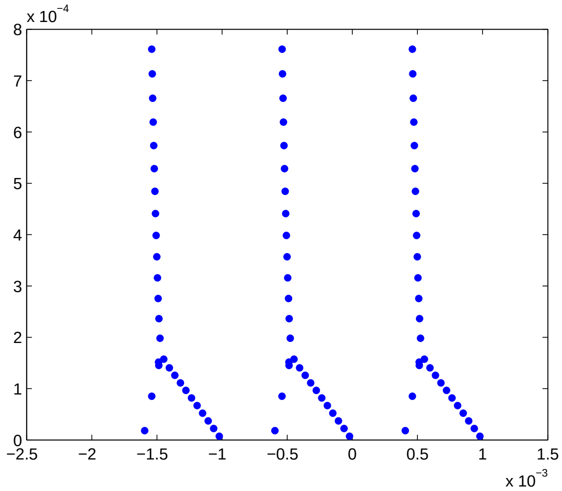

Figure 1: Numerical computation of the eigenvalues of the operator $P_\epsilon = hD_t + i\epsilon Q(x, hD_x)$, $t \in S^1$, $x \in \mathbf{R}$, where $Q(x, hD_x)$ is the one-dimensional double well Schrödinger operator perturbed by a complex even potential, $Q(x, hD_x) = (hD_x)^2 - x^2 + x^4 + i\widetilde{\epsilon}x^2$, $\widetilde{\epsilon} = 0.8$. We take $h = 0.001$ and $\epsilon = 0.02$ so that the assumption $h^2 < \epsilon < h^{1/2}$ is satisfied. When computing the eigenvalues of $Q(x, hD_x)$, following [32], we discretized the operator using the Chebyshev spectral method.

When starting this project, we underestimated the amount of ingredients needed, and in order to keep the work within a reasonable size, we decided to exclude from the paper the very interesting case when there is more than one saddle point on the same connected component of $(\operatorname{Re}\langle q\rangle)^{-1}(F_0)$ at real energy 0. The most important case here is probably the one with 2 saddle points arising because of an anti-symplectic involution (typically $(x, \xi) \mapsto (x, -\xi)$ in the Schrödinger case). We hope to take up at least the two saddle point case in a future work (having settled essentially all heavy technicalities in the present work). We might then also include the interesting case when (1.12) breaks down at isolated points, leading to orbifolds. See Colin de Verdière–Vũ Ngọc [4] in the selfadjoint case.

We also ran into a somewhat unexpected difficulty. Indeed, for the 1-dimensional operators Q, we cannot exclude a pseudospectral phenomenon leading to an exponential growth of the resolvent norm in important regions near the spectrum of these operators (*cf.* [8, 9]). This makes it very important to keep the errors in the reduction to the operator $\widehat{P}_\epsilon$ in (1.16) exponentially small, so that the accumulated error in the global resolvent constructions remains controlled. In [14, 15] we avoided that problem by working in naturally adapted norms where the pseudospectral problems disappeared, but that does not seem equally easy to do here. This refined reduction is carried out in Section 3 using quasi-norms from the theory of analytic pseudodifferential operators originally due to Boutet de Monvel–Krée [2], in the simplified variant of [26]. The price to pay is the apparent necessity to impose the condition

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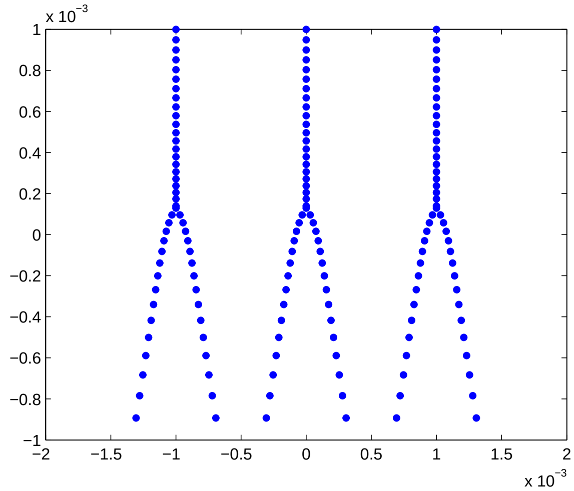

Figure 2: Numerical computation of the eigenvalues of the operator $P_\epsilon = hD_t + Q(x, hD_x)$, $t \in S^1$, $x \in \mathbf{R}$, where $Q(x, hD_x)$ is the one-dimensional double well Schrödinger operator perturbed by a complex odd potential, $Q(x, hD_x) = (hD_x)^2 - x^2 + x^4 + i\widetilde{\epsilon}x^3$, $\widetilde{\epsilon} = 0.8$, in the case when $h = 0.001$, $\epsilon = 0.02$.

(1.15) and the upper bound $\epsilon \ll h^{1/2}$, that should be compared to the bound $\epsilon \ll h^\delta$ for every fixed $\delta > 0$ in [14, 15] or even $\epsilon \ll 1$ in [29, 30].

2 Reduction to a One-Dimensional Pseudodifferential Operator

Let $H_0 \subset p^{-1}(0)$ be a hypersurface which is transversal to the H_p-directions and such that H_0 can be identified with $\operatorname{neigh}(\widetilde{\Lambda}_{0,0}, \Sigma_0)$. We can then identify $\widetilde{\Lambda}_{0,0}$ with a curve $\widetilde{H}_{0,0}$ in H_0.

Let $f = g^{-1} \circ p$, where g is the unique increasing analytic function with $g(0) = 0$ such that the H_f-flow is 2π periodic with 2π as its minimal period on $\Lambda_{0,0}$.

Let $\alpha\colon \operatorname{neigh}(K_{0,0}, \mathbf{R}^2) \to \operatorname{neigh}(\widetilde{H}_{0,0}, H_0)$ be a real-analytic canonical transformation, where $K_{0,0}$ is an ∞-shaped curve, as in the introduction. The existence of such a map with a suitable $K_{0,0}$ follows from [6, Theorem 5] (see also [25]), according to which if Ω_1 and Ω_2 are two compact real-analytic symplectic manifolds of dimension 2, possibly with boundary, which have the same area and such that there exists an orientation preserving analytic diffeomorphism between them, then Ω_1 can be mapped onto Ω_2 by an analytic canonical transformation.

We extend α to a canonical transformation

$$\kappa\colon \operatorname{neigh}(\{\tau = 0\}, T^*S^1)_{t,\tau}) \times \operatorname{neigh}(K_{0,0}, \mathbf{R}^2_{x,\xi}) \to \operatorname{neigh}(\Lambda_{0,0}, T^*M),$$

with $f \circ \kappa = \tau$ in the following way: extend H_0 to an analytic hypersurface H in the full phase space which intersects $p^{-1}(0)$ transversally along H_0. Let $\widetilde{t}$ be the grad-periodic function defined near $\Lambda_{0,0}$ which solves $H_f\widetilde{t} = 1$, with $\widetilde{t} = 0$ on H. Because

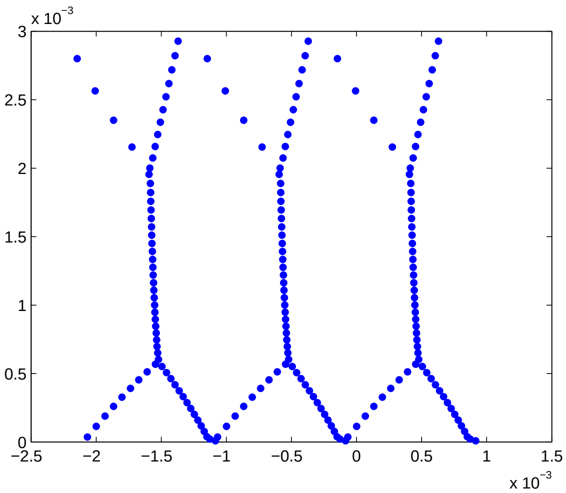

Figure 3: Numerical computation of the eigenvalues of the operator $P_\epsilon = hD_t + Q(x, hD_x)$, $t \in S^1$, $x \in \mathbf{R}$, $Q(x, hD_x) = (hD_x)^2 - x^2 + x^4 + i\widetilde{\epsilon}\left(x^2 + \delta x\right)$, in the case when $h = 0.001$, $\epsilon = 0.02$, $\widetilde{\epsilon} = 0.8$ and $\delta = 0.12$. Here δ is chosen so that the perturbation $x^2 + \delta x$ is of the same sign in both of the potential wells of the selfadjoint part of $Q(x, hD_x)$, although of different orders of magnitude.

of the 2π-periodicity of the H_f-flow, we see that $\widetilde{t}$ is well defined up to an integer multiple of 2π. Using that H_f and $H_{\widetilde{t}}$ commute, we notice that if ρ is a point close to $\Lambda_{0,0}$, then we can write $\rho = \exp(tH_f + \tau H_{\widetilde{t}})(\rho_0)$, $\rho_0 \in H_0$, where $\tau \in \mathbf{R}$ is small and $t \in \mathbf{R}$ is well defined modulo a multiple of 2π.

Put $\kappa(t, \tau; x, \xi) = \exp(tH_f + \tau H_{\widetilde{t}})(\alpha(x, \xi))$. Then κ has the desired properties.

As in [15, §2], we introduce the triple $S = (S_0, S_1, S_3) \in \mathbf{R}^3$ of action differences, with S_0 corresponding to a closed H_p-orbit $\subset p^{-1}(0)$, and S_1, S_2 corresponding to the left- and right-closed orbits of the ∞-shaped set $\widetilde{H}_{0,0}$. Let $\theta = (\theta_0, \theta_1, \theta_2)$, $\theta_j = S_j/(2\pi h) + k_j/4$, where $k_j \in \mathbf{Z}$ is a suitable Maslov index. Let $L^2_\theta(S^1 \times \mathbf{R})$ be the space of microlocally defined functions $u(t, x)$ in $\operatorname{neigh}(\{\tau = 0\}, (T^*S^1)_{t,\tau}) \times \operatorname{neigh}(K_{0,0}, \mathbf{R}^2)$ that are multivalued but θ-Floquet periodic as in [14, §6] (or as in [18, §3]. Let $U\colon L^2_\theta(S^1 \times \mathbf{R}) \to L^2(M)$ be a microlocally defined unitary Fourier integral operator as in the cited works.

Repeating the argument in the beginning of [14, §3], we will assume from now on that the leading perturbation q in (1.10) has already been averaged along the H_p-flow so that the leading symbol of P_ϵ becomes

$$p_\epsilon = p + i\epsilon\langle q\rangle + \mathcal{O}(\epsilon^2). \tag{2.1}$$

The operator $\widetilde{P}_\epsilon := U^{-1}P_\epsilon U$, has the principal symbol

$$\widetilde{p}_\epsilon = g(\tau) + i\epsilon\langle q\rangle(\tau, x, \xi) + \mathcal{O}(\epsilon^2).$$

At this stage, we get a complete analogue to the situation in [14, §3]. There [14, Proposition 3.2] extends, and we get a reduction of $\widetilde{P}_\epsilon$ to an operator $\widehat{P}_\epsilon = \widehat{P}_\epsilon(hD_t, x, hD_x; h)$, also acting on L^2_θ, with a complete symbol independent of t, and the principal symbol still of the form

$$\widehat{p}_\epsilon = g(\tau) + i\epsilon\langle q\rangle(\tau, x, \xi) + \mathcal{O}(\epsilon^2), \tag{2.2}$$

now completely independent of t.

As in [14, §5], we use Fourier series expansions in the t-variable and get a reduction of (2.2) to the family of operators:

$$\widehat{P}_\epsilon\Big(h\Big(k - \frac{k_0}{4}\Big) - \frac{S_0}{2\pi}, x, hD_x; h\Big), \quad k \in \mathbf{Z}, \tag{2.3}$$

where $k_0 \in \mathbf{Z}$ is a fixed Maslov index and it is understood that we only consider such values of k for which the first argument (τ) in $\widehat{P}_\epsilon$ is small.

(i) In the general case, without any assumption on the subprincipal symbol of $P_{\epsilon=0}$, we write the full symbol of $\widehat{P}_\epsilon$ as

$$\begin{aligned}\widehat{P}_\epsilon(\tau, x, \xi; h) \\ = g(\tau) + \epsilon\Big[i\langle q\rangle(\tau, x, \xi) + \mathcal{O}(\epsilon) + \frac{h}{\epsilon}p_1(\tau, x, \xi) + h\frac{h}{\epsilon}p_2(\tau, x, \xi) + \cdots\Big],\end{aligned} \tag{2.4}$$

and consider h/ϵ as an additional small parameter.

(ii) When the subprincipal symbol of $P_{\epsilon=0}$ vanishes, we have the same fact for $\widehat{P}_\epsilon$ by the improved Egorov property of U; see [14, §2]. Thus $(p_1)_{\epsilon=0} = 0$ and we can write

$$\frac{h}{\epsilon}p_1(\tau, x, \xi, \epsilon) = h\widetilde{p}_1(\tau, x, \xi, \epsilon).$$

Instead of (2.4), we get

$$\widehat{P}_\epsilon(\tau, x, \xi; h) = g(\tau) + \epsilon\Big[i\langle q\rangle(\tau, x, \xi) + \mathcal{O}(\epsilon) + \frac{h^2}{\epsilon}p_2(\tau, x, \xi) + h\widetilde{p}_1 + h^2\widetilde{p}_2 + \cdots\Big], \tag{2.5}$$

depending on the small parameters ϵ, h^2/ϵ.

Summing up the discussion so far, we have the following.

Proposition 2.1 *Let P_ϵ be as above satisfying the assumptions* (1.1), (1.2), (1.3), (1.4), (1.5), (1.6), (1.7), (1.8), (1.9), (1.11), (1.12), (1.13). *Also assume that* $0 \le \epsilon \le h^\delta$ *for some fixed* $\delta > 0$. *Then there exist $G_0(x, \xi)$ holomorphic in some fixed neighborhood of $p^{-1}(0)$, an elliptic Fourier integral operator U of order 0, with the associated canonical transformation κ as above, and an h-pseudodifferential operator $A = A(t, hD_t, x, hD_x; h)$ of order* 0 *with principal symbol $\mathcal{O}(\epsilon^2)$, such that the operator*

$$\widehat{P}_\epsilon = e^{\frac{i}{h}A}U^{-1}e^{-\frac{\epsilon}{h}G_0^w}P_\epsilon e^{\frac{\epsilon}{h}G_0^w}Ue^{-\frac{i}{h}A} = \mathrm{Ad}_{e^{\frac{i}{h}A}U^{-1}e^{-\frac{\epsilon}{h}G_0}}P_\epsilon \tag{2.6}$$

has a symbol $\widehat{P}_\epsilon(\tau, x, \xi, \epsilon; h)$ independent of t, modulo $\mathcal{O}(h^\infty)$. Here $G_0^w = G_0^w(x, hD_x)$.

In the general case, we have (2.4), *provided that $h \ll \epsilon \le h^\delta$, and when the subprincipal symbol of $P_{\epsilon=0}$ vanishes, we have* (2.5) *provided that $h^2 \ll \epsilon \le h^\delta$.*

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In this proposition all symbols and phase functions are holomorphic in fixed h, ϵ-independent domains. The weight G_0 in (2.6) is used to get a first reduction of the principal symbol to the form (2.1); see also (8.23) in Section 8.

3 Exponential Decoupling

Since some solution operators to the localized 1-dimensional problems later on will have some exponential growth, the decoupling result of Section 2 has to be sharpened in the sense that we get some exponential smallness control over the remainders.

First we need to recall some notions about classical analytic symbols and their associated quasi-norms. That was introduced in the pioneering work by L. Boutet de Monvel and P. Krée [2], but here we shall use the simplified quasi-norms of [26, Chapter 1]. If $\Omega \subset \mathbf{C}^{2n}_{x,\xi}$ is open, a classical analytic symbol of order 0 is given by the formal asymptotic expansion,

$$a(x,\xi;h) \sim \sum_{k=0}^{\infty} h^k a_k(x,\xi),$$

where a_k are holomorphic in Ω and satisfy the growth condition,

$$\forall K \Subset \Omega,\ \exists C = C_K > 0;\ |a_k(x,\xi)| \leq C^{k+1} k^k,\ (x,\xi) \in K.$$

To such an a, we associate the formal differential operator of infinite order,

$$A(x,\xi,D_x;h) = a(x,\xi + hD_x;h) \sim \sum_{k=0}^{\infty} h^k A_k(x,\xi,D_x), \tag{3.1}$$

where

$$A_k = \sum_{\ell+|\alpha|=k} \frac{1}{\alpha!} \partial_\xi^\alpha a_\ell(x,\xi) D_x^\alpha.$$

Let $\Omega_t \Subset \Omega$, $t_0 \leq t \leq t_1$ be an increasing family of open subsets with $t_0 < t_1$, such that $\operatorname{dist}_\infty(\Omega_s, \partial\Omega_t) \geq t - s$, for $t_0 \leq s < t \leq t_1$, with the distance associated with the ℓ^∞-norm. Let $f_j(A_j) \geq 0$ be the smallest constant such that

$$\|A_j\|_{s,t} \leq f_j(A_j) \Big(\frac{j}{t-s} \Big)^j,$$

where $\|\cdot\|_{s,t}$ is the operator norm from the space of bounded holomorphic functions on Ω_t to the same space on Ω_s. Then

$$|||a|||_\rho := \sum_0^\infty \rho^j f_j(A_j)$$

is finite for $\rho > 0$ small enough, and conversely, the finiteness of $|||a|||_\rho$ for some fixed $\rho > 0$ implies that $a = A(1)$ is a classical analytic symbol on Ω_{t_1}. If a, b are classical

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analytic symbols on Ω and we let $a(x, hD; h)$ and $b(x, hD; h)$ denote the associated h-pseudodifferential operators for the classical quantization, then the composition of these two operators has the symbol

$$a\#b \sim \sum_{k=0}^{\infty} h^k \sum_{|\alpha|=k} \frac{1}{\alpha!} \partial_\xi^\alpha a(x, \xi; h) D_x^\alpha b(x, \xi; h).$$

Using that the differential operators A, B compose correspondingly, it was shown very simply in [26, Chapter 1] that

$$|||a\#b|||_\rho \leq |||a|||_\rho |||b|||_\rho, \tag{3.2}$$

implying that the composed symbol is also a classical analytic symbol.

If we prefer to work with the Weyl quantization, then the same result and proof as in [26] remain valid, provided that we modify the choice of the associated infinite order differential operator to

$$A(x, \xi, D_{x,\xi}; h)u = \sum_{k=0}^{\infty} \frac{h^k}{k!} \Big(\Big(\frac{i}{2} \sigma(D_{x,\xi}; D_{y,\eta}) \Big)^k a(x, \xi; h) u(y, \eta) \Big) \Big|_{y=x, \eta=\xi}, \tag{3.3}$$

so that $A(x, \xi, D_{x,\xi}; h) = a(x - \frac{h}{2} D_\xi, \xi + \frac{h}{2} D_x; h)$.

In the following, we will consider symbols depending on additional parameters;

$$a(x, \xi, \epsilon, h; h) = \sum h^k a_k(x, \xi, \epsilon, h), \tag{3.4}$$

including h (which is then viewed as an independent parameter) and consequently the admissible values of ρ will depend on these parameters. When defining $|||a|||_\rho$ in the case when h is among the additional parameters, we have in mind some specific representation of the form (3.4). To have ρ tending to 0 as some power of h, and that is what we will encounter, means roughly that we consider Gevrey symbols.

Proposition 3.1 *Let $\ell(x, \xi)$ be affine and real and let $a(x, \xi; h)$ be an analytic symbol of order 0. Then*

$$|||[\ell(x, hD), a(x, hD; h)]|||_\rho \leq 2\rho \|\nabla \ell\|_\infty |||a\|_\rho.$$

Here and in the following we shall consider $[\ell, a]$ as an h-pseudodifferential operator of order 0.

Proof The symbol of $[\ell(x, hD), a(x, hD; h)]$ is equal to $\frac{h}{i}\{\ell, a\} = \frac{h}{i}\nu(\partial_{x,\xi})a$, where $\nu = H_\ell$ is the Hamilton field of ℓ. With a we associate the infinite order differential operator $A(x, \xi, D_x; h)$ as in (3.1). Similarly, we have

$$[\ell(x, hD), a(x, hD; h)] \leftrightarrow \sum_{j=1}^{\infty} h^j B_j(x, \xi, D_x) = \frac{h}{i} \sum_{k=0}^{\infty} h^k C_k,$$

with $C_k = \nu(\partial_{x,\xi})(A_k)$ in the sense that ν acts as a differential operator on the coefficients of A_k. Thus, $B_j = \frac{1}{i}C_{j-1} = \frac{1}{i}\nu(\partial_{x,\xi})(A_{j-1})$. We can also express this as

$$B_j = \frac{1}{i}\Big(\frac{\partial}{\partial r}\Big)_{r=0}\tau_{-r\nu}\circ A_{j-1}\circ\tau_{r\nu}, \tag{3.5}$$

where $\tau_{r\nu}$ denotes translation in the complex domain by the vector $r\nu$ and r may be complex.

Assume for simplicity that $|\nu| = \|\nu\|_\infty \le 1$. Then for $(x,\xi) \in \Omega_s$, $t_0 \le s + |r| < \widetilde{t} \le t_1 - |r|$ and for u holomorphic in a larger domain, we get

$$\begin{aligned}|(\tau_{-r\nu}\circ A_{j-1}\circ\tau_{r\nu}u)(x,\xi)| &= |(A_{j-1}\tau_{r\nu}u)((x,\xi)+r\nu)|\\ &\le \frac{f_{j-1}(A_{j-1})(j-1)^{j-1}}{(\widetilde{t}-(s+|r|))^{j-1}}\sup_{\Omega_{\widetilde{t}}}|\tau_{r\nu}u|\\ &\le f_{j-1}(A_{j-1})\frac{(j-1)^{j-1}}{(\widetilde{t}-(s+|r|))^{j-1}}\sup_{\Omega_{\widetilde{t}+|r|}}|u|.\end{aligned}$$

For $t_0 \le s < t \le t_1$, choose $\widetilde{t} = t - |r|$, $2|r| < t - s$:

$$|\tau_{-r\nu}\circ A_{j-1}\circ\tau_{r\nu}u(x,\xi)| \le f_{j-1}(A_{j-1})\frac{(j-1)^{j-1}}{(t-s-2|r|)^{j-1}}\sup_{\Omega_t}|u|.$$

In other words,

$$\|\tau_{r\nu}\circ A_{j-1}\circ\tau_{r\nu}\|_{s,t} \le \frac{f_{j-1}(A_{j-1})(j-1)^{j-1}}{(t-s-2|r|)^{j-1}},$$

and from the Cauchy inequality and (3.5), we get

$$\|B_j\|_{s,t} \le \frac{f_{j-1}(A_{j-1})(j-1)^{j-1}}{\delta(t-s-2\delta)^{j-1}} = \frac{2f_{j-1}(A_{j-1})}{(t-s)^j}\frac{(j-1)^{j-1}}{\frac{2\delta}{(t-s)}(1-\frac{2\delta}{t-s})^{j-1}},\quad 0 < 2\delta < t-s.$$

Here we choose δ so that $\theta := \frac{2\delta}{t-s}$ minimizes $\frac{1}{\theta(1-\theta)^{j-1}}$. We find

$$\theta = \frac{1}{j},\qquad \frac{1}{\frac{1}{j}(1-\frac{1}{j})^{j-1}} = \frac{j^j}{(j-1)^{j-1}}.$$

Hence,

$$\|B_j\|_{s,t} \le \frac{2f_{j-1}(A_{j-1})}{(t-s)^j}j^j,$$

so

$$f_j(B_j) \le 2f_{j-1}(A_{j-1}),\quad\text{and}\quad |||B|||_\rho \le 2\sum_{j=1}^\infty f_{j-1}(A_{j-1})\rho^j = 2\rho|||A|||_\rho. \qquad \blacksquare$$

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In the following, we allow a finite number of families $(\Omega_t^\nu)_{t_1^\nu \le t \le t_2^\nu}$, $\Omega_t^\nu \Subset \Omega$, $\nu = 1, \dots, N$ as above. If $|||a|||_\rho^\nu$ denotes the ρ-quasi-norm, defined with the help of the ν-th family, we define

$$|||a|||_\rho = \sum_\nu |||a|||_\rho^\nu. \tag{3.6}$$

Here it is understood that if a is parameter dependent with h among the parameters, then we use the *same* representation (3.4) when defining each of the quasi-norms $|||a|||_\rho^\nu$. Notice that we still have (3.2).

Proposition 3.2 *Let $g(x, \xi; h)$ be a classical analytic symbol of order 0, defined in a finite union $D = \bigcup_{\nu=1}^N D_\nu$ of polydiscs with $\overline{\Omega} \Subset D$, and let $g(x, hD; h)$ be the corresponding h-pseudodifferential operator. Let $|||\cdot|||_\rho$ be a quasi-norm of the form (3.6) with the corresponding family $\Omega_t^\nu \Subset D_\nu$. Then for ρ small enough, we have*

$$|||[g(x, hD; h), a(x, hD; h)]|||_\rho \le C(g)\rho |||a|||_\rho.$$

Proof Write $g = g_0(x, \xi) + hg_1(x, \xi; h)$, where g_1 is a classical analytic symbol of order 0. We notice that $|||hg_1|||_\rho \le C\rho |||g_1|||_\rho$, (where hg_1 is viewed as a symbol of order 0), so on the operator level, we have

$$|||[hg_1, a]|||_\rho \le 2|||hg_1|||_\rho |||a|||_\rho \le 2C\rho |||g_1|||_\rho |||a|||_\rho.$$

Hence it only remains to treat the contribution to the commutator from $g_0(x, hD)$. We may assume we work in a polydisc centered at (0,0) with the radii $r_1, r_2, \dots, r_n$, $s_1, \dots, s_n$. Then,

$$g_0(x, \xi) = \sum_{\alpha, \beta \in \mathbf{N}^n} g_0^{\alpha,\beta} x^\alpha \xi^\beta,$$

where $\sum |g_0^{\alpha,\beta}| r^\alpha s^\beta < \infty$. Now choose the classical quantization for simplicity. On the operator level,

$$g_0(x, hD) = \sum g_0^{\alpha,\beta} x^\alpha (hD)^\beta, \tag{3.7}$$

where the sum converges in the space of analytic symbols, since $|||x_j|||_\rho \le r_j$ and $|||hD_{x_j}|||_\rho \le s_j + \rho =: \widetilde{s}_j$, and we can allow some shrinking in r_j, s_j and choose $\rho > 0$ small enough.

Using that $|||[x_j, a]|||_\rho, |||[hD_{x_j}, a]|||_\rho \le 2\rho |||a|||_\rho$, in view of Proposition 3.1, we see that

$$\begin{aligned} |||[x^\alpha (hD_x)^\beta, a]|||_\rho \le \big(&\alpha_1 r^{\alpha - e_1} \widetilde{s}^\beta + \alpha_2 r^{\alpha - e_2} \widetilde{s}^\beta + \dots + \alpha_n r^{\alpha - e_n} \widetilde{s}^\beta \\ &+ \beta_1 r^\alpha \widetilde{s}^{\beta - e_1} + \dots + \beta_n r^\alpha \widetilde{s}^{\beta - e_n} \big) 2\rho |||a|||_\rho, \end{aligned}$$

which can be written more briefly as

$$|||[x^\alpha (hD_x)^\beta, a]|||_\rho \le (\partial_{r_1} + \dots + \partial_{r_n} + \partial_{\widetilde{s}_1} + \dots + \partial_{\widetilde{s}_n})(r^\alpha \widetilde{s}^\beta) 2\rho |||a|||_\rho.$$

Hence

$$|||[g_0(x,hD),a]|||_\rho \le \Big[(\partial_{r_1}+\cdots+\partial_{r_n}+\partial_{\widetilde{s}_1}+\cdots+\partial_{\widetilde{s}_n})\Big(\sum_{\alpha,\beta}|g_0^{\alpha,\beta}|r^\alpha\widetilde{s}^\beta\Big)\Big]\,2\rho|||a|||_\rho.$$

■

Remark The proposition remains valid for the Weyl quantization. Indeed, only (3.7) has to be modified, by adding a term $h\widetilde{g}_1(x,hD;h)$ to the right-hand side, where g_1 is an analytic symbol of order 0.

Returning to the considerations of Section 2, let us consider the analytic symbol,

$$P=g(\tau)+\epsilon q(\tau,x,\xi)+hr(t,\tau,x,\xi,\epsilon;h)=g(\tau)+\epsilon\widetilde{q},$$

defined in neigh$(t\in S^1,\tau=0;((S^1+i\mathbf{R})\times\mathbf{C})\times\Omega)$, where $\Omega\subset\mathbf{C}^2_{x,\xi}$ is open. Here we assume either that r is a classical analytic symbol of order 0 or simply that r has an asymptotic expansion in integral powers of h in the space of holomorphic functions. We have already seen in Section 2 that after a finite number of conjugations, we may assume that r is independent of t modulo $\mathcal{O}(h^N)$. We also assume that $g'\neq 0$.

In the general case, we have $\widetilde{q}=\mathcal{O}(1+\frac{h}{\epsilon})$, and when the subprincipal symbol of $P_{\epsilon=0}$ vanishes, we have $hr=\mathcal{O}(h^2+\epsilon h+\epsilon^2)$, $\widetilde{q}=\mathcal{O}(1+h^2/\epsilon+h+\epsilon)$. In the two cases, we shall assume respectively that

$$h\ll\epsilon\le h^\delta, \tag{3.8}$$

$$h^2\ll\epsilon\le h^\delta, \tag{3.9}$$

for some $\delta>0$. Then in both cases, we have $\widetilde{q}=\mathcal{O}(1)$.

We shall see how to eliminate the t-dependence by conjugation with a pseudo-differential operator up to an exponentially decaying error. The problem can be attacked directly, but it seems that we get better remainder estimates if we first reduce ourselves to the case when

$$g(\tau)=\tau. \tag{3.10}$$

This is possible by means of a holomorphic functional calculus. Indeed, let $f=g^{-1}$ be the inverse of the map g. It is easy to see that $f(P)$ is well defined in the sense of formal analytic h-pseudodifferential operators or equivalently in the sense of composition of classical analytic symbols. (When r is merely assumed to have an asymptotic expansion in powers of h, we consider those hs as additional parameters.) We also see that $f(P)$ as a symbol has the same properties as P above, but now with g given by (3.10). It will also be easy to return to the original P, for if $\mathrm{Ad}_A P=e^APe^{-A}$, then at least formally, $\mathrm{Ad}_A f(P)=f(\mathrm{Ad}_A P)$, and to say that a symbol is independent of t is equivalent to saying that the corresponding operator commutes with translations in t and this latter property is stable under taking holomorphic functions of the operator. Until further notice g will be given by (3.10).

Using Proposition 3.2, we get

$$[P, A] = h\frac{1}{i}\{g, A\}(t, x, hD_{t,x}; h) + R(A), \quad R(A) = \epsilon[\widetilde{q}, A], \tag{3.11}$$

where

$$|||R(A)|||_{\rho} \leq C(\widetilde{q})\epsilon\rho|||A|||_{\rho}, \tag{3.12}$$

assuming that the $|||\cdot|||_{\rho}$-quasi-norm is chosen as in Proposition 3.2.

Consider the map

$$A \mapsto \mathrm{Ad}_A(P) = e^A P e^{-A}, \tag{3.13}$$

where $|||A|||_{\rho}$ is supposed to be small. At least formally,

$$\mathrm{Ad}_A(P) = e^{\mathrm{ad}_A}(P). \tag{3.14}$$

We expand

$$\mathrm{Ad}_A(P) = \sum_0^{\infty} \frac{1}{k!}(\mathrm{ad}_A)^k(P), \tag{3.15}$$

and get the expression for the differential

$$\delta A \mapsto \mathrm{ad}_{\delta A}(P) + \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{\substack{\nu+\mu=k-1 \\ \nu,\mu\geq 0}} (\mathrm{ad}_A)^{\nu} \,\mathrm{ad}_{\delta A}(\mathrm{ad}_A)^{\mu}(P).$$

An application of Proposition 3.2 shows that the ρ quasi-norm of the last term can be estimated by

$$\begin{aligned} C(P)\rho \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{\nu+\mu=k-1} (2|||A|||_{\rho})^{\nu+\mu} |||\delta A|||_{\rho} &= C(P)\rho\Big(\sum_{k=2}^{\infty} \frac{1}{(k-1)!}(2|||A|||_{\rho})^{k-1}\Big)|||\delta A|||_{\rho} \\ &= C(P)\rho(e^{2|||A|||_{\rho}} - 1)|||\delta A|||_{\rho}. \end{aligned}$$

So, if we assume some fixed upper bound on $|||A|||_{\rho}$, we can represent the differential of (3.13) as

$$\delta A \mapsto \mathrm{ad}_{\delta A}(P) + K(A, \delta A), \quad |||K(A, \delta A)|||_{\rho} \leq \widetilde{C}(P)\rho|||A|||_{\rho}|||\delta A|||_{\rho},$$

and combining this with (3.11) and (3.12), we get the expression for the differential:

$$\begin{aligned} \delta A \mapsto -\frac{h}{i}\{g, \delta A\} + \widetilde{K}(A, \delta A) &= -\frac{h}{i}g'(\tau)\partial_t \delta A + \widetilde{K}(A, \delta A), \\ |||\widetilde{K}(A, \delta A)|||_{\rho} &\leq C\rho(|||A|||_{\rho} + \epsilon)|||\delta A|||_{\rho}. \end{aligned} \tag{3.16}$$

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Consider the linear problem $\frac{1}{i}g'(\tau)\partial_t A = B - \langle B\rangle$, where B is a classical analytic symbol of order 0 and

$$\langle B\rangle(\tau, x, \xi; h) = \frac{1}{2\pi}\int_0^{2\pi} B(t, \tau, x, \xi; h)\, dt.$$

It has the solution

$$A = \mathcal{L}(B) = \mathcal{L}(B - \langle B\rangle), \quad \mathcal{L}(B) = -\frac{i}{g'(\tau)}\int_0^{2\pi}\Big(\frac{s}{2\pi} - \frac{1}{2}\Big) B(t - s, \tau, x, \xi)\, ds.$$

Clearly (with a convenient choice of the families Ω_s),

$$|||\mathcal{L}(B)|||_\rho \leq C_0 |||B - \langle B\rangle|||_\rho \leq \widetilde{C}_0 |||B|||_\rho.$$

Choose

$$\rho \ll \frac{h}{\epsilon}. \tag{3.17}$$

We look for A of the form $\sum_0^\infty A_j$ such that $\mathrm{Ad}_A(P)$ is independent of t. We shall do the construction by successive approximations in a such a way that uniformly during all the steps, $|||A|||_\rho \leq \mathcal{O}(1)$, $\rho|||A|||_\rho \ll h$, and hence

$$|||\widetilde{K}(A, \delta A)|||_\rho \leq h\theta |||\delta A|||_\rho, \text{ where } \theta \ll 1. \tag{3.18}$$

To start with, we may assume that $r - \langle r\rangle = \mathcal{O}(h^2)$ in the sense of ordinary symbols. Choose $A_0 = \mathcal{L}(r)$, $|||A_0|||_\rho \leq C_0 |||r - \langle r\rangle|||_\rho = \mathcal{O}(h^2)$. Then, using (3.16) and (3.18),

$$\mathrm{Ad}_{A_0}(P) = P - hr + h\langle r\rangle + hr_1,$$

where $P - hr + h\langle r\rangle$ is independent of t and $|||r_1|||_\rho \leq \theta |||r - \langle r\rangle|||_\rho$.

Put $A_1 = \mathcal{L}(r_1)$, $|||A_1|||_\rho \leq C_0\theta |||r - \langle r\rangle|||_\rho$. Then

$$\mathrm{Ad}_{A_0 + A_1}(P) = \mathrm{Ad}_{A_0}(P) - h(r_1 - \langle r_1\rangle) + hr_2 = g + \epsilon q + h\langle r\rangle + h\langle r_1\rangle + hr_2,$$

$$|||r_2|||_\rho \leq \theta |||r_1|||_\rho \leq \theta^2 |||r - \langle r\rangle|||_\rho.$$

Since $\theta \ll 1$, the procedure will converge geometrically and we get a formal solution A with $|||A|||_\rho \leq C_1 |||r - \langle r\rangle|||_\rho$.

By construction $|||A|||_\rho = \mathcal{O}(h^2)$ for $0 < \rho \ll \min(1, h/\epsilon)$ and we have defined $\mathrm{Ad}_A(P)$ by (3.15). We define e^{tA} as a formal analytic symbol of order 0, by

$$e^{tA} = \sum_{k=0}^\infty \frac{t^k A^k}{k!},$$

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so that $|||e^{tA}|||_\rho \le e^{|||A|||_\rho}$, $\partial_t e^{tA} = Ae^{tA} = e^{tA}A$, $e^{0A} = 1$ in the space of formal symbols. Similarly, we see that

$$F_t = \mathrm{Ad}_{tA} = e^{t\,\mathrm{ad}_A} = \sum_0^\infty \frac{t^k\,\mathrm{ad}_A^k}{k!}$$

satisfies $\partial_t F_t = \mathrm{ad}_A \circ F_t$, $F_0 = \mathrm{id}$. We then verify (3.14) simply by noticing that

$$\partial_t(e^{tA}Pe^{-tA}) = \mathrm{ad}_A(e^{tA}Pe^{-tA}).$$

Notice that from the fact that

$$|||A|||_\rho = \mathcal{O}(h^2), \quad A = \sum_0^\infty h^\nu A_\nu,\ a_\nu = A_\nu(1),$$

we infer that locally $|a_\nu| \le f_\nu(A_\nu)(C\nu)^\nu$, with $\sum f_\nu(A_\nu)\rho^\nu < \mathcal{O}(h^2)$. Hence $|a_\nu| \le C_0 h^2 (C\nu/\rho)^\nu$, so

$$|h^\nu a_\nu| \le C_0 h^2 (C\nu)^\nu \Big(\frac{h}{\rho}\Big)^\nu,$$

so A is an analytic symbol with h replaced by h/ρ and can be realized with an uncertainty $\mathcal{O}(h^2)e^{-1/(Ch/\rho))} = \mathcal{O}(h^2)e^{-\rho/(Ch)}$; see [26, Chapter 1]. Taking ρ as large as possible respecting (3.17), and recalling that we also assume that ρ is bounded, we get the uncertainty

$$\mathcal{O}(h^2)e^{-1/C(\epsilon+h)}. \tag{3.19}$$

This discussion can also be applied with A replaced by e^A. Let B be such a realization of e^A. Then we can view A as an analytic symbol of order 0 by declaring that h is an independent parameter. Let B^{-1} be a parametrix, so that $B\#B^{-1} = 1 + \mathcal{O}(e^{-1/(Ch)})$ if we also denote by B^{-1} a realization. From the construction, it follows that $B \circ P \circ B^{-1}$ has a symbol which is t-independent up to an error of the size (3.19). With this in mind, we can state the following.

Proposition 3.3 *We can construct A in Proposition* 2.1*, such that the symbol* $\widehat{P}_\epsilon$ *there is independent of t up to an error which is* $\mathcal{O}(1)\exp(-1/C(\epsilon+h))$. *Here we assume* (3.8) *in the general case and* (3.9) *in the case when the subprincipal symbol of* $P_{\epsilon=0}$ *vanishes.*

We end this section with a heuristic discussion explaining why we eventually will assume that the subprincipal symbol of $P_{\epsilon=0}$ is zero. After decoupling by eliminating the t-dependence as above, we get a family of 1-dimensional operators (2.3) which we consider at the branching level. If we first consider the general case without any assumptions on the subprincipal symbol, we can expect to have an estimate on the inverse of these operators or on the associated Grushin problems roughly of the order

$$\exp\Big(C\Big(\frac{\epsilon}{h} + \frac{h/\epsilon}{h}\Big)\Big) = \exp\Big(C\Big(\frac{\epsilon}{h} + \frac{1}{\epsilon}\Big)\Big).$$

In order to combine everything, we would like this quantity times (3.19) to be $\ll 1$. This is clearly not the case.

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In the case when the subprincipal symbol of $P_{\epsilon=0}$ vanishes we expect to improve the estimate on the 1-dimensional resolvents or inverses of Grushin problems to roughly

$$\exp\Big(C\Big(\frac{\epsilon}{h}+\frac{h^2/\epsilon}{h}\Big)\Big) = \exp\Big(C\Big(\frac{\epsilon}{h}+\frac{h}{\epsilon}\Big)\Big).$$

This leads to the condition

$$\frac{\epsilon}{h}+\frac{h}{\epsilon} \ll \frac{1}{\epsilon+h},$$

which simplifies to the condition $\epsilon^2 \ll h,\ h^2 \ll \epsilon$, which in addition to (3.9) gives $h^2 \ll \epsilon \ll h^{1/2}$. This is the condition on ϵ stated in Theorem 1.1.

4 Transition Matrix at a Branching Level

In this section and the following one, we study certain model problems. Much of the material is standard and close, for instance, to [10, 13, 21] (see also [3] for the C^∞-case), but our setup is somewhat different, and we need to recollect some of the basic facts before returning to our operator P_ϵ.

Consider

$$\Big(\frac{1}{2}(xD_x + D_x x) - \alpha\Big)u = 0, \tag{4.1}$$

or equivalently the equation

$$x\frac{\partial}{\partial x}u = \Big(i\alpha - \frac{1}{2}\Big)u.$$

From [21, Proposition 11] we recall that the solutions of (4.1) in $\mathcal{D}'(\mathbf{R})$ form a 2-dimensional subspace of $\mathcal{S}'(\mathbf{R})$.

- For $x > 0$, we express u as $u_1 x^{i\alpha-\frac{1}{2}}$,
- For $x < 0$, we express u as $u_3|x|^{i\alpha-\frac{1}{2}}$,
- For $\xi > 0$, we express $\widehat{u}(\xi)$ as $u_2\xi^{-i\alpha-\frac{1}{2}}$,
- For $\xi < 0$, we express $\widehat{u}$ as $u_4|\xi|^{-i\alpha-\frac{1}{2}}$.

Here $\widehat{u}(\xi) = \mathcal{F}u(\xi) = \frac{1}{\sqrt{2\pi}}\int e^{-ix\xi}u(x)\,dx$ is the Fourier transform and we observe that (4.1) is equivalent to

$$\Big(\frac{1}{2}(\xi D_\xi + D_\xi \xi) + \alpha\Big)\widehat{u} = 0.$$

If $|\operatorname{Im}\alpha| < 1/2$, we have two solutions $u = U_\pm$ of (4.1), given by

$$\widehat{U}_+(\xi) = H(\xi)\xi^{-\frac{1}{2}-i\alpha} \quad \text{and} \quad \widehat{U}_-(\xi) = H(-\xi)|\xi|^{-\frac{1}{2}-i\alpha},$$

where $H = 1_{]0,+\infty[}$ is the Heaviside function, and the general solution to (4.1) becomes a linear combination, $u = u_2U_+ + u_4U_-$. We see that U_+ is the boundary value

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of a holomorphic function in the upper half-plane that we also denote by U_+, and for $x = iy$, $y > 0$, we get

$$U_+(iy) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-y\xi} \xi^{-\frac{1}{2}-i\alpha}\, d\xi = \frac{1}{\sqrt{2\pi}} \Gamma\Big(\frac{1}{2} - i\alpha\Big)\, y^{i\alpha - \frac{1}{2}}.$$

Thus for real x,

$$U_+(x) = \frac{1}{\sqrt{2\pi}} \Gamma\Big(\frac{1}{2} - i\alpha\Big) \Big(\frac{x + i0}{i}\Big)^{i\alpha - \frac{1}{2}},$$

which gives

$$U_+(x) = \frac{1}{\sqrt{2\pi}} \Gamma\Big(\frac{1}{2} - i\alpha\Big) \times \begin{cases} e^{\frac{\pi}{2}\alpha + i\frac{\pi}{4}} x^{i\alpha - \frac{1}{2}}, & x > 0, \\ e^{-\frac{\pi}{2}\alpha - i\frac{\pi}{4}} |x|^{i\alpha - \frac{1}{2}}, & x < 0. \end{cases} \tag{4.2}$$

Similarly, $U_-(x) = U_-(x - i0)$, with

$$U_-(-iy) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{y\xi} |\xi|^{-i\alpha - \frac{1}{2}}\, d\xi = \frac{1}{\sqrt{2\pi}} \Gamma\Big(\frac{1}{2} - i\alpha\Big)\, y^{i\alpha - \frac{1}{2}},\ y > 0,$$

so

$$U_-(x) = \frac{1}{\sqrt{2\pi}} \Gamma\Big(\frac{1}{2} - i\alpha\Big) \big(i(x - i0)\big)^{i\alpha - \frac{1}{2}},$$

$$U_-(x) = \frac{1}{\sqrt{2\pi}} \Gamma\Big(\frac{1}{2} - i\alpha\Big) \times \begin{cases} e^{-\frac{\pi}{2}\alpha - i\frac{\pi}{4}} x^{i\alpha - \frac{1}{2}}, & x > 0, \\ e^{\frac{\pi}{2}\alpha + i\frac{\pi}{4}} |x|^{i\alpha - \frac{1}{2}}, & x < 0. \end{cases} \tag{4.3}$$

Now let u be a solution of (4.1), so that

$$u = u_1 H(x) x^{-\frac{1}{2}+i\alpha} + u_3 H(-x) |x|^{-\frac{1}{2}+i\alpha} = u_2 U_+ + u_4 U_-.$$

Using (4.2) and (4.3), we get

$$\begin{aligned} u_1 &= \frac{1}{\sqrt{2\pi}} \Gamma\Big(\frac{1}{2} - i\alpha) e^{\frac{\pi}{2}\alpha + i\frac{\pi}{4}} u_2 + \frac{1}{\sqrt{2\pi}} \Gamma\Big(\frac{1}{2} - i\alpha) e^{-\frac{\pi\alpha}{2} - i\frac{\pi}{4}} u_4, \\ u_3 &= \frac{1}{\sqrt{2\pi}} \Gamma\Big(\frac{1}{2} - i\alpha\Big) e^{-\frac{\pi}{2}\alpha - i\frac{\pi}{4}} u_2 + \frac{1}{\sqrt{2\pi}} \Gamma\Big(\frac{1}{2} - i\alpha) e^{\frac{\pi\alpha}{2} + i\frac{\pi}{4}} u_4. \end{aligned} \tag{4.4}$$

Here we want to express u_2, u_1 in terms of u_3, u_4. From (4.4), we get

$$\begin{pmatrix} u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2} - i\alpha)} e^{\frac{\pi}{2}\alpha + i\frac{\pi}{4}} & -e^{\pi\alpha + i\frac{\pi}{2}} \\ e^{\pi\alpha + i\frac{\pi}{2}} & \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2} + i\alpha)} e^{\frac{\pi}{2}\alpha - i\frac{\pi}{4}} \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}, \tag{4.5}$$

where we also used the reflection identity,

$$\Gamma\Big(\frac{1}{2} + iz\Big) \Gamma\Big(\frac{1}{2} - iz\Big) = \frac{\pi}{\cosh \pi z}.$$

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Recall that $\Gamma(z)$ is meromorphic with simple poles at $-k$, for $k \in \mathbf{N}$, and no other poles. The reflection identity above can also be written

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

and implies that if $\Gamma(z) = 0$, then $1-z$ has to be pole, so $1-z = -k$ for some $k \in \mathbf{N}$, $z = k+1$, which is impossible since we also know that $\Gamma(k+1) = k! \neq 0$. Hence $\Gamma(z)$ has no zeros, and $\frac{1}{\Gamma(z)}$ is entire. The transition matrix in (4.5) has determinant 1 and is an entire holomorphic function of α. The relation (4.5) remains valid therefore for all $\alpha \in \mathbf{C}$.

We next compute the transition matrix analogous to the one in (4.5) in the semi-classical case, for solutions of

$$\Big(\frac{1}{2}(xhD_x + hD_x x) - \mu\Big) u = 0. \tag{4.6}$$

This is, of course, the same equation as (4.1) with $\alpha = \mu/h$. We now require

$$u = \begin{cases} u_1 x^{i\frac{\mu}{h}-\frac{1}{2}} & x > 0, \\ u_3 |x|^{i\frac{\mu}{h}-\frac{1}{2}} & x < 0, \end{cases}$$

$$\mathcal{F}_h u(\xi) = \begin{cases} u_2 \xi^{-i\frac{\mu}{h}-\frac{1}{2}} & \xi > 0, \\ u_4 |\xi|^{-i\frac{\mu}{h}-\frac{1}{2}} & \xi < 0, \end{cases}$$

where

$$\mathcal{F}_h u(\xi) = \frac{1}{\sqrt{2\pi h}} \int e^{-ix\xi/h} u(x)\, dx = \frac{1}{\sqrt{h}} \widehat{u}\Big(\frac{\xi}{h}\Big). \tag{4.7}$$

A simple computation gives
(4.8)

$$\begin{pmatrix} u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2\pi}\, h^{i\frac{\mu}{h}}}{\Gamma(\frac{1}{2}-i\frac{\mu}{h})} e^{\frac{\pi}{2}\frac{\mu}{h}+i\frac{\pi}{4}} & -e^{\pi\frac{\mu}{h}+i\frac{\pi}{2}} \\ e^{\pi\frac{\mu}{h}+i\frac{\pi}{2}} & \frac{\sqrt{2\pi}\, h^{-i\frac{\mu}{h}}}{\Gamma(\frac{1}{2}+i\frac{\mu}{h})} e^{\frac{\pi}{2}\frac{\mu}{h}-i\frac{\pi}{4}} \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} a_{2,3} & a_{2,4} \\ a_{1,3} & a_{1,4} \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}.$$

We summarize the discussion above in the following proposition.

Proposition 4.1 *Let $\mu \in \mathbf{C}$ be such that $|\operatorname{Im} \mu| < h/2$. If $u \in \mathcal{D}'(\mathbf{R})$ is a solution of* (4.6), *then $u \in \mathcal{S}'(\mathbf{R})$ and there exist $u_1, u_2, u_3, u_4 \in \mathbf{C}$ such that*

$$\begin{aligned} u &= u_1 H(x) x^{i\frac{\mu}{h}-\frac{1}{2}} + u_2 \mathcal{F}_h^{-1}(H(\xi)\xi^{-i\frac{\mu}{h}-\frac{1}{2}}) \\ &= u_3 H(-x)\,|x|^{i\frac{\mu}{h}-\frac{1}{2}} + u_4 \mathcal{F}_h^{-1}(H(-\xi)\,|\xi|^{-i\frac{\mu}{h}-\frac{1}{2}}). \end{aligned}$$

Here $\mathcal{F}_h$ is the semiclassical Fourier transform defined in (4.7) *and the coefficients u_2, u_1 can be expressed in terms of u_3, u_4 by* (4.8). *The transition matrix which occurs in* (4.8) *is entire holomorphic in μ and has determinant* 1.

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We finish this section by the following observation which will be useful in Section 6. The operator $P_0 = \frac{1}{2}(xhD_x + hD_x x)$ has the principal symbol $p_0(x,\xi) = x\xi$. For $\mu \in \mathbf{C}$, $|\mu| \ll 1$, define $\rho_j \in p_0^{-1}(\mu)$ by $\rho_1 = (1,\mu)$, $\rho_2 = (\mu, 1)$, $\rho_3 = (-1,-\mu)$, $\rho_4 = (-\mu,-1)$. Working in the semiclassical limit, define the microlocal null solutions e_j of $P_0 - \mu$, for $j = 1,\dots,4$ by $e_1 = x^{i\frac{\mu}{h}-\frac{1}{2}}$ near ρ_1, $e_1 = 0$ near ρ_3, $\mathcal{F}_h e_2 = \xi^{-i\frac{\mu}{h}-\frac{1}{2}}$ near $\rho_1 \approx \kappa_{\mathcal{F}_h}\rho_2$, $e_2 = 0$ near ρ_4, $e_j(x) = e_{j-2}(-x)$, $j = 3, 4$. Here $\kappa_{\mathcal{F}_h}$ is the map $(x,\xi) \mapsto (\xi,-x)$ associated with $\mathcal{F}_h$. Then a general null solution of $P_0 - \mu$ can be written either as $u_2e_2 + u_1e_1$ or as $u_3e_3 + u_4e_4$, where (4.8) holds.

5 Asymptotics of the Transition Matrix

In this section, we shall derive asymptotic formulas for the entries of the transition matrix (4.8). We shall use the following version of Stirling's formula [20],

$$\frac{\Gamma(z)}{\sqrt{2\pi}} = e^{-z}z^{z-\frac{1}{2}}\left(1 + \frac{1}{12z} + \cdots\right),\ |z| \to \infty,\ |\arg z| \le \pi - \delta, \tag{5.1}$$

for every fixed $\delta > 0$. We apply this in two cases.

Case A. We have $|\mu|/h \gg 1$ and $\mu \notin$ a conic neighborhood of the negative imaginary axis. Then $\frac{1}{2} - i\frac{\mu}{h}, -i\frac{\mu}{h}$ avoid a conic neighborhood of $\mathbf{R}_-$ and we can apply (5.1), to get

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\Gamma\Big(\frac{1}{2} - i\frac{\mu}{h}\Big) &= e^{-\frac{1}{2}+i\frac{\mu}{h}}\Big(\frac{1}{2} - i\frac{\mu}{h}\Big)^{-i\frac{\mu}{h}} e^{\mathcal{O}(\frac{h}{\mu})} \\
&= \exp\Big[-\frac{1}{2} + i\frac{\mu}{h} - i\frac{\mu}{h}\ln\Big(\frac{1}{2} - i\frac{\mu}{h}\Big) + \mathcal{O}\Big(\frac{h}{\mu}\Big)\Big] \\
&= \exp\Big[-\frac{1}{2} + i\frac{\mu}{h} - i\frac{\mu}{h}\Big(\ln\Big(-i\frac{\mu}{h}\Big) - \frac{h}{2i\mu}\Big) + \mathcal{O}\Big(\frac{h}{\mu}\Big)\Big] \\
&= \exp\Big[-\frac{1}{2} + i\frac{\mu}{h} - i\frac{\mu}{h}\ln\Big(-i\frac{\mu}{h}\Big) + \frac{1}{2} + \mathcal{O}\Big(\frac{h}{\mu}\Big)\Big],
\end{aligned}$$

so in this case we have

$$\frac{1}{\sqrt{2\pi}}\Gamma\Big(\frac{1}{2} - i\frac{\mu}{h}\Big) = \exp\Big[\frac{i\mu}{h} - i\frac{\mu}{h}\ln(-i\mu) + \frac{i\mu}{h}\ln h + \mathcal{O}_-\Big(\frac{h}{\mu}\Big)\Big]. \tag{5.2}$$

Here and in what follows ln always stands for the principal branch of the logarithm.

Case B. We have $|\mu|/h \gg 1$ and μ avoids some conic neighborhood of the positive imaginary axis. Then we can apply the earlier results with μ replaced by $-\mu$ and get

$$\frac{1}{\sqrt{2\pi}}\Gamma\Big(\frac{1}{2} + i\frac{\mu}{h}\Big) = \exp\Big[-\frac{i\mu}{h} + i\frac{\mu}{h}\ln(i\mu) - \frac{i\mu}{h}\ln h + \mathcal{O}_+\Big(\frac{h}{\mu}\Big)\Big]. \tag{5.3}$$

If $\operatorname{Re}\mu \ge \frac{1}{C}|\operatorname{Im}\mu|$, we combine this with the reflection identity

$$\frac{1}{2\pi}\Gamma\Big(\frac{1}{2} - i\frac{\mu}{h}\Big)\Gamma\Big(\frac{1}{2} + i\frac{\mu}{h}\Big) = \frac{1}{2\cosh\frac{\pi\mu}{h}} = e^{-\frac{\pi\mu}{h} + \mathcal{O}(e^{-2\pi\operatorname{Re}\mu/h})}$$

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and the fact that $\ln(i\mu) - \ln(-i\mu) = i\pi$ in this region, to conclude that

$$\mathcal{O}_+\Big(\frac{h}{\mu}\Big) + \mathcal{O}_-\Big(\frac{h}{\mu}\Big) = \mathcal{O}(e^{-2\pi \operatorname{Re}\mu/h}).$$

It is straightforward to establish a corresponding estimate in the region $\operatorname{Re}\mu \leq -\frac{1}{C}|\operatorname{Im}\mu|$, and we can summarize both cases in

$$\mathcal{O}_+\Big(\frac{h}{\mu}\Big) + \mathcal{O}_-\Big(\frac{h}{\mu}\Big) = \mathcal{O}(e^{-2\pi \operatorname{Re}\mu/h}), \quad |\operatorname{Re}\mu| \geq \tfrac{1}{C}|\operatorname{Im}\mu|. \tag{5.4}$$

Remark When μ is real we have

$$\operatorname{Re}\mathcal{O}_+\Big(\frac{h}{\mu}\Big) = \operatorname{Re}\mathcal{O}_-\Big(\frac{h}{\mu}\Big) = \mathcal{O}(e^{-2\pi|\mu|/h}). \tag{5.5}$$

In fact, if we first assume that $\mu \gg h$, we get from (5.2) and (5.3):

$$\frac{\Gamma\big(\frac{1}{2} - i\frac{\mu}{h}\big)}{\sqrt{\pi}} = \exp\Big[i\frac{\mu}{h} - i\frac{\mu}{h}\ln\mu + i\frac{\mu}{h}\ln h - \frac{\pi\mu}{2h} + \mathcal{O}_-\Big(\frac{h}{\mu}\Big)\Big],$$

$$\frac{\Gamma\big(\frac{1}{2} + i\frac{\mu}{h}\big)}{\sqrt{\pi}} = \exp\Big[-i\frac{\mu}{h} + i\frac{\mu}{h}\ln\mu - i\frac{\mu}{h}\ln h - \frac{\pi\mu}{2h} + \mathcal{O}_+\Big(\frac{h}{\mu}\Big)\Big],$$

and using that the second quantity is equal to the complex conjugate of the other, we conclude that (5.5) holds in this case. In the case $\mu \ll -h$, we can use the same argument. In this case (5.2)and (5.3) give

$$\frac{\Gamma(\frac{1}{2} - i\frac{\mu}{h})}{\sqrt{\pi}} = \exp\Big[i\frac{\mu}{h} - i\frac{\mu}{h}\ln|\mu| + i\frac{\mu}{h}\ln h + \frac{\pi\mu}{2h} + \mathcal{O}_-\Big(\frac{h}{\mu}\Big)\Big],$$

$$\frac{\Gamma(\frac{1}{2} + i\frac{\mu}{h})}{\sqrt{\pi}} = \exp\Big[- i\frac{\mu}{h} - i\mu h\ln|\mu| - i\frac{\mu}{h}\ln h + \frac{\pi\mu}{2h} + \mathcal{O}_+\Big(\frac{h}{\mu}\Big)\Big],$$

and we can again conclude that (5.5) holds.

In any closed sector away from $i\mathbf{R}_-$, we can use (4.8) and (5.2) to get

$$a_{2,3} = \exp\Big(\frac{i}{h}\Big(\mu\ln(-i\mu) - i\frac{\pi\mu}{2} - \mu + \frac{h\pi}{4} + ih\mathcal{O}_-\Big(\frac{h}{\mu}\Big)\Big)\Big). \tag{5.6}$$

In any closed sector away from $i\mathbf{R}_+$, we can use (4.8), the reflection identity and (5.3), to get

$$a_{2,3} = 2\cosh\Big(\frac{\pi\mu}{h}\Big)\, e^{\frac{i}{h}(\mu\ln(i\mu) - \frac{i\pi\mu}{2} - \mu + \frac{\pi h}{4} - ih\mathcal{O}_+(\frac{h}{\mu}))} \tag{5.7}$$

Using (5.3) and (4.8), we get for μ away from a sector around $i\mathbf{R}_+$:

$$a_{1,4} = e^{\frac{i}{h}(-\mu\ln(i\mu) - i\frac{\pi\mu}{2} + \mu - \frac{h\pi}{4} + ih\mathcal{O}_+(h\mu))}. \tag{5.8}$$

In a sector $\operatorname{Im}\mu > -C|\operatorname{Re}\mu|$, we use the reflection identity

$$\frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2}+i\frac{\mu}{h})} = \frac{\Gamma(\frac{1}{2}-i\frac{\mu}{h})}{\sqrt{2\pi}} 2\cosh\pi\frac{\mu}{h},$$

to get

$$a_{1,4} = 2\cosh\Big(\frac{\pi\mu}{h}\Big)e^{\frac{i}{h}(-\mu\ln(\frac{\mu}{i})-i\frac{\pi\mu}{2}+\mu-\frac{h\pi}{4}-ih\mathcal{O}_-(\frac{h}{\mu}))}. \tag{5.9}$$

Combining the asymptotic formulae (5.6)–(5.9), we may summarize the discussion in this section in the following proposition.

Proposition 5.1 *We have the following tableau for the coefficients of the transition matrix* (4.8)*, when* $|\mu|/h \gg 1$*. In all cases:*

$$a_{2,4} = -e^{\pi\frac{\mu}{h}+i\frac{\pi}{2}},\ a_{1,3} = e^{\pi\frac{\mu}{h}+i\frac{\pi}{2}}$$

For $\operatorname{Re}\mu > C^{-1}|\operatorname{Im}\mu|$*:*

$$a_{2,3} = e^{\frac{i}{h}(\mu\ln\mu-i\pi\mu-\mu+\pi\frac{h}{4}+ih\mathcal{O}_-(\frac{h}{\mu}))},$$
$$a_{1,4} = e^{\frac{i}{h}(-\mu\ln\mu-i\pi\mu+\mu-\pi\frac{h}{4}+ih\mathcal{O}_+(\frac{h}{\mu}))}.$$

For $\operatorname{Im}\mu > -C|\operatorname{Re}\mu|$*:*

$$a_{2,3} = e^{\frac{i}{h}(\mu\ln\frac{\mu}{i}-i\frac{\pi\mu}{2}-\mu+\frac{\pi h}{4}+ih\mathcal{O}_-(\frac{h}{\mu}))},$$
$$a_{1,4} = e^{\frac{i}{h}(-\mu\ln\frac{\mu}{i}-i\frac{\pi\mu}{2}\pm i\pi\mu+\mu-\frac{\pi h}{4}-ih\mathcal{O}_-(\frac{h}{\mu}))}.$$

For $\operatorname{Re}\mu < -C^{-1}|\operatorname{Im}\mu|$*:*

$$a_{2,3} = e^{\frac{i}{h}(\mu\ln(-\mu)-\mu+\frac{\pi h}{4}+ih\mathcal{O}_-(\frac{h}{\mu}))},$$
$$a_{1,4} = e^{\frac{i}{h}(-\mu\ln(-\mu)+\mu-\frac{\pi h}{4}+ih\mathcal{O}_+(\frac{h}{\mu}))}.$$

For $\operatorname{Im}\mu < C|\operatorname{Re}\mu|$*:*

$$a_{2,3} = e^{\frac{i}{h}(\mu\ln(i\mu)-i\frac{\pi\mu}{2}\pm i\pi\mu-\mu+\frac{\pi h}{4}-ih\mathcal{O}_+(\frac{h}{\mu}))},$$
$$a_{1,4} = e^{\frac{i}{h}(-\mu\ln(i\mu)-i\frac{\pi\mu}{2}+\mu-\frac{\pi h}{4}+ih\mathcal{O}_+(\frac{h}{\mu}))}.$$

Here the terms with $\pm i\pi\mu$ *in the exponents indicate that we should take the sum of the two possible terms with the same remainders* $\mathcal{O}_+$ *or* $\mathcal{O}_-$ *in the exponent for each of the two terms.*

6 The One-Dimensional Spectral Problem

We now return to Proposition 2.1, which shows (when combined with the exponential remainder estimates of Section 3) that the study of P_ϵ near $\Lambda_{0,0}$ (considered in a suitable weighted space) can be reduced by conjugation to that of $\widehat{P}_\epsilon$ acting on the space $L^2_\theta(S^1 \times \mathbf{R})$ of functions defined microlocally in some fixed neighborhood of $\tau = 0$, $(x, \xi) \in K_{0,0} \subset \mathbf{R}^2$ in $T^*S^1 \times T^*\mathbf{R}$, with a $\theta = (\theta_0, \theta_1, \theta_2)$ Floquet periodicity condition. Here $K_{0,0}$ is an ∞-shaped curve as in the introduction, and θ was defined in the beginning of Section 2. In order to fix the ideas, we assume that we are in the case when the subprincipal symbol of $P_{\epsilon=0}$ vanishes, so that the symbol of $\widehat{P}_\epsilon$ is given by (2.5), and we are then in the parameter range $h^2 \ll \epsilon \leq h^\delta$. At least formally, the study of $\widehat{P}_\epsilon$ can be reduced to a family of 1-dimensional problems by a Fourier series expansion in the t-variable and (as noted in (2.3)) we get the 1-dimensional operators

$$\widehat{P}_\epsilon(h(k-\theta_0), x, hD_x; h), \quad k \in \mathbf{Z}, \tag{6.1}$$

where $\theta_0 = S_0/(2\pi h) + k_0/4$, and we restrict the range of k by requiring that $h(k-\theta_0)$ be small. The operators (6.1) should be considered as acting on the microlocal space $L^2_{\theta'}(\mathbf{R})$ defined similarly to $L^2_\theta(S^1 \times \mathbf{R})$, with $\theta' = (\theta_1, \theta_2)$. Using (2.5), we see that (6.1) becomes

$$g(\tau) + i\epsilon Q\Big(\tau, x, hD_x, \epsilon, \frac{h^2}{\epsilon}; h\Big), \quad \tau = h(k-\theta_0),$$

where

$$Q\Big(\tau, x, \xi, \epsilon, \frac{h^2}{\epsilon}; h\Big) \sim Q_0(\tau, x, \xi, \epsilon, \frac{h^2}{\epsilon}) + hQ_1 + h^2 Q_2 + \cdots$$

is holomorphic with respect to (τ, x, ξ) in a fixed complex neighborhood of $\{0\} \times K_{0,0}$ and depends smoothly on the other parameters. We further notice that

$$Q_0(\tau, x, \xi, 0, 0) = \langle q \rangle(\tau, x, \xi)$$

is equal to the trajectory average of q, expressed in suitable real symplectic coordinates, and we know by construction that $\langle q \rangle(0, x, \xi) = 0$ on $K_{0,0}$.

We also recall the assumptions (1.11) and (1.13), which say that

$$\langle q \rangle(\tau, x, \xi) = f(\tau, \operatorname{Re}\langle q \rangle(\tau, x, \xi)), \quad \operatorname{Re} f(\tau, r) = r,\ f(0,0) = 0, \tag{6.2}$$
$$\operatorname{Re}\langle q \rangle''_{(x,\xi),(x,\xi)}(0,0,0) \text{ is non-degenerate of signature } 0.$$

Here we assume for simplicity that $(0, 0) \in K_{0,0}$ is the branching point.

In the following we shall discuss the spectrum of the 1-dimensional operator

$$Q = Q_\tau = Q\Big(\tau, x, hD_x, \epsilon, \frac{h^2}{\epsilon}; h\Big). \tag{6.3}$$

Since this operator is only defined microlocally and up to an error $\mathcal{O}\big(e^{-\frac{1}{C(\epsilon+h)}}\big)$, we must keep in mind that for the moment the eigenvalues will be defined only formally and up to errors of at least the same size.

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A First Localization of the Spectrum

Assume first that $\langle q\rangle$ is real-valued. Then from the sharp Gårding inequality, we see that the spectrum of the operator (6.3) in the band $|\operatorname{Re} z| < 1/\mathcal{O}(1)$ is contained in

$$\Big\{ z \in \mathbf{C}; |\operatorname{Re} z| < 1/\mathcal{O}(1), |\operatorname{Im} z| \le \mathcal{O}(1)\Big(h + \epsilon + \frac{h^2}{\epsilon}\Big)\Big\}. \tag{6.4}$$

Under the more general assumption (6.2), we see that (6.4) can be applied to $g(\tau, Q_\tau)$, where $g(\tau, \cdot\,) = f^{-1}(\tau, \cdot\,)$. So in the general case, we see that the spectrum of the operator (6.3) in the band $|\operatorname{Re} z| < 1/\mathcal{O}(1)$ is contained in

$$\Sigma_\tau := \Big\{ z \in \mathbf{C}; |\operatorname{Re} z| < 1/\mathcal{O}(1), |z - f(\tau, \operatorname{Re} z)| \le \mathcal{O}(1)\Big(h + \epsilon + \frac{h^2}{\epsilon}\Big)\Big\}. \tag{6.5}$$

We also have

$$\|(Q_\tau - z)^{-1}\| \le \frac{\mathcal{O}(1)}{\operatorname{dist}(z, \Sigma_\tau)}, \text{ for } |\operatorname{Re} z| < 1/\mathcal{O}(1), \quad z \notin \Sigma_\tau.$$

Normal Forms near the Branching Points

Let $Q_0 = Q_0(\tau, x, \xi, \epsilon, \frac{h^2}{\epsilon})$ be the principal symbol of Q. Following [14] (and [13, Appendix B]), we get the following adaptation of [14, Proposition 5.3].

Proposition 6.1 *We can find a canonical transformation:* $(x, \xi) \mapsto \kappa_{\tau,\epsilon,h^2/\epsilon}(x, \xi)$ *depending analytically on τ and smoothly on $\epsilon, h^2/\epsilon$ with values in the holomorphic canonical transformations:* $\operatorname{neigh}((0,0), \mathbf{C}^2) \to \operatorname{neigh}((0,0), \mathbf{C}^2)$*, and an analytic function $k_{\epsilon,h^2/\epsilon}(\tau, q)$ depending smoothly on $\epsilon, h^2/\epsilon$, such that*

$$\kappa_{\tau,\epsilon,h^2/\epsilon}(0,0) = (x(\tau, \epsilon, h^2/\epsilon), \xi(\tau, \epsilon, h^2/\epsilon))$$

is the unique critical point close to $(0,0)$ *of* $(x, \xi) \mapsto Q_0(\tau, x, \xi, \epsilon, h^2/\epsilon)$ *and with*

$$Q_0(\tau, \kappa_{\tau,\epsilon,h^2/\epsilon}(x, \xi), \epsilon, h^2/\epsilon) = k_{\epsilon,h^2/\epsilon}(\tau, x\xi).$$

Moreover, $\kappa_{\tau,0,0}$ is real when τ is real and

$$\frac{\partial}{\partial q} \operatorname{Re} k_{\epsilon,h^2/\epsilon}(\tau, 0) > 0.$$

After a conjugation by an elliptic Fourier integral operator associated to the canonical transformation $\kappa_{\tau,\epsilon,h^2/\epsilon}$ we may assume that the leading symbol of Q_τ,

$$Q_0 = Q_0(\tau, x, \xi, \epsilon, h^2/\epsilon)$$

is a function of τ, ϵ, h^2/ϵ and $x\xi$. We can get a complete normal form by making further conjugations by analytic pseudodifferential operators of order 0 in such a way that the complete symbol also becomes a function of τ, ϵ, h^2/ϵ and $x\xi$. This is carried out in Appendix A. We get the following result which is very close to one of the main results of [13, Appendix B].

Proposition 6.2 *We can quantize $\kappa_{\tau,\epsilon,h^2/\epsilon}$ by an elliptic Fourier integral operator $U = U_{\tau,\epsilon,h^2/\epsilon}$ with an analytic symbol, depending holomorphically on τ and smoothly on $\epsilon, h^2/\epsilon$, such that*

$$U^{-1}QU = K_{\epsilon,h^2/\epsilon}(\tau, I; h) + \mathcal{O}(e^{-\frac{1}{Ch}}),\ I = P_0 = \tfrac{1}{2}(x \circ hD_x + hD_x \circ x), \tag{6.6}$$

where $K_{\epsilon,h^2/\epsilon}(\tau,\iota;h)$ is a classical analytic symbol of order 0 depending holomorphically on τ and smoothly on $\epsilon, h^2/\epsilon$. The leading part of K is $k_{\epsilon,h^2/\epsilon}(\tau,\iota)$ appearing in Proposition 6.1.

The Quantization Condition

We start with a side remark about normalization. When $P(x, hD_x)$ is a selfadjoint 1-dimensional h-pseudodifferential operator of principal type, and $z \in \mathbf{R}$, then we can normalize microlocally defined solutions of $(P - z)u = 0$, $z \in \mathbf{R}$, by imposing that $(i/h[P, \chi]u|u) = 1$. Here $\chi = \chi(x, \xi)$ is defined near a piece of the real characteristics and has the property that $\nabla\chi$ is of compact support near the characteristics of $P - z$ and χ increases from 0 to 1 when we progress in the Hamilton flow direction. (See [13, (4.28)].) It is easy to check that if we view u as a solution of $(f(P) - f(z))u = 0$, then we get the corresponding normalization

$$\Big(\frac{i}{h}[f(P), \chi]u|u\Big) = g(z, z)\Big(\frac{i}{h}[P, \chi]u|u\Big) = g(z, z),$$

where $f(P) - f(z) = (P - z)g(P, z)$. If we drop the requirement that P be selfadjoint or just let z become complex, there is no obvious normalization of null-solutions of $P-z$, but we still have a well-defined sesqui-linear form on $\mathcal{N}(P-z)\times\mathcal{N}(P^*-\overline{z})$, given by $(i/h[P, \chi]u|v)$. If we have some additional information allowing us to identify the two null-spaces, then this can still be used to normalize null-solutions of $P-z$. In the following we abandon the attempt to normalize completely the null-solutions, since already the operator $Q_{\epsilon=0,h^2/\epsilon=0}$ is not necessarily selfadjoint.

By Proposition 6.2 we have an analytic symbol $f(\,\cdot\,; h)$ depending analytically on τ and smoothly on ϵ, h^2/ϵ, such that

$$U^{-1}f(Q; h)U = P_0. \tag{6.7}$$

Notice that if u is a null-solution of $P_0 - \mu$ in a full

neighborhood of $(0, 0)$, then $(Q - z)Uu = 0$ near $(0, 0)$, where the spectral parameters are related by

$$f(z; h) = \mu. \tag{6.8}$$

Recall from the end of Section 4 that $P_0 - \mu$ has the four characteristic points ρ_j, $j = 1, 2, 3, 4$ and that this operator has the microlocal null solutions e_j described after Proposition 4.1. When μ is real, we check that e_j is normalized near ρ_j. If v is a global null-solution of $P_0 - \mu$, with $v = v_j e_j$ near ρ_j, then by Proposition 4.1 we have:

$$\begin{pmatrix} v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} a_{2,3} & a_{2,4} \\ a_{1,3} & a_{1,4} \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix}. \tag{6.9}$$

Assume for simplicity sake that $\kappa = \kappa_{\tau,\epsilon,h^2/\epsilon}$ is defined in a suitable domain, containing ρ_j, $j = 1, \dots, 4$. Let $\alpha_j = \kappa(\rho_j)$, $f_j = Ue_j$. Then if u is a null-solution of $Q - z$ near the branching point $\kappa((0,0))$, equal to $v_j f_j$ near α_j, then (6.9) still holds. We may wish to renormalize the f_j by putting

$$f_j = e^{\frac{i}{h} d_j} g_j. \tag{6.10}$$

Then a straightforward calculation shows that if u is a null-solution of $Q - z$ near the branching point, and $u = u_j g_j$ near α_j, then

$$\begin{pmatrix} u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} c_{2,3} & c_{2,4} \\ c_{1,3} & c_{1,4} \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix},\ c_{j,k} = e^{-\frac{i}{h}(d_j - d_k)} a_{j,k}. \tag{6.11}$$

Here is a natural example of such a renormalization. Assume for the sake of simplicity that near α_j the set $Q_0^{-1}(z)$ takes the form $\xi = \lambda_j(x)$, where λ_j is analytic and depends analytically on the parameters $\epsilon, h^2/\epsilon, \tau, z$. Then choose g_j so that microlocally near α_j we have the standard WKB-form:

$$g_j = b_j(x; h) e^{\frac{1}{h}\psi_j(x)},$$

where b_j is a classical elliptic analytic symbol of order 0. The function ψ_j solves the eikonal equation

$$\frac{\partial \psi_j}{\partial x} - \lambda_j(x) = 0, \qquad \text{with the extra condition } \psi_j(\pi_x(\alpha_j)) = 0,$$

and b_j, ψ_j depend analytically on the additional parameters τ, z and smoothly on $\epsilon, h^2/\epsilon$. Using an explicit representation of U we write near α_j for $j = 1, 3$:

$$f_j(x) = h^{-\frac{1+N}{2}} \iint e^{i/h(\psi(x,y,\theta) + \phi_j(y))} A(x, y, \theta) a_j(y)\, dy d\theta = e^{i/h \widetilde{\psi}_j} \widetilde{b}_j(x; h). \tag{6.12}$$

Here the last equality follows from stationary phase, ψ is a non-degenerate phase function generating κ and near ρ_j we write e_j in the WKB-form

$$e_j(y) = a_j(y) e^{\frac{i}{h}\phi_j(y)} = |y|^{\frac{i\mu}{h} - \frac{1}{2}} = |y|^{-\frac{1}{2}} e^{\frac{i}{h}\mu \ln |y|}.$$

The function $\widetilde{\psi}_j(x)$ in (6.12) appears as the critical value in the stationary phase expansion of (6.12) and solves the same eikonal equation as ψ_j.

For $j = 2, 4$, we get near α_j:

$$f_j(x) = h^{-\frac{2+N}{2}} \iiint e^{i/h(\psi(x,y,\theta) + y\eta + \phi_j(\eta))} A(x, y, \theta) a_j(\eta)\, dy d\eta d\theta = e^{\frac{i}{h}\widetilde{\psi}_j} \widetilde{b}_j(x; h),$$

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where a_j and ϕ_j appear when writing $\mathcal{F}_h e_j$ on WKB-form near $\kappa_{\mathcal{F}}(\rho_j)$.

In this case we see that $d_j = d_j(h)$ is a classical analytic symbol of order 0, depending analytically on the additional parameters and with the imaginary part of the leading symbol vanishing when $\operatorname{Im}\mu = 0$, $\epsilon = 0$, $h^2/\epsilon = 0$. The leading part of d_j can be further described in terms of symplectic geometry.

Put

$$\theta_{j,k} := d_j - d_k. \tag{6.13}$$

We have the obvious relation $\theta_{2,3} + \theta_{1,4} = \theta_{1,3} + \theta_{2,4}$.

Now we work in a full neighborhood of $K_{0,0}$ (introduced after (1.14)). Recall that we have the points α_1, α_2, α_3, α_4 on the four crossing branches of $K_{0,0}$ distributed with positive orientation around the branching point. We may assume that α_3, α_4 are situated close to the left closed curve γ^1 of $K_{0,0}$ and that α_1, α_2 are situated close to the right closed curve γ^2 of $K_{0,0}$. Start with a microlocal null-solution to $Q - z$ near $\kappa((0,0))$, of the form $u_4 g_4$ near α_4 and of the form $u_3 g_3$ near α_3. Here for the moment u_3, u_4 can be prescribed arbitrarily, and we then know that $u = u_j g_j$ near α_j for $j = 2, 1$, where u_2, u_1 are given by (6.11). We require temporarily that u is a well-defined single-valued null-solution along the whole left closed component γ^1 of $K_{0,0}$. Then if we follow u around the exterior part of γ^1 from α_4 to α_3, we get

$$u_3 = e^{i/h S_{3,4}} u_4, \text{ where } S_{3,4} = \int_{\gamma_{3,4}} \xi\, dx + \mathcal{O}(h) = S^0_{3,4} + \mathcal{O}(h), \tag{6.14}$$

with $\gamma_{3,4}$ denoting the exterior part of γ^1 which joins α_4 to α_3.

Now recall that we really want $u \in L^2_{\theta'}$, $\theta' = (\theta_1, \theta_2)$, where $\theta_j = \frac{S_j}{2\pi h} + \frac{k_j}{4}$ where S_j is a real action difference related to the reduction in Proposition 2.1 and $k_j \in \mathbf{Z}$ a corresponding Maslov index. This means that u should be multivalued, but Floquet periodic along γ^1 in the sense that

$$\gamma^1_* u = e^{-2\pi i \theta_1} u, \tag{6.15}$$

where $\gamma^1_* u$ denotes the extension of u along one loop of γ^1 which we assume to be oriented in the following way: $\alpha_4 \to \alpha_3 \to (0,0) \to \alpha_4$. Starting near α_3, we get $\gamma^1_* u$ near the same point in two steps:

$$u_3 g_3 \to u_4 g_4 \to e^{\frac{i}{h} S_{3,4}} u_4 g_3.$$

The Floquet condition (6.15) therefore becomes $e^{-2\pi i \theta_1} u_3 = e^{i S_{3,4}/h} u_4$, or equivalently

$$u_3 = e^{2\pi i \theta_1 + \frac{i}{h} S_{3,4}} u_4, \tag{6.16}$$

instead of (6.14).

Similarly, let γ^2 be the right-hand loop in $K_{0,0}$ with the orientation: $\alpha_2 \to \alpha_1 \to (0,0) \to \alpha_2$. Then, if we want u to extend to a null-solution in $L^2_{\theta'}$ near γ^2, we get the analogue of (6.16):

$$u_1 = e^{2\pi i \theta_2 + \frac{i}{h} S_{1,2}} u_2, \tag{6.17}$$

with $S_{1,2}$ defined as in (6.14) with $\gamma_{3,4}$ there replaced by $\gamma_{1,2}$, the exterior segment in γ^2 that joins α_2 to α_1.

Start near α_4 with $u_4 g_4$, use (6.16) to get u_3 and then (6.11) to get u_2, u_1:

$$u_2 = (c_{2,3}e^{2\pi i\theta_1+\frac{i}{h}S_{3,4}} + c_{2,4})u_4, \quad u_1 = (c_{1,3}e^{2\pi i\theta_1+\frac{i}{h}S_{3,4}} + c_{1,4})u_4,$$

and in order to get a global solution in $L^2_{\theta'}$, we also need to apply (6.17), which gives our global one-dimensional quantization condition

$$0 = c_{2,3}e^{2\pi i(\theta_1+\theta_2)+\frac{i}{h}(S_{3,4}+S_{1,2})} + c_{2,4}e^{2\pi i\theta_2+\frac{i}{h}S_{1,2}} - c_{1,3}e^{2\pi i\theta_1+\frac{i}{h}S_{3,4}} - c_{1,4}, \tag{6.18}$$

where we took $u_4 = 1$.

In this relation, we substitute (6.11) and (6.13) and get after multiplication with $e^{i\theta_{1,4}/h}$:

$$0 = a_{2,3}e^{\frac{i}{h}(\widehat{S}_{3,4}+\widehat{S}_{1,2})} + a_{2,4}e^{\frac{i}{h}\widehat{S}_{1,2}} - a_{1,3}e^{\frac{i}{h}\widehat{S}_{3,4}} - a_{1,4},$$

with

$$\begin{aligned}\widehat{S}_{1,2} &= S_{1,2} + \theta_{1,2} + 2\pi h\theta_2 = S_{1,2} + \theta_{1,2} + S_2 + hk_2\frac{\pi}{2}\\ \widehat{S}_{3,4} &= S_{3,4} + \theta_{3,4} + 2\pi h\theta_1 = S_{3,4} + \theta_{3,4} + S_1 + hk_1\frac{\pi}{2},\end{aligned} \tag{6.19}$$

where we recall that $\theta_j = \frac{S_j}{2\pi} + \frac{k_j}{4}$. With

$$\widetilde{S}_{j,k} = \widehat{S}_{j,k} + h\frac{\pi}{2}, \tag{6.20}$$

we get

$$0 = a_{2,3}e^{\frac{i}{h}(\widetilde{S}_{3,4}+\widetilde{S}_{1,2})-i\frac{\pi}{2}} + a_{2,4}e^{\frac{i}{h}\widetilde{S}_{1,2}} - a_{1,3}e^{\frac{i}{h}\widetilde{S}_{3,4}} - a_{1,4}e^{i\frac{\pi}{2}}. \tag{6.21}$$

Proposition 6.3 *Assume that*

$$Q(\tau, x, \xi, \epsilon, h^2/\epsilon; h) \sim Q_0(\tau, x, \xi, \epsilon, h^2/\epsilon) + hQ_1(\tau, x, \xi, \epsilon, h^2/\epsilon) + \cdots$$

is holomorphic in $(\tau, (x,\xi)) \in \operatorname{neigh}(0, \mathbf{C}) \times \operatorname{neigh}(K_{0,0}, \mathbf{C}^2)$ *and depends smoothly on* $\epsilon, \frac{h^2}{\epsilon} \in \operatorname{neigh}(0, \mathbf{R})$*. Here* $K_{0,0}$ *is an* ∞*-shaped curve with the self-crossing at* $(0,0)$*. Assume furthermore that*

$$Q_0(\tau, x, \xi, 0, 0) = \langle q\rangle(\tau, x, \xi) = f(\tau, \operatorname{Re}\langle q\rangle(\tau, x, \xi)),$$

where f *is an analytic function with* $f(0,0) = 0$*. We assume next that along* $K_{0,0}$, $\operatorname{Re}\langle q\rangle(0,x,\xi) = 0$ *and that* $\operatorname{Re}\langle q\rangle(0,x,\xi)$ *has a unique critical point on* $K_{0,0}$, $(0,0)$, *which is a non-degenerate saddle point. When* $z \in \operatorname{neigh}(0, \mathbf{C})$*, put* $\mu = f(z;h)$ *where* $f(z;h)$ *is an analytic symbol introduced in Propositions* 6.1 *and* 6.2*, and in* (6.8)*. Then* z *is a quasi-eigenvalue of the operator* $Q(\tau, x, hD_x, \epsilon, h^2/\epsilon; h)$ *acting on* $L^2_{\theta'}(\mathbf{R})$ *if and only if the corresponding* μ *satisfies* (6.21)*. In* (6.21)*, the coefficients* $a_{1,3}$, $a_{1,4}$, $a_{2,3}$, *and* $a_{2,4}$ *are introduced in Proposition* 4.1*, and the quantities* $\widetilde{S}_{1,2}$ *and* $\widetilde{S}_{3,4}$ *are defined in* (6.19) *and* (6.20)*. They depend holomorphically on* μ *with* $\partial_\mu \widetilde{S}_{j,k} = \mathcal{O}(1)$*, and when* μ *is real, we have* $\operatorname{Im}\widetilde{S}_{j,k} = \mathcal{O}(\epsilon + h^2/\epsilon)$*.*

In the formulation of the proposition, we leave the notion of a quasi-eigenvalue undefined and refer the reader to Section 11 for a complete justification of this terminology.

7 Zeros of Sums of Exponential Functions

Here we elaborate on arguments in [7], and a related and even more general discussion can be found in Hager [11, Proposition 8.1]. The results established in this section will be used in Section 10.

Let $\gamma_1, \gamma_2, \dots, \gamma_N$ be compact C^1 segments in $\mathbf{C}$ such that γ_j starts at $s_{j-1} \in D(z_{j-1}, r_{j-1}/2)$ and ends at $e_j \in D(z_j, r_j/2)$, where we use the cyclic convention and view the index j as an element of $\mathbf{Z}/N\mathbf{Z}$. We assume that $N \in \{1, 2, \dots\}$ is fixed, but allow $\gamma_j, z_j, r_j, s_j, e_j$ to vary with the semi-classical parameter h while all estimates below will be uniform in h. Let f be a holomorphic function defined in $\bigcup_{j=0}^{N-1}(D(z_j, r_j) \cup \operatorname{neigh}(\gamma_{j+1}))$, such that

$$f = e^{\frac{i}{h}S_j(z) + \mathcal{O}(1)} \text{ on } \gamma_j, \quad |f| \le e^{1/h(-\operatorname{Im} S_j(z)) + \mathcal{O}(1)} \text{ on } D(z_j, r_j),$$

where S_j is holomorphic in $\operatorname{neigh}(\gamma_j) \cup D(z_{j-1}, r_{j-1}) \cup D(z_j, r_j)$ and

$$\operatorname{Im}(S_{j+1} - S_j) = \mathcal{O}(h) \text{ on } D(z_j, r_j).$$

In $D(z_j, r_j)$ we can write $f(z) = e^{\frac{i}{h}S_j(z)} g_j(z)$, $|g_j(z)| \le \mathcal{O}(1)$. We further know that $|g_j(e_j)| \ge 1/\mathcal{O}(1)$. Standard arguments (see for instance [28, §5]), including Jensen's formula, imply that the number of zeros of g_j in $D(z_j, r_j/2)$ is $\mathcal{O}(1)$ and if α_j is a segment in $D(z_j, r_j/2)$ from e_j to s_j which avoids the zeros $w_1, \dots, w_M$ of g_j in $D(z_j, r_j/2)$ such that $|\operatorname{var} \arg_{\alpha_j}(z - w_k)| < 2\pi$ for every k, then

$$\operatorname{Re} \frac{1}{2\pi i} \int_{\alpha_j} \frac{g_j'}{g_j}\, dz = \mathcal{O}(1),$$

and consequently

$$\begin{aligned} \operatorname{Re} \frac{1}{2\pi i} \int_{\alpha_j} \frac{f'}{f}\, dz &= \mathcal{O}(1) + frac12\pi h \int_{\alpha_j} \operatorname{Re} S_j'(z)\, dz \\ &= \mathcal{O}(1) + \frac{1}{2\pi h} \operatorname{Re}(S_j(s_j) - S_j(e_j)). \end{aligned} \tag{7.1}$$

Let γ be the closed contour given by $\gamma_1 \cup \alpha_1 \cup \gamma_2 \cup \dots \cup \gamma_N \cup \alpha_0$. We want to study

$$N(f, \gamma) := \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi} \operatorname{var} \arg_{\gamma}(f).$$

When γ is the oriented boundary of a bounded domain Γ, where f is holomorphic, then $N(f, \gamma)$ is the number of zeros of f inside Γ.

Along γ_j, we write $f = e^{\frac{i}{h}\widetilde{S}_j(z)}$, $\widetilde{S}_j(z) = S_j(z) + \mathcal{O}(h)$. Then,

$$\frac{1}{2\pi i} \int_{\gamma_j} \frac{f'}{f}\, dz = \frac{1}{2\pi h} \int_{\gamma_j} \widetilde{S}_j'(z)\, dz = \frac{1}{2\pi h}(\widetilde{S}_j(e_j) - \widetilde{S}_j(s_{j-1})),$$

so

$$\sum_{j=1}^{N} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f'}{f}\, dz = \frac{1}{2\pi h} \sum_{j=0}^{N-1} (\widetilde{S}_j(e_j) - \widetilde{S}_{j+1}(s_j))$$
$$= \frac{1}{2\pi h} \sum_{j=0}^{N-1} (S_j(e_j) - S_{j+1}(s_j)) + \mathcal{O}(1),$$

and hence in view of (7.1) and the uniform boundedness of N:

$$N(f,\gamma) = \frac{1}{2\pi h} \sum_{j=0}^{N-1} (S_j(s_j) - S_{j+1}(s_j)) + \mathcal{O}(1).$$

Here we recall that $\operatorname{Im}(S_j - S_{j+1}) = \mathcal{O}(h)$ in $D(z_j, r_j)$. It follows that $\nabla(S_j - S_{j+1}) = \mathcal{O}(h/r_j)$ in $D(z_j, r_j/2)$ and consequently that

$$S_j(s_j) - S_{j+1}(s_{j+1}) = S_j(z) - S_{j+1}(z) + \mathcal{O}(h),$$

for any other point $z \in D(z_j, r_j/2)$. Thus finally,

$$N(f,\gamma) = \frac{1}{2\pi h} \sum_{j=0}^{N-1} (S_j(w_j) - S_{j+1}(w_j)) + \mathcal{O}(1),$$

with $w_j \in D(z_j, r_j/2)$ chosen arbitrarily. Here we can further replace $S_j(w_j) - S_{j+1}(w_j)$ by its real part, since $\operatorname{Im} S_j - S_{j+1} = \mathcal{O}(h)$ in $D(z_j, r_j)$.

8 Skeleton in the Region $|\mu| \gg h$.

We now return to the situation in Section 6. We are interested in the solutions μ of (6.21). In the following, we will write $S_{j,k}$ instead of $\widetilde{S}_{j,k}$, so we are interested in the zeros of the function $F_0(\mu; h)$ appearing in (6.21), given by

$$F_0(\mu; h) = e^{\frac{i}{h}(S_{1,2}+S_{3,4}) - i\frac{\pi}{2}} a_{2,3} + e^{\frac{i}{h}S_{1,2}} a_{2,4} - a_{1,3} e^{\frac{i}{h}S_{3,4}} - a_{1,4} e^{\frac{i\pi}{2}}.$$

Pulling out a factor $e^{-i\pi/2}$, we get the new equivalent function

$$F(\mu; h) = e^{\frac{i}{h}(S_{1,2}+S_{3,4})} a_{2,3} + e^{\frac{i}{h}S_{1,2} + \pi\frac{\mu}{h}} + e^{\frac{i}{h}S_{3,4} + \pi\frac{\mu}{h}} + a_{1,4}, \tag{8.1}$$

which has the same zeros as F_0. Here we have also used the explicit formulae for $a_{2,4}$, $a_{1,3}$ in (4.8).

Using the results of Section 5, we shall now look at the asymptotics of $F(\mu; h)$, when $|\mu|/h \gg 1$.

Case 1: Assume that

$$Ch \le |\mu| \ll 1, \ \left|\arg \mu - \tfrac{\pi}{2}\right| \le \pi - \tfrac{1}{C}. \tag{8.2}$$

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(Case 2, given by $|\arg\mu + \frac{\pi}{2}| < \pi - \frac{1}{\mathcal{O}(1)}$ (see page 615) will be reduced to Case 1 by a symmetry argument.) In this region, we have (5.9):

$$a_{1,4} = e^{\mathcal{O}_-(\frac{h}{\mu})}\left(e^{\frac{i}{h}(-\mu\ln\frac{\mu}{i}+\mu-\frac{\pi h}{4}+i\frac{\pi\mu}{2})} + e^{\frac{i}{h}(-\mu\ln\frac{\mu}{i}+\mu-\frac{\pi h}{4}-i\frac{3\pi\mu}{2})}\right),$$

and using (8.1) and (5.6) we also get,

$$\begin{aligned} F(\mu;h) &= e^{\frac{i}{h}(S_{1,2}+S_{3,4})}e^{\frac{i}{h}(\mu\ln\frac{\mu}{i}-\mu+\frac{\pi h}{4}-i\frac{\pi\mu}{2})-\mathcal{O}_-(\frac{h}{\mu})} + e^{\frac{i}{h}S_{1,2}+\pi\frac{\mu}{h}} + e^{\frac{i}{h}S_{3,4}+\pi\frac{\mu}{h}} \\ &\quad + e^{\mathcal{O}_-(\frac{h}{\mu})+\frac{i}{h}(-\mu\ln\frac{\mu}{i}+\mu-\frac{\pi h}{4}+i\frac{\pi\mu}{2})} + e^{\mathcal{O}_-(\frac{h}{\mu})+\frac{i}{h}(-\mu\ln\frac{\mu}{i}+\mu-\frac{\pi h}{4}-i\frac{3\pi\mu}{2})} \\ &= e^{\frac{\pi\mu}{2h}}G(\mu;h), \end{aligned}$$

where

$$\begin{aligned} G(\mu;h) &= a_1 + a_2 + a_3 + a_4,\ a_4 = a_{4^+} + a_{4^-}, \\ a_1 &= e^{\frac{i}{h}(S_{1,2}+S_{3,4}+\mu\ln\frac{\mu}{i}-\mu+\frac{\pi h}{4})-\mathcal{O}_-(\frac{h}{\mu})}, \\ a_2 &= e^{\frac{i}{h}S_{1,2}+\frac{\pi\mu}{2h}}, \\ a_3 &= e^{\frac{i}{h}S_{3,4}+\frac{\pi\mu}{2h}}, \\ a_{4\pm} &= e^{\mathcal{O}_-(\frac{h}{\mu})+\frac{i}{h}(-\mu\ln\frac{\mu}{i}+\mu-\frac{\pi h}{4})\pm\frac{\pi\mu}{h}}. \end{aligned} \tag{8.3}$$

We have $|a_j| = e^{r_j/h}$, $j = 1, 2, 3, 4^\pm$, where

$$\begin{aligned} r_1 &:= -\operatorname{Im} S_{1,2} - \operatorname{Im} S_{3,4} + (\operatorname{Im}\mu)\ln\frac{1}{|\mu|} - \operatorname{Re}\mu\,\arg\frac{\mu}{i} + \operatorname{Im}\mu - h\operatorname{Re}\mathcal{O}_-\Big(\frac{h}{\mu}\Big), \\ r_2 &:= -\operatorname{Im} S_{1,2} + \frac{\pi}{2}\operatorname{Re}\mu, \\ r_3 &:= -\operatorname{Im} S_{3,4} + \frac{\pi}{2}\operatorname{Re}\mu, \\ r_4^\pm &:= -(\operatorname{Im}\mu)\ln\frac{1}{|\mu|} + \operatorname{Re}\mu\,\arg\frac{\mu}{i} - \operatorname{Im}\mu \pm \pi\operatorname{Re}\mu + h\operatorname{Re}\mathcal{O}_-\Big(\frac{h}{\mu}\Big). \end{aligned}$$

Notice that $a_{4\pm}$ is dominating over $a_{4\mp}$ when $\pm\operatorname{Re}\mu \geq 0$, and in each half-plane $\pm\operatorname{Re}\mu > 0$, we may associate a_4 to the dominating term, modulo an error which is $\mathcal{O}(e^{-2\pi|\operatorname{Re}\mu|/h})$ times the leading term. Also notice that the last equations take the form

$$\begin{aligned} r_1 &= (\operatorname{Im}\mu)\ln\frac{1}{|\mu|} - \operatorname{Im} S_{1,2} - \operatorname{Im} S_{3,4} - Y(\mu), \\ r_2 &= -\operatorname{Im} S_{1,2} + \frac{\pi}{2}\operatorname{Re}\mu, \\ r_3 &= -\operatorname{Im} S_{3,4} + \frac{\pi}{2}\operatorname{Re}\mu, \\ r_{4\pm} &= -(\operatorname{Im}\mu)\ln\frac{1}{|\mu|} \pm \pi\operatorname{Re}\mu + Y(\mu), \end{aligned}$$

where

$$Y(\mu) = (\operatorname{Re}\mu)\arg\Big(\frac{\mu}{i}\Big) - \operatorname{Im}\mu + h\operatorname{Re}\mathcal{O}_-\Big(\frac{h}{\mu}\Big). \tag{8.4}$$

Following the general principles, as explained for example in [7] (see also [1]), we shall now look for the curves $\Gamma_{j,k}$, $j,k = 1,2,3,4^{\pm}$, where $|a_j| = |a_k|$, and we shall especially be interested in those parts of $\Gamma_{j,k}$ where $|a_j| = |a_k|$ is dominating over the other $|a_\nu|$. In doing so, let us remark first that we will not see any zeros of G generated by the zeros of a_2+a_3 as a dominating part of G, for when $\operatorname{Re}\mu > 0$, then r_4^+ dominates over r_4^- and $r_1+r_4^+ = r_2+r_3$ and clearly we cannot have $r_2 = r_3 \geq \max(r_1, r_4^+)+\text{Const}$. Now when $\operatorname{Re}\mu < 0$, r_4^- dominates over r_4^+ and $r_2 + r_3 - \pi\operatorname{Re}\mu = r_1 + r_4^-$, so that $r_2 + r_3 < r_1 + r_4^-$ in this case, leading to an even stronger conclusion.

We now begin look at the location of zeros of a_4 and of sums of two of the a_j.
Zeros of a_4 are of the form $\mu = i(k + \frac{1}{2})h$, $k = 0, 1, 2, \dots$.
Zeros of $a_3 + a_{4^\pm}$ are contained in the region $\Gamma_{3,4^\pm}$:

$$(\operatorname{Im}\mu)\ln\frac{1}{|\mu|} = \operatorname{Im}S_{3,4}(\mu) + Y(\mu) - \frac{\pi}{2}\operatorname{Re}\mu \pm \pi\operatorname{Re}\mu.$$

Similarly the zeros of $a_2 + a_{4^\pm}$ are contained in $\Gamma_{2,4^\pm}$:

$$(\operatorname{Im}\mu)\ln\frac{1}{|\mu|} = \operatorname{Im}S_{1,2}(\mu) + Y(\mu) - \frac{\pi}{2}\operatorname{Re}\mu \pm \pi\operatorname{Re}\mu.$$

The zeros of $a_1 + a_3$ are contained in $\Gamma_{1,3}$:

$$(\operatorname{Im}\mu)\ln\frac{1}{|\mu|} = \operatorname{Im}S_{1,2}(\mu) + \frac{\pi}{2}\operatorname{Re}\mu + Y(\mu).$$

The zeros of $a_1 + a_2$ are contained in $\Gamma_{1,2}$:

$$(\operatorname{Im}\mu)\ln\frac{1}{|\mu|} = \operatorname{Im}S_{3,4}(\mu) + \frac{\pi}{2}\operatorname{Re}\mu + Y(\mu).$$

The zeros of $a_1 + a_{4^\pm}$ are contained in $\Gamma_{1,4^\pm}$:

$$(\operatorname{Im}\mu)\ln\frac{1}{|\mu|} = \frac{1}{2}(\operatorname{Im}S_{1,2} + \operatorname{Im}S_{3,4}) \pm \frac{\pi}{2}\operatorname{Re}\mu + Y(\mu).$$

Put

$$X = \frac{\pi}{2}\operatorname{Re}\mu + (\operatorname{Re}\mu)\arg\Big(\frac{\mu}{i}\Big) - \operatorname{Im}\mu + h\operatorname{Re}\mathcal{O}_-\Big(\frac{h}{\mu}\Big) = \frac{\pi}{2}\operatorname{Re}\mu + Y(\mu).$$

When $\operatorname{Re}\mu > 0$, a_{4^+} dominates over a_{4^-} and we shall only consider $\Gamma_{3,4^+} = \Gamma_{1,2}$, $\Gamma_{2,4^+} = \Gamma_{1,3}$, $\Gamma_{1,4^+}$ given by

$$(\operatorname{Im}\mu)\ln\frac{1}{|\mu|} = \begin{cases} \operatorname{Im}S_{3,4} + X, & \text{on } \Gamma_{3,4^+} = \Gamma_{1,2} \\ \operatorname{Im}S_{1,2} + X & \text{on } \Gamma_{2,4^+} = \Gamma_{1,3}, \\ \frac{1}{2}(\operatorname{Im}S_{1,2} + \operatorname{Im}S_{3,4}) + X & \text{on } \Gamma_{1,4^+}. \end{cases} \tag{8.5}$$

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Recall now from Section 5 that $\mathcal{O}_-(h/\mu)$ in (8.4) appears as a remainder in Stirling's formula, so that $\partial_\mu \mathcal{O}_-(h/\mu) = \mathcal{O}(h/\mu^2)$, and hence X is uniformly Lipschitz for $|\mu| \geq h$. Proposition B.1 can therefore be applied to get the approximate behavior of the $\Gamma_{j,k}$. This will be exploited later.

In the left half-plane, a_{4^-} dominates over a_{4^+} and we consider all the 5 curves $\Gamma_{3,4^-}, \Gamma_{2,4^-}, \Gamma_{1,4^-}, \Gamma_{1,2}$, and $\Gamma_{1,3}$, given by:

$$(8.6) \qquad (\operatorname{Im}\mu)\ln\frac{1}{|\mu|} = \begin{cases} \operatorname{Im} S_{3,4} - 2\pi \operatorname{Re}\mu + X & \text{on } \Gamma_{3,4^-} \\ \operatorname{Im} S_{1,2} - 2\pi \operatorname{Re}\mu + X & \text{on } \Gamma_{2,4^-} \\ \operatorname{Im} S_{1,2} + X & \text{on } \Gamma_{1,3}, \\ \operatorname{Im} S_{3,4} + X & \text{on } \Gamma_{1,2}, \\ \frac{1}{2}(\operatorname{Im} S_{1,2} + \operatorname{Im} S_{3,4}) - \pi \operatorname{Re}\mu + X & \text{on } \Gamma_{1,4^-}. \end{cases}$$

Again Proposition B.1 can be applied to give the approximate shape of $\Gamma_{j,k}$. Recall that we are in Case 1 with $|\mu| \gg h$, so that (8.2) holds.

8.1 Skeleton in the Region $\operatorname{Re}\mu \geq 0$

(We will implicitly use that $|a_2||a_3| = |a_1||a_{4^+}|$.) The region $|\operatorname{Re}\mu| \leq \mathcal{O}(h)$ will require a special discussion. In the region $\operatorname{Re}\mu \geq Ch$, we have $|a_4^+| = e^{2\pi \operatorname{Re}\mu/h}|a_{4^-}| \gg |a_4^-|$, so $|a_4| \sim |a_4^+|$ and in this region we see from an earlier observation that the zeros of $a_2 + a_3$ will not play any essential role. In this region we shall therefore use the $\Gamma_{j,k}$ appearing in (8.5) and, as pointed out earlier, we are interested here in the part of each such $\Gamma_{j,k}$, where $|a_j| = |a_k|$ dominates over the other $|a_\nu|$.

It follows from Proposition B.1 that the curves in (8.5) (as well as the ones in (8.6)), are of the form $\operatorname{Im}\mu = \gamma_{j,k}(\operatorname{Re}\mu)$, with $|\gamma_{j,k}|, |\gamma'_{j,k}| \ll 1$. Notice that every crossing point of two of the curves $\Gamma_{1,4^+}, \Gamma_{1,2}, \Gamma_{2,4^+}$ is also a crossing point for all three. This follows from (8.5) or even more trivially from the observation that two of the three equations $|a_1| = |a_{4^+}|$, $|a_1| = |a_2|$, $|a_2| = |a_{4^+}|$, imply the third one. Also notice that if we draw the two curves $\Gamma_{1,2} = \Gamma_{3,4^+}, \Gamma_{1,3} = \Gamma_{2,4^+}$, then $\Gamma_{1,4^+}$ is between the two; see Figure 4.

For $\mu > 0$ we have $X(\mu) = h \operatorname{Re}\mathcal{O}_-(\frac{h}{\mu}) = h\mathcal{O}(e^{-2\pi\mu/h})$ by (5.5) and hence $\gamma_{j,k}(\operatorname{Re}\mu)$ is described as in Proposition B.1 with

(8.7)

$$F = F_{j,k}(\operatorname{Re}\mu) = h\mathcal{O}(e^{-2\pi \operatorname{Re}\mu/h}) + \begin{cases} \operatorname{Im} S_{3,4}(\operatorname{Re}\mu),\ (j,k) = (3,4^+),\ (1,2), \\ \operatorname{Im} S_{1,2}(\operatorname{Re}\mu),\ (j,k) = (2,4^+),\ (1,3), \\ \frac{1}{2}(\operatorname{Im} S_{1,2} + \operatorname{Im} S_{3,4})(\operatorname{Re}\mu),\ (j,k) = (1,4^+). \end{cases}$$

In the region $\operatorname{Im}\mu < \min(\gamma_{2,4^+}(\operatorname{Re}\mu), \gamma_{3,4^+}(\operatorname{Re}\mu))$ we have

$$|a_{4^+}| \geq \max(|a_1|, |a_2|, |a_3|, |a_{4^-}|),$$

and if we restrict further to

$$(8.8) \qquad \operatorname{Im}\mu < \min(\gamma_{2,4^+}(\operatorname{Re}\mu), \gamma_{3,4^+}(\operatorname{Re}\mu)) - \frac{Ch}{\ln\frac{1}{|\mu|}},\ \operatorname{Re}\mu > Ch,$$

with $C \gg 1$, we see that a_{4^+} is dominating in the sense that

$$|a_{4^+}| \geq 2|a_1 + a_2 + a_3 + a_{4-}|,$$

and hence $G(\mu; h)$ has no zeros in that region. Similarly in the region $\operatorname{Im}\mu > \max(\gamma_{1,2}, \gamma_{1,3})(\operatorname{Re}\mu)$, we have $|a_1| \geq |a_2|, |a_3|, |a_{4\pm}|$, and if

$$\operatorname{Im}\mu > \max(\gamma_{1,2}, \gamma_{1,3})(\operatorname{Re}\mu) + \frac{Ch}{\ln\frac{1}{|\mu|}}, \quad \operatorname{Re}\mu \geq 0, \tag{8.9}$$

with $C \gg 1$, then a_1 is dominating in the sense that $|a_1| \geq 2|a_2 + a_3 + a_4|$, and again $G(\mu; h)$ has no zeros there.

Now consider a point $\mu \in \Gamma_{3,4^+}$, where $\gamma_{3,4^+} \leq \gamma_{2,4^+}$, so that $\operatorname{Im} S_{3,4} \leq \operatorname{Im} S_{1,2}$, with $\operatorname{Re}\mu \gg h$. Going down (*i.e.,* decreasing $\operatorname{Im}\mu$ while keeping $\operatorname{Re}\mu$ constant) by a distance $\gg h/\ln\frac{1}{|\mu|}$, we reach the region (8.8), where a_{4^+} dominates.

Case a: $0 \leq \operatorname{Im} S_{1,2} - \operatorname{Im} S_{3,4} \leq \mathcal{O}(h)$. Going up by a distance $\gg h/\ln\frac{1}{|\mu|}$, we cross $\Gamma_{2,4^+} = \Gamma_{1,3}$ and reach the region (8.9), where a_1 is dominating.

Case b: $\operatorname{Im} S_{1,2} - \operatorname{Im} S_{3,4} \geq Ch$ for $C \gg 1$. Going up by a distance $\sim h/\ln\frac{1}{|\mu|}$, we reach the region, where a_3 is dominating, $|a_3| \geq 2|a_1 + a_2 + a_4|$, and continuing to go up, a_3 remains dominating until we reach a $h/\ln\frac{1}{|\mu|}$-neighborhood of $\Gamma_{1,3} = \Gamma_{2,4^+}$. After crossing that curve and going up by another amount $Ch/\ln\frac{1}{|\mu|}$, we reach the region, where a_1 is dominating.

Our discussion shows the following.

Proposition 8.1 *We work in the region* (8.2) *and assume in addition that* $\operatorname{Re}\mu > 0$. *If* $\operatorname{Im}\mu \leq \min(\gamma_{2,4^+}, \gamma_{3,4^+})(\operatorname{Re}\mu)$, *then*

$$|a_{4^+}| \geq \max(|a_1|, |a_2|, |a_3|, |a_{4-}|), \tag{8.10}$$

If $\operatorname{Im}\mu \geq \max(\gamma_{1,2}, \gamma_{1,3})(\operatorname{Re}\mu)$, *then*

$$|a_1| \geq \max(|a_2|, |a_3|, |a_{4\pm}|). \tag{8.11}$$

If $\gamma_{3,4^+}(\operatorname{Re}\mu)(= \gamma_{1,2}(\operatorname{Re}\mu)) \leq \operatorname{Im}\mu \leq \gamma_{1,3}(\operatorname{Re}\mu)(= \gamma_{2,4^+}(\operatorname{Re}\mu))$, *then*

$$|a_3| \geq \max(|a_1|, |a_2|, |a_{4\pm}|). \tag{8.12}$$

If $\gamma_{2,4^+}(\operatorname{Re}\mu)(= \gamma_{1,3}(\operatorname{Re}\mu)) \leq \operatorname{Im}\mu \leq \gamma_{3,4^+}(\operatorname{Re}\mu)(= \gamma_{1,2}(\operatorname{Re}\mu))$, *then*

$$|a_2| \geq \max(|a_1|, |a_3|, |a_{4\pm}|). \tag{8.13}$$

If the distance from μ *to* $(\Gamma_{1,2} = \Gamma_{3,4^+}) \cup (\Gamma_{1,2} = \Gamma_{3,4^+})$ *is* $\geq Ch/\ln\frac{1}{|\mu|}$, *with* $C \gg 1$, *then in the respective cases* (8.11), (8.12), *and* (8.13) *can be sharpened to the dominance in the sense explained above. In particular, G has no zeros in this region. If, in addition,* $\operatorname{Re}\mu \geq Ch$ *with* $C \gg 1$, *then we have the same conclusion in the case of* (8.10).

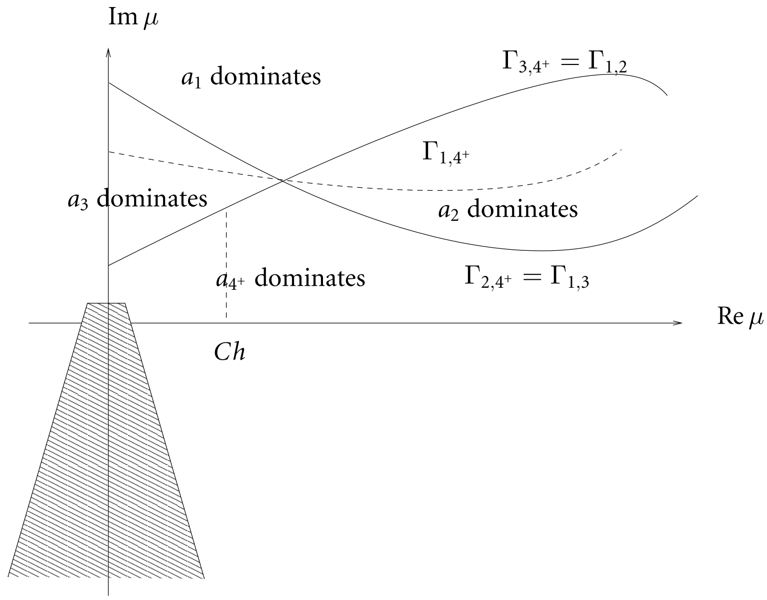

Figure 4: The union of the solid curves in the figure gives a schematic representation of the skeleton S' in the right half-plane intersected with the region (8.2). Proposition 8.1 shows that the zeros of G in this region are inside the union of the thickened skeleton, obtained by placing a disc of radius $Ch/\,|\ln|\mu||$ around each point $\mu \in S'$, and the set of all μ with $0 \le \operatorname{Re}\mu \le Ch$ below S'. Proposition 8.3 gives a more precise description of the location of the zeros of G with $|\operatorname{Re}\mu| \le \mathcal{O}(h)$.

In the region (8.2), intersected with the right half-plane $\operatorname{Re}\mu > 0$ we define the skeleton to be the union of the curves $\operatorname{Im}\mu = \max(\gamma_{1,2}, \gamma_{1,3})(\operatorname{Re}\mu)$ and $\operatorname{Im}\mu = \min(\gamma_{2,4^+}, \gamma_{3,4^+})(\operatorname{Re}\mu)$. The proposition shows that the zeros of G in the region under consideration are contained in the union of all discs $D(\mu, Ch/\ln\frac{1}{|\mu|})$ with μ in the skeleton just defined, and the set of all μ below the skeleton, with $0 \le \operatorname{Re}\mu < Ch$, for $C \gg 1$.

8.2 Skeleton in the Region $\operatorname{Re}\mu \le 0$.

Again the region $|\operatorname{Re}\mu| \le \mathcal{O}(h)$ will require a separate discussion so we restrict our attention to $\operatorname{Re}\mu \le -Ch$ and we will use $|a_{4^-}| = e^{-2\pi\mu/h}|a_{4^+}| \gg |a_{4^+}|$, so that $|a_4| \sim |a_{4^-}|$. We therefore concentrate our attention on the curves in (8.6). As before we notice that every crossing point of two of the three curves $\Gamma_{1,2}, \Gamma_{2,4^-}, \Gamma_{1,4^-}$ is a crossing point of all three. The same holds for $\Gamma_{1,3}, \Gamma_{3,4^-}, \Gamma_{1,4^-}$.

Now use that $|\operatorname{Im}\mu|$ is small and hence that $\operatorname{Im} S_{1,2}, \operatorname{Im} S_{3,4}$ and their derivatives with respect to $\operatorname{Re}\mu$ are small. We can therefore consider the two curves:

$$A : -2\pi\operatorname{Re}\mu = \operatorname{Im} S_{1,2} - \operatorname{Im} S_{3,4}, \quad B : -2\pi\operatorname{Re}\mu = \operatorname{Im} S_{3,4} - \operatorname{Im} S_{1,2}.$$

They are of the form $-\operatorname{Re}\mu = \gamma_A(\operatorname{Im}\mu)$ and $-\operatorname{Re}\mu = \gamma_B(\operatorname{Im}\mu)$, where γ_A, γ_B are

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small with small derivatives and satisfy

$$|\gamma_A(\operatorname{Re}\mu)| \sim |\gamma_B(\operatorname{Re}\mu)|, \quad \gamma_A\gamma_B \le 0. \tag{8.14}$$

The curve $\Gamma_{1,4^-}$ will play a central role. It crosses A, B at the points μ_A, μ_B (unless these points are hidden in the forbidden region), and we have

$$(\operatorname{Re}\mu_A)(\operatorname{Re}\mu_B) \le 0, \ |\operatorname{Re}\mu_A| \sim |\operatorname{Re}\mu_B|.$$

We notice that μ_A is the unique crossing point for $\Gamma_{1,3}, \Gamma_{3,4^-}, \Gamma_{1,4^-}$ while μ_B is the unique crossing point for $\Gamma_{1,2}, \Gamma_{2,4^-}, \Gamma_{1,4^-}$. More precisely, $\gamma_{1,3}(t) - \gamma_{1,4^-}(t)$, $\gamma_{1,4^-}(t) - \gamma_{3,4^-}(t)$ vanish precisely for $t = \operatorname{Re}\mu_A$ and have the same sign as $t - \operatorname{Re}\mu_A$. Similarly $\gamma_{1,2}(t) - \gamma_{1,4^-}(t)$, $\gamma_{1,4^-}(t) - \gamma_{2,4^-}(t)$ vanish precisely for $t = \operatorname{Re}\mu_B$ and have the same sign as $t - \operatorname{Re}\mu_B$. We also notice that if μ belongs to one of the three curves $\Gamma_{1,3}, \Gamma_{1,4^-}, \Gamma_{3,4^-}$, then the distance from μ to any of the two other curves among these three is $\ge C^{-1}|\operatorname{Re}\mu - \operatorname{Re}\mu_A|/\ln\frac{1}{|\mu|}$. The same observation holds for $\Gamma_{1,2}, \Gamma_{1,4^-}, \Gamma_{2,4^-}$ with μ_B instead of μ_A.

For $\mu < 0$, we have $X - \pi\operatorname{Re}\mu = h\operatorname{Re}\mathcal{O}_-(\frac{h}{\mu}) = h\mathcal{O}(e^{-2\pi|\mu|/h})$ by (5.5), and hence $\gamma_{1,4^-}$ is described as in Proposition B.1 with

$$F = F_{1,4^-}(\operatorname{Re}\mu) = \tfrac{1}{2}(\operatorname{Im}S_{1,2} + \operatorname{Im}S_{3,4})(\operatorname{Re}\mu) + h\mathcal{O}(e^{-2\pi|\operatorname{Re}\mu|/h}).$$

To fix the ideas we now assume that $\operatorname{Re}\mu_A \le 0$ (the case $\operatorname{Re}\mu_B \le 0$ can be treated similarly). Considering the three curves $\Gamma_{1,3}, \Gamma_{1,4^-}, \Gamma_{3,4^-}$, we see that for $\operatorname{Re}\mu \le 0$:

- If $\operatorname{Im}\mu \le \min(\gamma_{1,4^-}, \gamma_{3,4^-})(\operatorname{Re}\mu)$, then $|a_{4^-}| \ge |a_1|, |a_2|, |a_3|, |a_{4^+}|$. (In this case, we also have $\operatorname{Im}\mu \le \gamma_{2,4^-}(\operatorname{Re}\mu)$.)
- If $\operatorname{Im}\mu \ge \max(\gamma_{1,4^-}, \gamma_{1,3})(\operatorname{Re}\mu)$, then $|a_1| \ge |a_2|, |a_3|, |a_{4\pm}|$. (In this case, we also have $\operatorname{Im}\mu \ge \gamma_{1,2}(\operatorname{Re}\mu)$.)
- If $\gamma_{3,4^-}(\operatorname{Re}\mu) \le \operatorname{Im}\mu \le \gamma_{1,3}(\operatorname{Re}\mu)$, then $|a_3| \ge |a_1|, |a_2|, |a_{4\pm}|$.

This covers all possible cases with $\operatorname{Re}\mu \le 0$. Notice that the last case can appear only when $\operatorname{Re}\mu_A \le \operatorname{Re}\mu \le 0$. When $\operatorname{Re}\mu_B \le 0$, we get the analogous discussion after a permutation of the indices 2 and 3.

- If $\operatorname{Im}\mu \le \min(\gamma_{1,4^-}, \gamma_{2,4^-})(\operatorname{Re}\mu)$, then $|a_{4^-}| \ge |a_1|, |a_2|, |a_3|, |a_{4^+}|$.
- If $\operatorname{Im}\mu \ge \max(\gamma_{1,4^-}, \gamma_{1,2})(\operatorname{Re}\mu)$, then $|a_1| \ge |a_2|, |a_3|, |a_{4\pm}|$.
- If $\gamma_{2,4^-}(\operatorname{Re}\mu) \le \operatorname{Im}\mu \le \gamma_{1,2}(\operatorname{Re}\mu)$, then $|a_2| \ge |a_1|, |a_3|, |a_{4\pm}|$.

Moreover, if in addition all the inequalities for $\operatorname{Im}\mu$ are valid with an extra margin $Ch/\ln\frac{1}{|\mu|}$, $C \gg 1$, then in the various cases, we have dominance of a_1, a_2, a_3 respectively, in the sense explained before. For the dominance of a_{4^-}, we also need the assumption that $\operatorname{Re}\mu \le -Ch$ with $C \gg 1$.

8.2.1 Exponential Localization to the Skeleton

Recall that we are still working in the region (8.2). In this region we define the skeleton to be $S = S' \cup \Gamma_4$, where we define S' to be the union of the following two sets in the left and the right half-planes, respectively:

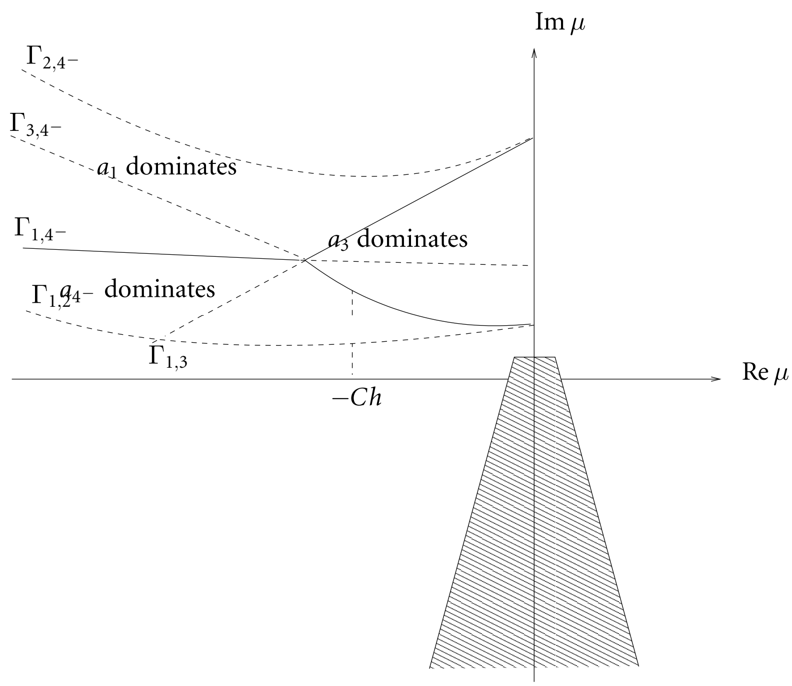

Figure 5: The union of the solid curves in the figure gives a schematic representation of the skeleton S' in the left half-plane intersected with the region (8.2). Proposition 8.2 shows that the zeros of G are inside the union of the thickened skeleton, obtained by placing a disc of radius $Ch/\left|\ln|\mu|\right|$ around each point $\mu \in S'$, and the set of all μ with $|\operatorname{Re}\mu| \le Ch$ below S'.

- In the closed left half-plane (intersected with (8.2)), assume $\operatorname{Re}\mu_A \le 0$ to fix the ideas. Then this part of S' is given by all points of the form $\operatorname{Im}\mu = \gamma_{1,4^-}(\operatorname{Re}\mu)$ with $\operatorname{Re}\mu \le \operatorname{Re}\mu_A$, all points of the form $\operatorname{Im}\mu = \gamma_{3,4^-}(\operatorname{Re}\mu)$ or of the form $\operatorname{Im}\mu = \gamma_{1,3}(\operatorname{Re}\mu)$, with $\operatorname{Re}\mu_A \le \operatorname{Re}\mu \le 0$.
- In the closed right half-plane the corresponding part of S' is defined to be the union of the two curves:

$$\operatorname{Im}\mu = \max(\gamma_{1,3}, \gamma_{1,2})(\operatorname{Re}\mu) \quad \text{and} \quad \operatorname{Im}\mu = \min(\gamma_{2,4^+}, \gamma_{3,4^+})(\operatorname{Re}\mu).$$

Γ_4 is defined to be the part of the imaginary axis given by

$$\operatorname{Re}\mu = 0,\ Ch \le \operatorname{Im}\mu \le \min(\gamma_{2,4^+}, \gamma_{3,4^+})(0),$$

where C is the same constant as in (8.2). Notice that this part may be empty. The earlier discussion shows that we have the following.

Proposition 8.2 *The zeros of G in the domain* (8.2) *are contained in the set*

$$\Big(\bigcup_{\mu\in S'} D\Big(\mu, \frac{Ch}{\ln\frac{1}{|\mu|}}\Big)\Big) \cup \{\mu \text{ below } S';\ |\operatorname{Re}\mu| < Ch\}.$$

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Remark The localization result of Proposition 8.2 improves if we are far from the branch points of the skeleton. Thus for instance, if μ is below S' and $\operatorname{dist}(\mu, S') \geq Ch/\ln\frac{1}{|\mu|}$, then

$$\max|a_{4\pm}| \geq e^{\frac{\ln\frac{1}{|\mu|}}{Ch}\operatorname{dist}(\mu,S')}(|a_1| + |a_2| + |a_3|),$$

so the zeros of G are exponentially small perturbations of those of $a_4 = a_{4^+} + a_{4^-}$. In this region there is a bijection b between the zeros of a_4 and those of G, with

$$|b(\mu) - \mu| \leq \mathcal{O}(h)\exp[-(Ch)^{-1}\ln\Big(\frac{1}{|\mu|}\Big)\operatorname{dist}(\mu, S')].$$

Similarly let μ_0 be a point of S' in the right half-plane, say $\mu_0 \in \Gamma_{1,3}$ with a_1, a_3 dominating above and below this point respectively. If

$$\operatorname{dist}(\mu_0, \Gamma_{3,4^+} = \Gamma_{1,2}) \geq \frac{C^2h}{\ln\frac{1}{|\mu_0|}},$$

then in $D\big(\mu_0, Ch/(\ln(1/|\mu|_0|))\big)$, we have

$$\max(|a_1|, |a_3|) \geq e^{(Ch)^{-1}(\ln\frac{1}{|\mu_0|})\operatorname{dist}(\mu_0,\Gamma_{1,2})}\max(|a_2|, |a_4|),$$

and we conclude that the zeros of G are exponentially close to those of $a_1 + a_3$. Essentially the same statement holds when μ_0 belongs to the lower part of S', but here the size of $\operatorname{Re}\mu$ also matters, so near a point $\mu_0 \in \Gamma_{3,4^+} \cap S'$ we get a bijection b between the zeros of $a_3 + a_{4^+}$ and those of G with

$$b(\mu) - \mu = \mathcal{O}(1)\Big(\frac{h}{\ln\frac{1}{|\mu_0|}}\Big)(e^{-\frac{\ln\frac{1}{|\mu_0|}}{Ch}\operatorname{dist}(\mu,\Gamma_{1,3})} + e^{-\frac{2\pi|\operatorname{Re}\mu|}{h}}),$$

and we have to assume both that $\operatorname{dist}(\mu_0, \Gamma_{1,3}) \gg h/\ln\frac{1}{|\mu_0|}$ and that $|\operatorname{Re}\mu| \gg h$.

The analogous statements hold in the left half-plane.

8.2.2 More Refined Analysis in the Region $|\operatorname{Re}\mu| = \mathcal{O}(h)$

The study of the upper part ($\Gamma_{1,3}$ or $\Gamma_{1,2}$) of the skeleton is unchanged in this region, while the lower part requires more attention in view of the fact that $|a_4|$ may be considerably smaller than $\max(|a_{4^+}|, |a_{4^-}|)$ when we are close to a zero of a_4. In order to fix the ideas we assume that $\gamma_{3,4^+}(0) \leq \gamma_{2,4^+}(0)$.

After multiplication of $a_4, a_{4\pm}, a_3$ by the same exponential factor, we arrive at

$$\widetilde{a}_4 = \widetilde{a}_{4^+} + \widetilde{a}_{4^-} = 2\cosh\frac{\pi\mu}{h}, \quad \widetilde{a}_{4\pm} = e^{\frac{\pm\pi\mu}{h}}, \quad \widetilde{a}_3 = e^{\frac{i}{h}\phi(\mu;h)},$$

and we shall drop the tildes in the following discussion. Here

$$\phi(\mu; h) = S_{3,4}(\mu) - i\frac{\pi\mu}{2} + ih\mathcal{O}_-\Big(\frac{h}{\mu}\Big) + \mu\ln\frac{\mu}{i} + \frac{\pi h}{4} - \mu,$$

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with

$$-\operatorname{Im}\phi(\mu;h) = -\operatorname{Im} S_{3,4}(\mu) + \operatorname{Re}\frac{\pi\mu}{2} - h\operatorname{Re}\mathcal{O}_-\Big(\frac{h}{\mu}\Big) + \operatorname{Im}\mu \ln\frac{1}{|\mu|} - \operatorname{Re}\mu\arg\Big(\frac{\mu}{i}\Big) + \operatorname{Im}\mu.$$

We have

$$\partial_{\operatorname{Im}\mu}(-\operatorname{Im}\phi) = \ln\frac{1}{|\mu|} + \mathcal{O}(1), \quad \partial_{\operatorname{Re}\mu}(-\operatorname{Im}\phi) = \mathcal{O}(1),$$
$$\nabla_\mu^\alpha(-\operatorname{Im}\phi) = \mathcal{O}(|\mu|^{1-|\alpha|}), \quad |\alpha| \geq 2.$$

Recall that we work in the region $|\mu| \geq Ch$. Notice that $h\nabla_\mu \ln|a_3| = \nabla_\mu(-\operatorname{Im}\phi)$. Similarly, we look at

$$h\partial_\mu \ln a_4 = \pi\frac{\sinh\frac{\pi\mu}{h}}{\cosh\frac{\pi\mu}{h}} = \pi\sqrt{1 - \frac{1}{(\cosh\frac{\pi\mu}{h})^2}},$$

for a suitable branch of the square root. Also $h\partial_{\overline{\mu}} \ln a_4 = 0$, so this relation gives a bound for $h\nabla_\mu \ln a_4$ and its real part $h\nabla_\mu \ln|a_4|$. We have the general estimate

$$|\cosh z| \geq \frac{1}{C}\operatorname{dist}(z, \cosh^{-1}(0))e^{|\operatorname{Re} z|}, \quad \text{for } |\operatorname{Re} z| \leq \text{Const},$$

so we get

$$h\partial_\mu \ln a_4 = \pi\operatorname{sgn}(\operatorname{Re}\mu) + \mathcal{O}(1)\frac{e^{-\frac{\pi|\operatorname{Re}\mu|}{h}}}{\operatorname{dist}(\frac{\pi\mu}{h}, \cosh^{-1}(0))}.$$

Assuming

$$\operatorname{dist}\Big(\mu, \frac{h}{\pi}\cosh^{-1}(0)\Big) \geq \frac{Ch}{\ln\frac{1}{|\mu|}}, \quad \text{with } C \gg 1,$$

we conclude that $\big|h\nabla_\mu \ln|a_4|\big| \ll h\big|\nabla_\mu \ln|a_3|\big|$, and consequently,

$$h\partial_{\operatorname{Im}\mu}\ln\frac{|a_3|}{|a_4|} = (1 + o(1))\ln\frac{1}{|\mu|} + \mathcal{O}(1),\ h\partial_{\operatorname{Re}\mu}\ln\frac{|a_3|}{|a_4|} = \mathcal{O}(1) + o(1)\ln\frac{1}{|\mu|},$$

where $\mathcal{O}(1)$ denotes terms that are uniformly bounded and $o(1)$ denotes terms that tend to 0, when

$$\frac{\operatorname{dist}(\mu, \frac{h}{\pi}\cosh^{-1}\{0\})}{(h/\ln\frac{1}{|\mu|})} \to \infty.$$

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For each zero μ_j of $\cosh \frac{\pi\mu}{h}$, we introduce the diamond shaped neighborhood

$$D_j = \Big\{ \mu; |\operatorname{Re}\mu| + |\operatorname{Im}\mu - \operatorname{Im}\mu_j| \leq \frac{Ch}{\ln \frac{1}{|\mu|}} \Big\}, \tag{8.15}$$

with C large enough so that the preceding estimates apply away from the union of all the D_j. Define $\Gamma_{3,4}$ to be the set of points with $|a_3|/|a_4| = 1$ away from the union of all the D_j, with D_{j_0} added, if D_{j_0} has the property that the distance from this diamond to the points just defined, is zero. If it exists, D_{j_0} is unique since the other points of $\Gamma_{3,4}$ form a curve $\operatorname{Im}\mu = \gamma_{3,4}(\operatorname{Re}\mu)$, with $|\gamma'_{3,4}| \ll 1$. From the above estimates we get

$$\begin{aligned} \gamma_{3,4\pm}(\operatorname{Re}\mu) - \mathcal{O}(h)\frac{\ln\ln\frac{1}{|\mu|}}{\ln\frac{1}{|\mu|}} &\leq \gamma_{3,4}(\operatorname{Re}\mu) \\ &\leq \gamma_{3,4\pm}(\operatorname{Re}\mu) + \frac{\mathcal{O}(h)}{\ln\frac{1}{|\mu|}}, \quad \pm\operatorname{Re}\mu \geq 0. \end{aligned} \tag{8.16}$$

(If $\Gamma_{3,4}$ stays away from an h/C-neighborhood of the zeros of $\cosh \frac{\pi\mu}{h}$, then the agreement is better:

$$\gamma_{3,4}(\operatorname{Re}\mu) = \gamma_{3,4\pm}(\operatorname{Re}\mu) + \frac{\mathcal{O}(h)}{\ln\frac{1}{|\mu|}}, \quad \pm\operatorname{Re}\mu \geq 0.)$$

In fact, we can get an even more precise estimate for the distance between $\Gamma_{3,4}$ and $\Gamma_{3,4\pm}$. Let $\mu_0 \in \Gamma_{3,4}$, and put

$$d(\mu_0) = \max\Big(\frac{h}{\ln\frac{1}{|\mu_0|}}, \operatorname{dist}(\mu_0, a_4^{-1}(0)) \Big).$$

Then

$$\frac{d(\mu_0)}{Ch} \leq \frac{|a_4(\mu_0)|}{|a_{4\pm}(\mu_0)|} \leq C,$$

away from the diamonds. We therefore get the following estimate for the vertical distance from μ_0 to $\Gamma_{3,4\pm}$:

$$|\gamma_{3,4}(\operatorname{Re}\mu_0) - \gamma_{3,4\pm}(\operatorname{Re}\mu_0)| \leq \frac{Ch}{\ln\frac{1}{|\mu_0|}} \ln\frac{h}{d(\mu_0)},$$

(assuming for simplicity that $d(\mu_0) \leq h/2$). This is a refinement of the lower bound in (8.16), and the argument also gives the upper bound there.

We reach the following conclusion about the location of the zeros in the region $|\operatorname{Re}\mu| \leq Ch$.

Proposition 8.3 • *Above S' and at distance $\geq Ch/\ln\frac{1}{|\mu|}$ from S', $a_1(\mu)$ is dominating.*

- *a_4 is dominating if μ is below S', at distance $\geq Ch/\ln\frac{1}{|\mu|}$ from $a_4^{-1}(0)$ and at distance $\geq Ch\frac{\ln\ln}{\ln}\big(\frac{1}{|\mu|}\big)$ from S'.*
- *In between (for instance below $\Gamma_{1,3}$ but above $\Gamma_{3,4\pm}$), a_3 (or a_2) is dominating if the distance to the skeleton is $\geq Ch/\ln\frac{1}{|\mu|}$.*

8.2.3 Improvement in the Region $\operatorname{Re}\mu \gg h$

Let us recall that $a_1 a_{4^+} = a_2 a_3$. Therefore,

$$a_1 + a_2 + a_3 + a_{4^+} = a_{4^+}\Big(1 + \frac{a_2}{a_{4^+}}\Big)\Big(1 + \frac{a_3}{a_{4^+}}\Big).$$

The zeros of $1 + (a_2/a_{4^+})$ are situated on $\Gamma_{2,4^+}$ and are given by the explicit quantization condition

$$\mu \ln \mu - \mu + \frac{\pi h}{4} + S_{1,2} + ih\mathcal{O}_-\Big(\frac{h}{\mu}\Big) = 2\pi h\Big(k + \frac{1}{2}\Big),\ k \in \mathbf{Z}.$$

The distance between successive zeros is $\sim h/\ln \frac{1}{|\mu|}$. If μ_0 is such a zero, then in a disc $D(\mu_0, r)$ with $r \ll h/\ln \frac{1}{|\mu_0|}$, we have

$$\Big|1 + \frac{a_2}{a_{4^+}}\Big| \sim |\mu - \mu_0| \frac{\ln \frac{1}{|\mu_0|}}{h}.$$

Away from the union of all such discs, we have

$$\Big|1 + \frac{a_2}{a_{4^+}}\Big| \geq \frac{1}{\mathcal{O}(1)}.$$

Similarly, the zeros of $1 + (a_3/a_{4^+})$ are situated on the curve $\Gamma_{3,4^+}$ and given by the quantization condition

$$\mu \ln \mu - \mu + \frac{\pi h}{4} + S_{3,4} + ih\mathcal{O}_-\Big(\frac{h}{\mu}\Big) = 2\pi h\Big(k + \frac{1}{2}\Big), \quad k \in \mathbf{Z},$$

and the other statements about $1 + (a_2/a_{4^+})$ carry over to $1 + (a_3/a_{4^+})$.

Now consider

$$\begin{aligned} G(\mu; h) &= a_{4^+}\Big[\Big(1 + \frac{a_2}{a_{4^+}}\Big)\Big(1 + \frac{a_3}{a_{4^+}}\Big) + \frac{a_{4-}}{a_{4^+}}\Big] \\ &= a_{4^+}\Big[\Big(1 + \frac{a_2}{a_{4^+}}\Big)\Big(1 + \frac{a_3}{a_{4^+}}\Big) + e^{-\frac{2\pi\mu}{h}}\Big]. \end{aligned}$$

We get the following.

Proposition 8.4 *In the region $\operatorname{Re}\mu \gg h$, there is a bijection b from the union of the zeros of $1 + a_2/a_{4^+}$ and of $1 + a_3/a_{4^+}$ to the zeros of G with*

$$b(\mu) - \mu = \mathcal{O}(1)\frac{h}{\ln \frac{1}{|\mu|}} e^{-\frac{\pi \operatorname{Re}\mu}{h}}.$$

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So in the region $\operatorname{Re}\mu \gg h$, and modulo an exponentially small error, we can identify the zeros of G with the union of the zeros of $1 + a_2/a_{4^+}$ and of $1 + a_3/a_{4^+}$.

This finishes the analysis of the skeleton in the first case (8.2).

Case 2: Assume that

$$\Big|\arg\mu + \frac{\pi}{2}\Big| \le \pi - \frac{1}{C}, \quad h \ll |\mu| \ll 1.$$

In this case from (5.7) and (5.8), respectively, we recall that

$$a_{2,3} = 2\cosh\Big(\frac{\pi\mu}{h}\Big)\exp\Big[\mathcal{O}_+\Big(\frac{h}{\mu}\Big) + \frac{i}{h}\Big(\mu\ln(i\mu) - \mu + \frac{\pi h}{4}\Big) + \frac{\pi\mu}{2h}\Big],$$

$$a_{1,4} = \exp\Big[-\mathcal{O}_+\Big(h\mu\Big) - \frac{i}{h}\Big(\mu\ln(i\mu) - \mu + \frac{\pi h}{4}\Big) + \frac{\pi\mu}{2h}\Big].$$

Using (8.1), we get

$$F(\mu;h) = e^{\frac{\pi\mu}{2h}}G(\mu;h), \quad G(\mu;h) = a_1 + a_2 + a_3 + a_4, \tag{8.17}$$

where a_j, $j = 1,2,3,4$ are the same as in Case 1, but now with a partially different representation:

$$\begin{aligned} a_1 &= a_{1^+} + a_{1^-}, \\ a_{1^\pm} &= e^{\frac{i}{h}(S_{1,2}+S_{3,4}) + \mathcal{O}_+(\frac{h}{\mu}) + \frac{i}{h}(\mu\ln(i\mu) - \mu + \frac{\pi h}{4}) \pm \frac{\pi\mu}{h}}, \\ a_2 &= e^{\frac{i}{h}S_{1,2} + \frac{\pi\mu}{2h}}, \\ a_3 &= e^{\frac{i}{h}S_{3,4} + \frac{\pi\mu}{2h}}, \\ a_4 &= e^{-\mathcal{O}_+(\frac{h}{\mu}) - \frac{i}{h}(\mu\ln(i\mu) - \mu + \frac{\pi h}{4})} \end{aligned}$$

Again we consider h times the real parts of the different exponents of the a_j, $j = 1^\pm, 2, 3, 4$:

$$\begin{aligned} r_{1^\pm} &= -\operatorname{Im} S_{1,2} - \operatorname{Im} S_{3,4} + (\operatorname{Im}\mu)\ln\frac{1}{|\mu|} - \widetilde{Y}(\mu) \pm \pi\operatorname{Re}\mu, \\ r_2 &= -\operatorname{Im} S_{1,2} + \frac{\pi}{2}\operatorname{Re}\mu, \\ r_3 &= -\operatorname{Im} S_{3,4} + \frac{\pi}{2}\operatorname{Re}\mu, \\ r_4 &= -(\operatorname{Im}\mu)\ln\frac{1}{|\mu|} + \widetilde{Y}(\mu), \end{aligned}$$

where

$$\widetilde{Y}(\mu) = (\operatorname{Re}\mu)\arg(i\mu) - \operatorname{Im}\mu - h\operatorname{Re}\mathcal{O}_+\Big(\frac{h}{\mu}\Big).$$

By a symmetry argument, we shall now see that $(r_{1\pm}, r_2, r_3, r_4)$ plays the same role in the present Case 2 as $(r_{4\pm}, r_3, r_2, r_1)$ in Case 1, provided that we perform the following transformations on $(r_{1\pm}, r_2, r_3, r_4)$:

- Add $\operatorname{Im}(S_{1,2}+S_{3,4})$ to each of the five terms.
- Replace μ by $\overline{\mu}$.

Then we get $\widetilde{r}_j(\overline{\mu}) = (\operatorname{Im}(S_{1,2}+S_{3,4}) + r_j)(\mu)$:

$$\begin{aligned}
\widetilde{r}_{1\pm}(\overline{\mu}) &= -(\operatorname{Im}\overline{\mu})\ln\frac{1}{|\overline{\mu}|} + (\operatorname{Re}\overline{\mu})\arg\Big(\frac{\overline{\mu}}{i}\Big) - \operatorname{Im}\overline{\mu} + h\operatorname{Re}\mathcal{O}_+\Big(\frac{h}{\mu}\Big) \pm \pi\operatorname{Re}\overline{\mu},\\
\widetilde{r}_2(\overline{\mu}) &= \operatorname{Im}S_{3,4} + \frac{\pi}{2}\operatorname{Re}\overline{\mu},\\
\widetilde{r}_3(\overline{\mu}) &= \operatorname{Im}S_{1,2} + \frac{\pi}{2}\operatorname{Re}\overline{\mu},\\
\widetilde{r}_4(\overline{\mu}) &= \operatorname{Im}S_{1,2} + \operatorname{Im}S_{3,4} + (\operatorname{Im}\overline{\mu})\ln\frac{1}{|\overline{\mu}|} - (\operatorname{Re}\overline{\mu})\arg\Big(\frac{\overline{\mu}}{i}\Big)\\
&\qquad + \operatorname{Im}\overline{\mu} - h\operatorname{Re}\mathcal{O}_+\Big(\frac{h}{\mu}\Big).
\end{aligned}$$

This is analogous with $(r_{4\pm}, r_3, r_2, r_1)$ in Case 1 except for the fact that $\operatorname{Re}\mathcal{O}_+(\frac{h}{\mu})$ here corresponds to $\operatorname{Re}\mathcal{O}_-(\frac{h}{\mu})$ in Case 1.

Remark Using (5.4), it is easy to check that

$$\widetilde{Y}(\mu) - Y(\mu) = \pm\pi\operatorname{Re}\mu + h\mathcal{O}(e^{-2\pi|\operatorname{Re}\mu|/h}),$$

when

(8.18) $$|\arg(\pm\mu)| \le \frac{\pi}{2} - \frac{1}{C}, \quad |\mu| \ge h.$$

It follows that

(8.19) $$r_j(\mu) = r_{j\pm}(\mu) + h\mathcal{O}(e^{-2\pi|\operatorname{Re}\mu|/h}), \quad j = 1, 4,$$

when μ satisfies (8.18), and hence if μ belongs to the skeleton S' defined according to Case 1, the distance from μ to the corresponding skeleton S' defined according to Case 2 is

$$\mathcal{O}\Big(\frac{h}{\ln\frac{1}{|\mu|}} e^{-2\pi|\operatorname{Re}\mu|/h}\Big).$$

We end this section by some general considerations that will be useful in Section 13. We see from (8.7) that the spectrum will have a genuinely 2-dimensional structure if $|\operatorname{Im}S_{3,4}(0) - \operatorname{Im}S_{1,2}(0)| \gg h$, or if $\operatorname{Im}S_{3,4}(0)$ and $\operatorname{Im}S_{1,2}(0)$ have the same sign and $\min(|\operatorname{Im}S_{3,4}(0)|, |\operatorname{Im}S_{1,2}(0)|) \gg h\ln\frac{1}{h}$. In the latter case, we even have some eigenvalues on the imaginary μ-axis, related to 1-dimensional barrier top resonances. It is therefore important to have a sufficiently invariant and direct description of $\operatorname{Im} S_{3,4}(0)$, $\operatorname{Im} S_{1,2}(0)$.

The final definition of $S_{3,4}$ in the beginning of Section 6 is simply that we start with the null-solution f_4 of Q near α_4 and extend it along the exterior part of $K_{0,0}$ until we reach a neighborhood of α_3, where we get $\exp(\frac{1}{h}S_{3,4})f_3$. (Here we neglected the real Floquet parameter θ_1, since we are only interested in the imaginary part of $S_{3,4}$). The definition of $S_{1,2}$ is similar.

Now take $\mu = 0$ (see (6.7) and (6.8)) and represent the operator Q as acting in $H_\Phi^{\mathrm{loc}}(\Omega)$, where Φ is strictly plurisubharmonic, with $\Lambda_\Phi \simeq$ a neighborhood of $K_{0,0}$ in $T^*\mathbf{R}$. From the construction of e_j, f_j, we see that f_j is near α_j a normalized null-solution of Q in $H_{\Phi_0}^{\mathrm{loc}}(\Omega)$, where $\Phi - \Phi_0$ is small and Φ_0 is defined in a sufficiently large neighborhood of the projection of the branching point. Here if κ_T is the canonical transformation associated to some standard FBI-Bargmann transform, then $\Lambda_\Phi = \kappa_T(\mathbf{R}^2)$, $\Lambda_{\Phi_0} = \kappa_T \circ \kappa_U(\mathbf{R}^2)$, with U as in (6.7). Since $\Lambda_{\Phi_0} = \{\xi = \frac{2}{i}\frac{\partial \Phi_0}{\partial x}\}$ is naturally identified with $T^*\mathbf{R}$, where $p_0 = x\xi$, so (since $\mu = 0$), we know that the null set of Q_0 intersects Λ_{Φ_0} along two crossing "real" curves that we can identify with "the interior part" of $K_{0,0}$.

Extend Φ_0 to be defined in Ω with $\Phi - \Phi_0$ small. If $S_{3,4} \sim S_{3,4}^0 + hS_{3,4}^1 + \cdots$, then

$$-\operatorname{Im} S_{3,4}^0 = \int_{\gamma_{3,4}} (-\operatorname{Im}(\xi \cdot dx) - d\Phi_0), \tag{8.20}$$

where $\gamma_{3,4}$ now (see (6.14)) is a real curve from α_4 to α_3 in $Q_0^{-1}(z(0))$ close to the exterior part of the "left loop" of $K_{0,0}$. Here we let $z(0)$ denote the z-value in (6.8) corresponding to $\mu = 0$. Let this left loop be denoted by γ^1 and let us consider it (after slight deformation) as a closed curve in $Q_0^{-1}(z(0))$ joining the critical point of Q_0 to itself staying close to the left loop of $K_{0,0}$. Here, we may assume that the interior part of γ^1 (joining α_3 to α_4) is contained in Λ_{Φ_0}, so $-\operatorname{Im}(\xi \cdot dx) = d\Phi_0$ there. Hence (8.20) becomes

$$\operatorname{Im} S_{3,4}^0 = \int_{\gamma^1} (\operatorname{Im}(\xi \cdot dx) + d\Phi_0) = \int_{\gamma^1} (\operatorname{Im} \xi \cdot dx). \tag{8.21}$$

Here $\operatorname{Im}(\xi \cdot dx)$ in the last integral can be replaced by $\operatorname{Im}(\xi \cdot dx) + d\Phi$, which by Stokes' formula can be further replaced by any other real 1-form ω with $d\omega = \operatorname{Im} \sigma$, $\omega|_{\Lambda_\Phi} = 0$. This means that we can reinterpret the last integral in (8.21) as the corresponding one along the corresponding closed curve in the complexification of $\mathbf{R}^2$.

To simplify things further, recall that ϵ, h^2/ϵ are small perturbative parameters for Q_0 and that $Q_0^{-1}(z(0))$ is real when $\epsilon = h^2/\epsilon = 0$. In general, if $q = q_s$ depends smoothly on a real parameter s, if E is not a critical value and $\gamma = \gamma(s, E)$ is a simple closed curve in $q_s^{-1}(E)$, then for E fixed,

$$\frac{\partial}{\partial s} \int_\gamma \xi \, dx = -\int_0^{T(E,s)} \frac{\partial q}{\partial s}(x(t), \xi(t)) \, dt,$$

where $[0, T(E, s)] \ni t \mapsto \exp tH_{q_s}(\rho(0))$ is a natural parametrization. This can be

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applied to the case $q_s = q - E(s)$, so if E also depends on s, we get

$$\frac{\partial}{\partial s}\int_{\gamma}\xi\cdot\, dx = \int_0^{T(E,s)}\Big(\frac{\partial E(s)}{\partial s} - \frac{\partial q}{\partial s}(x(t),\xi(t))\Big)\, dt.$$

In this form, we can treat a loop like $\gamma_1(s) \subset q_s^{-1}(E(s))$, starting and ending at the critical points $\rho_c(s)$ of q_s, parametrized by $]-\infty,+\infty[\ \mapsto \exp(tH_{q_s})(\rho(0))$, provided that we take $E(s)$ equal to the critical value $q_s(\rho_c(s))$:

$$\frac{\partial}{\partial s}\int_{\gamma_1(s)}\xi\, dx = \int_{-\infty}^{+\infty}\Big(\frac{\partial}{\partial s}(q_s(\rho_c(s))) - (\frac{\partial}{\partial s}q_s)(x(t),\xi(t))\Big)\, dt. \tag{8.22}$$

This can be proved by a limiting procedure, approaching $\gamma_1(s)$ by closed curves at non-critical levels.

Taking the imaginary parts, this means that we have a fairly simple way of computing $\operatorname{Im} S_{1,2}$, $\operatorname{Im} S_{3,4}$ perturbatively. From this computation, we see that it is of interest to compute the ϵ^2 contribution to the averaged principal symbol. This computation was carried out in of [15, S2] under the assumption that $\langle q\rangle = 0$ and it works essentially the same way without that assumption. We start with the principal symbol $p_\epsilon = p + i\epsilon q + \epsilon^2 r + \mathcal{O}(\epsilon^3)$. The function

$$G_0 = \frac{1}{T(E)}\int_0^{T(E)}\Big(t - \frac{T(E)}{2}\Big)\, q\circ\exp(tH_p)\, dt, \quad \text{on } p^{-1}(E) \tag{8.23}$$

introduced in Proposition 2.1, satisfies $H_pG_0 = q - \langle q\rangle$. Put $G = G_0 + i\epsilon G_1 + \mathcal{O}(\epsilon^2)$, where G_1 remains to be determined. As in [15] we get at a general point $\exp(i\epsilon H_G)(\rho) \in \Lambda_{\epsilon G}$, $(\rho \in T^*M)$:

$$\begin{aligned} p_{\epsilon|_{\Lambda_\epsilon}} &\simeq p_\epsilon(\exp(i\epsilon H_G)(\rho)) = \sum_0^\infty \frac{(i\epsilon H_G)^k}{k!}p_\epsilon(\rho)\\ &= p + i\epsilon\langle q\rangle + \epsilon^2(r + H_pG_1 - H_{G_0}(tfrac12(q+\langle q\rangle))) + \mathcal{O}(\epsilon^3),\end{aligned}$$

where we used that $H_{G_0}^2 p = -H_{G_0}(q - \langle q\rangle)$.

Letting G_1 solve

$$H_pG_1 = H_{G_0}(\tfrac12(q+\langle q\rangle)) - \langle H_{G_0}(\tfrac12(q+\langle q\rangle))\rangle - (r - \langle r\rangle),$$

we get with $G = G_0 + i\epsilon G_1$,

$$p_\epsilon(\exp(i\epsilon H_G)(\rho)) = p + i\epsilon\langle q\rangle + \epsilon^2\langle s\rangle + \mathcal{O}(\epsilon^3). \tag{8.24}$$

Now assume for simplicity that $T(E) = T$ is constant. Then

$$\begin{aligned}\langle s\rangle &= \langle r\rangle - \frac{1}{2T}\int_0^T\{G_0, q+\langle q\rangle\}\circ\exp(tH_p)\, dt\\ &= \langle r\rangle - \frac{1}{2T^2}\int_0^T\int_0^T\Big(s - \frac{T}{2}\Big)\{q\circ\exp(t+s)H_p, (q+\langle q\rangle)\circ\exp(tH_p)\}\, dtds\\ &= \langle r\rangle - \frac{1}{2T}\int_0^T\Big(s - \frac{T}{2}\Big)\langle\{q\circ\exp(sH_p), q+\langle q\rangle\}\rangle\, ds.\end{aligned}$$

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Here we notice that

$$\langle \{ q \circ \exp(sH_p), \langle q \rangle \} \rangle = \{ \langle q \circ \exp(sH_p), \langle q \rangle \} = \{ \langle q \rangle, \langle q \rangle \} = 0,$$

so finally:

$$\langle s \rangle = \langle r \rangle - \frac{1}{2T} \int_0^T \left(s - \frac{T}{2} \right) \langle \{ q \circ \exp(sH_p), q \} \rangle \, ds. \tag{8.25}$$

The formulas (8.24) and (8.25) will be used in Section 13 together with the following remark. If we put

$$\operatorname{Cor}(q_1, q_2; s) = \langle \{ q_1 \circ \exp(sH_p), q_2 \} \rangle, \tag{8.26}$$

then a simple computation shows that

$$\operatorname{Cor}(q_1, q_2; s) = - \operatorname{Cor}(q_2, q_1; -s). \tag{8.27}$$

If we put

$$C(q_1, q_2) = \frac{1}{T} \int_0^T \left(s - \frac{T}{2} \right) \operatorname{Cor}(q_1, q_2; s) \, ds, \tag{8.28}$$

then combining (8.27) and the T periodicity of $\operatorname{Cor}(q_1, q_2; s)$ with the change of variables $T/2 - s = \widetilde{s} - T/2$, we get $C(q_1, q_2) = C(q_2, q_1)$.

9 Skeleton for $|\mu| \le \mathcal{O}(h)$

In this section we shall consider the case $|\mu| \le \mathcal{O}(1)h$. In doing so, we will use (4.8) more directly.

Case 1: We will first work in a region

$$\{ \mu \in \mathbf{C} \,;\, |\mu| < rh \} \cup \left\{ \mu \in \mathbf{C} \setminus \{0\} \,;\, |\arg \mu - \tfrac{\pi}{2}| < \pi - 1/C \right\},$$

where $0 < r < 1/2$, $C > 0$. (The corresponding region with $|\arg \mu + \frac{\pi}{2}| \le \pi - 1/C$, can be treated with a symmetry argument as in the end of Section 8, and this argument will be given later.) It follows from (4.8) that here $a_{2,3} \neq 0$ and $\ln \Gamma(\frac{1}{2} - i\frac{\mu}{h})$ is well defined, while $\Gamma(\frac{1}{2} + i\frac{\mu}{h})^{-1}$ may have zeros. Consequently we use the reflection identity to get

$$a_{1,4} = \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2} + i\frac{\mu}{h})} h^{-i\frac{\mu}{h}} e^{\frac{\pi\mu}{2h} - \frac{i\pi}{4}} = \frac{\Gamma(\frac{1}{2} - i\frac{\mu}{h})}{\sqrt{2\pi}} h^{-i\frac{\mu}{h}} e^{\frac{\pi\mu}{2h} - \frac{i\pi}{4}} 2\cosh\left(\frac{\pi\mu}{h}\right).$$

Now using (8.1) we get

$$\begin{aligned} F(\mu; h) &= e^{-i\frac{\mu}{h}\ln\frac{1}{h} - \ln\frac{\Gamma(\frac{1}{2} - i\frac{\mu}{h})}{\sqrt{2\pi}} + \frac{\pi\mu}{2h} + \frac{i\pi}{4} + \frac{i}{h}(S_{1,2} + S_{3,4})} + e^{\frac{i}{h}S_{1,2} + \pi\frac{\mu}{h}} + e^{\frac{i}{h}S_{3,4} + \pi\frac{\mu}{h}} \\ &\quad + e^{\ln\frac{\Gamma(\frac{1}{2} - i\frac{\mu}{h})}{\sqrt{2\pi}} + i\frac{\mu}{h}\ln\frac{1}{h} - \frac{i\pi}{4} + \frac{\pi\mu}{2h} + \ln 2\cosh\frac{\pi\mu}{h}} \\ &= e^{\frac{\pi\mu}{2h}} G(\mu; h), \end{aligned}$$

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where

$$G(\mu;h)=e^{\frac{i}{h}(S_{1,2}+S_{3,4})-i\frac{\mu}{h}\ln\frac{i}{h}-\ln\frac{\Gamma(\frac{1}{2}-i\frac{\mu}{h})}{\sqrt{2\pi}}+\frac{i\pi}{4}}+e^{\frac{i}{h}S_{1,2}+\pi\frac{\mu}{2h}}+e^{\frac{i}{h}S_{3,4}+\pi\frac{\mu}{2h}} \tag{9.1}$$
$$+e^{i\frac{\mu}{h}\ln\frac{1}{h}+\ln\frac{\Gamma(\frac{1}{2}-i\frac{\mu}{h})}{\sqrt{2\pi}}-\frac{i\pi}{4}}2\cosh\frac{\pi\mu}{h}.$$
$$=a_1+a_2+a_3+a_4,\qquad a_4=a_{4^+}+a_{4^-},$$

where the terms are the same as in (8.3), although we shall now use different asymptotic approximations.

Again we introduce h times the real parts of the different exponents:

$$r_1=-\operatorname{Im}S_{1,2}-\operatorname{Im}S_{3,4}+(\operatorname{Im}\mu)\ln\frac{1}{h}-h\operatorname{Re}\ln\frac{\Gamma(\frac{1}{2}-i\frac{\mu}{h})}{\sqrt{2\pi}},$$
$$r_2=-\operatorname{Im}S_{1,2}+\frac{\pi}{2}\operatorname{Re}\mu,$$
$$r_3=-\operatorname{Im}S_{3,4}+\frac{\pi}{2}\operatorname{Re}\mu,$$
$$r_{4\pm}=-(\operatorname{Im}\mu)\ln\frac{1}{h}+h\operatorname{Re}\ln\frac{\Gamma(\frac{1}{2}-i\frac{\mu}{h})}{\sqrt{2\pi}}\pm\pi\operatorname{Re}\mu.$$

As before, we have $r_2+r_3=r_1+r_{4^+}$.

Again, we define the different curves $\Gamma_{j,k}$ by $|a_j|=|a_k|$, for $j\neq k\in\{1,2,3,4^{\pm}\}$ with the exception of $(j,k)=(2,3)$ and $(j,k)=(4^+,4^-)$. (The segment Γ_4 is now defined to be the segment of the positive imaginary axis joining 0 to the lower part of S', provided that this lower part is not hidden in the forbidden region, in which case we let Γ_4 be empty). More explicitly, we get:

$$(\operatorname{Im}\mu)\ln\frac{1}{h}=\begin{cases}\operatorname{Im}S_{3,4}\pm\pi\operatorname{Re}\mu-\frac{\pi}{2}\operatorname{Re}\mu+X\text{ on }\Gamma_{3,4^\pm}\\ \operatorname{Im}S_{1,2}\pm\pi\operatorname{Re}\mu-\frac{\pi}{2}\operatorname{Re}\mu+X\text{ on }\Gamma_{2,4^\pm}\\ \operatorname{Im}S_{1,2}+\frac{\pi}{2}\operatorname{Re}\mu+X\text{ on }\Gamma_{1,3},\\ \operatorname{Im}S_{3,4}+\frac{\pi}{2}\operatorname{Re}\mu+X\text{ on }\Gamma_{1,2},\\ \frac{1}{2}(\operatorname{Im}S_{1,2}+\operatorname{Im}S_{3,4})\pm\frac{\pi\mu}{2}+X,\text{ on }\Gamma_{1,4^\pm},\end{cases} \tag{9.2}$$

with

$$X=h\operatorname{Re}\ln\Big(\frac{\Gamma(\frac{1}{2}-i\frac{\mu}{h})}{\sqrt{2\pi}}\Big). \tag{9.3}$$

The function X now differs from that of Section 8 by a term $-\frac{\pi}{2}\operatorname{Re}\mu$. The definition of $\Gamma_{1,3}=\Gamma_{2,4^+}$, $\Gamma_{1,2}=\Gamma_{3,4^+}$ coincides with that in Section 8 in the overlap region.

We shall also define a set $\Gamma_{j,4}$ for $j=1,2,3$ as in the preceding section. To do so, we check that

$$h\Big|\frac{\nabla\cosh\frac{\pi\mu}{h}}{\cosh\frac{\pi\mu}{h}}\Big|\ll\ln\frac{1}{h},$$

if

$$\operatorname{dist}\Big(\mu, \frac{h}{\pi}\cosh^{-1}(0)\Big) \gg \frac{h}{\ln\frac{1}{h}}. \tag{9.4}$$

In this region, we also have

$$\frac{1}{\mathcal{O}(1)\ln\frac{1}{h}} \le \Big|\frac{a_4}{a_{4\pm}}\Big| \le \mathcal{O}(1). \tag{9.5}$$

In the region (9.4) we can define $\Gamma_{j,4}$ by $|a_j| = |a_4|$, and see that we get a curve

$$\operatorname{Im}\mu = \gamma_{j,4}(\operatorname{Re}\mu),$$

with $|\gamma'_{j,4}| \ll 1$. Using (9.5), we also see that if we represent $\Gamma_{j,4\pm}$ by $\operatorname{Im}\mu = \gamma_{j,4\pm}(\operatorname{Re}\mu)$, then

$$(\gamma_{j,4} - \gamma_{j,4\pm})(\operatorname{Re}\mu) = \mathcal{O}(1)h\Big(\frac{\ln\ln}{\ln}\Big)\Big(\frac{1}{h}\Big).$$

Actually the upper bound can here be improved to $\mathcal{O}(h)/\ln\frac{1}{h}$. In analogy with Section 8, we define a diamond shaped neighborhood of each zero μ_j of a_4 by

$$D_j = \Big\{\mu\,;\,|\operatorname{Re}\mu| + |\operatorname{Im}\mu - \operatorname{Im}\mu_j| \le \frac{Ch}{\ln\frac{1}{h}}\Big\}, \tag{9.6}$$

The previously defined $\Gamma_{j,4}$ can hit at most one of the D_ν and if that happens, we add that diamond to the set $\Gamma_{j,4}$.

Now define the skeleton as before: $S = S' \cup \Gamma_4$, and as before we can describe the regions of dominance: a_4 dominates at distance $\gg h(\frac{\ln\ln}{\ln})(\frac{1}{h})$ below $\inf_{j=1,2,3}\gamma_{j,4^\pm}$, for $\pm\operatorname{Re}\mu \ge 0$, intersected with the complement of the union of the diamonds.

The other a_j dominate according to the earlier rules in their respective regions at a distance $\gg h/\ln\frac{1}{h}$ from the skeleton.

Case 2: We now consider the case when μ belongs to the set

$$\{\mu \in \mathbf{C};\ |\mu| < rh\} \cup \Big\{\mu \in \mathbf{C}\setminus\{0\};\ \Big|\arg\mu + \frac{\pi}{2}\Big| < \pi - \frac{1}{C}\Big\},$$

where $0 < r < \frac{1}{2}, C > 0$. From (4.8), we get

$$a_{1,3} = -a_{2,4} = e^{\frac{\pi\mu}{h} + i\frac{\pi}{2}}, \quad a_{1,4} = \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2} + i\frac{\mu}{h})} h^{-i\frac{\mu}{h}} e^{\frac{\pi\mu}{2h} - \frac{i\pi}{4}},$$

where $a_{1,4}$ is non-vanishing, while

$$a_{2,3} = \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2} - i\frac{\mu}{h})} h^{i\frac{\mu}{h}} e^{\frac{\pi\mu}{2h} + \frac{i\pi}{4}}$$

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may have zeros, so we use the reflection identity to write

$$a_{2,3} = \frac{\Gamma(\frac{1}{2} + i\frac{\mu}{h})}{\sqrt{2\pi}} h^{i\frac{\mu}{h}} e^{\frac{\pi\mu}{2h} + \frac{i\pi}{4}} 2\cosh\frac{\pi\mu}{h}.$$

We then use (8.1) to get $F(\mu; h) = e^{\frac{\pi\mu}{2h}} G(\mu; h)$, with $G(\mu; h) = a_1 + a_2 + a_3 + a_4$ where

$$a_1 = a_{1^+} + a_{1-},$$

$$a_{1\pm} = \exp\Big[\frac{i}{h}(S_{1,2} + S_{3,4}) + \ln\Big(\frac{\Gamma(\frac{1}{2} + i\frac{\mu}{h})}{\sqrt{2\pi}}\Big) + i\frac{\mu}{h}\ln h + \frac{i\pi}{4} \pm \frac{\pi\mu}{h}\Big],$$

$$a_2 = \exp\Big[\frac{i}{h}S_{1,2} + \frac{\pi\mu}{2h}\Big], \quad a_3 = \exp\Big[\frac{i}{h}S_{3,4} + \frac{\pi\mu}{2h}\Big],$$

$$a_4 = \exp\Big[-\ln\Big(\frac{\Gamma(\frac{1}{2} + i\frac{\mu}{h})}{\sqrt{2\pi}}\Big) - i\frac{\mu}{h}\ln h - \frac{i\pi}{4}\Big].$$

Again, we introduce h times the real parts of the different exponents:

$$r_{1\pm} = -\operatorname{Im} S_{1,2} - \operatorname{Im} S_{3,4} + (\operatorname{Im}\mu)\ln\frac{1}{h} + h\operatorname{Re}\ln\Big(\frac{\Gamma(\frac{1}{2} + i\frac{\mu}{h})}{\sqrt{2\pi}}\Big) \pm \pi\operatorname{Re}\mu,$$

$$r_2 = -\operatorname{Im} S_{1,2} + \frac{\pi}{2}\operatorname{Re}\mu, \quad r_3 = -\operatorname{Im} S_{3,4} + \frac{\pi}{2}\operatorname{Re}\mu,$$

$$r_4 = -(\operatorname{Im}\mu)\ln\frac{1}{h} - h\operatorname{Re}\ln\Big(\frac{\Gamma(\frac{1}{2} + i\frac{\mu}{h})}{\sqrt{2\pi}}\Big).$$

We shall now make the same symmetry transformations as in Section 8, to see that the functions $r_{1\pm}$, r_2, r_3, and r_4 play the same role as $r_{4\pm}$, r_3, r_2, and r_1 respectively, in the previously considered case:

- Add $\operatorname{Im}(S_{1,2} + S_{3,4})$ to each of the r_j.
- Consider the r_j as functions of $\overline{\mu}$.

Then we get

$$\widetilde{r}_{1\pm}(\overline{\mu}) = -(\operatorname{Im}\overline{\mu})\ln\frac{1}{h} + h\operatorname{Re}\ln\Big(\frac{\Gamma(\frac{1}{2} - i\frac{\overline{\mu}}{h})}{\sqrt{2\pi}}\Big) \pm \pi\operatorname{Re}\overline{\mu},$$

$$\widetilde{r}_2(\overline{\mu}) = \operatorname{Im} S_{3,4} + \frac{\pi}{2}\operatorname{Re}\overline{\mu}, \quad \widetilde{r}_3 = \operatorname{Im} S_{1,2} + \frac{\pi}{2}\operatorname{Re}\overline{\mu},$$

$$\widetilde{r}_4 = \operatorname{Im} S_{1,2} + \operatorname{Im} S_{3,4} + (\operatorname{Im}\overline{\mu})\ln\frac{1}{h} - h\operatorname{Re}\ln\Big(\frac{\Gamma(\frac{1}{2} - i\frac{\overline{\mu}}{h})}{\sqrt{2\pi}}\Big).$$

Thus apart from a change of sign in $\operatorname{Im} S_{1,2}$, $\operatorname{Im} S_{3,4}$, we see that $(\widetilde{r}_{1\pm}, \widetilde{r}_2, \widetilde{r}_3, \widetilde{r}_4)$ has the same properties as $(r_{4\pm}, r_3, r_2, r_1)$ in the previously considered case.

Remark In the overlap region

$$D(0, rh) \cup \Big\{\mu\,;\, |\mu| \le Ch,\ |\arg\mu| \le \frac{\pi}{2} - \frac{1}{C},\ \text{or } |\arg(-\mu)| \le \frac{\pi}{2} - \frac{1}{C}\Big\},$$

where both cases apply, we notice that trivially $r_j = r_{j\pm} + \mathcal{O}(h)$ for $\pm \operatorname{Re} \mu \geq 0$, $j = 1, 4$, (and these estimates improve by (8.19) when $|\mu|/h$ increases). As in the remark at the end of Section 8,the distance between the two skeletons, defined according to Case 1 and Case 2, is therefore $\mathcal{O}(h/\ln \frac{1}{h})$.

10 Eigenvalue Counting

In each of Cases 1 and 2 of Sections 8 and 9, we defined a skeleton S consisting of a horizontal part S', possibly with a vertical part (Γ_4 in Case 1 and Γ_1 in Case 2) added. We notice that the definitions in the two sections agree for each of Cases 1 and 2 in the overlap regions for the two sections, and we saw in the remarks at the end of the sections, that if we compare the skeletons for the two cases in the overlap region

$$\Big\{ |\operatorname{Re} \mu| > \frac{1}{C} |\operatorname{Im} \mu| \Big\} \cup D(0, rh),$$

then the distance between the corresponding skeletons is

$$\mathcal{O}\Big(\frac{h}{\ln \langle \mu \rangle_h} e^{-2\pi |\operatorname{Re} \mu|/h} \Big), \quad \langle \mu \rangle_h := \sqrt{h^2 + |\mu|^2}.$$

Now define the body by widening the skeleton:

$$B = \Big(\bigcup_{\mu \in S'} D\Big(\mu, \frac{Ch}{\ln \frac{1}{\langle \mu \rangle_h}} \Big) \Big) \cup B_v \cup B_e.$$

Here B_v, B_e may be empty and will now be defined. They are non-empty if S' stays entirely in the admissible regions for one of the cases, and in order to fix the ideas, we assume that this is Case 1, and S' does not intersect the negative imaginary half axis. If so, we have a non-empty segment Γ_4 in the imaginary axis, joining 0 to the closest imaginary point of S'. Recall that we have defined the diamonds D_j around the zeros of a_4 in Γ_4, by (8.15), and (9.6). We define B_v to be the union of Γ_4 (in Case 1, and Γ_1 in Case 2) and the corresponding diamonds. (In Case 2 we do the corresponding definition with 4 replaced by 1 and Γ_1 is then the segment in the negative imaginary axis, joining 0 to the closest part of S'.) Thus B_e is non-empty precisely when Γ_4 or Γ_1 is. In Case 1, it is defined to be the set of points μ below S' at distance at most

$$Ch \frac{\ln \ln}{\ln} \Big(\frac{1}{\langle \mu \rangle_h} \Big)$$

from S' with $C > 0$ sufficiently large and with $|\operatorname{Re} \mu| < h$. Here the upper bound h in the last estimate may be replaced by h/C_0 for any fixed $C_0 > 0$, and we could decrease B_e further by a more detailed discussion. In Case 2 we have the analogous definition.

We next define what we mean by an *admissible curve*. It should be a piecewise C^1-curve $\gamma\colon [a, b] \to \mathbf{C}$ without self-intersections, parametrized by arc-length. It is tacitly assumed that we consider a family of such curves, which is uniformly bounded in the sense that we have uniform bounds on the number of jump discontinuities of $\dot{\gamma}$, the continuity of $\dot{\gamma}$ between the discontinuities, and on the length $b-a$. It is also required that $\gamma(t)$ may belong to B only for $t \in I_j$, $j = 1, 2, \ldots, M$, where I_j are disjoint intervals of length $\leq Ch/\ln \frac{1}{\langle\mu\rangle_h}$, for some $\mu = \gamma(t) \in I_j$, if $\gamma(I_j) \cap B_e = \varnothing$ and of length $\leq Ch\frac{\ln\ln}{\ln}(\frac{1}{\langle\mu\rangle_h})$ otherwise. We also assume that we have a uniform bound on the number M of such intervals.

Assume for simplicity that $a, b \notin \bigcup I_j$ and let us partition $[a,b]$ into intervals in increasing order: $[a,b] = J_0 \cup I_1 \cup J_1 \cup I_2 \cup \cdots \cup I_M \cup J_M$. For each J_k, let $a_{\nu(k)}$ be the corresponding dominant term along $\gamma(J_k)$. For simplicity we shall assume that the image of γ is entirely contained in the admissible region for one of Cases 1 or 2, so that $\nu(k)$ is either in $\{1, 2, 3, 4^{\pm}\}$ (Case 1), or in $\{1^{\pm}, 2, 3, 4\}$ (Case 2). Let

$$\mu_{k,e} = \mu_{k+1,s} = \gamma(t_{k+1}) \quad \text{for some } t_{k+1} \in I_{k+1},\ k = 0, \ldots, M-1,$$

and put $\mu_{0,s} = \gamma(a)$, $\mu_{M,e} = \gamma(b)$. Then with $a_j = e^{i\phi_j/h}$, we have the following theorem (see (8.3), (8.17), (9.1)).

Theorem 10.1 *Let γ be an admissible curve as above. Then*

(10.1)

$$\operatorname{Re} \frac{1}{2\pi i} \int_\gamma \frac{G'}{G} d\mu =$$

$$\operatorname{Re} \frac{1}{2\pi h}\Big(\Big(\phi_{\nu(M)}(\mu_{M,e}) + \sum_{k=0}^{M-1} (\phi_{\nu(k)}(\mu_{k,e}) - \phi_{\nu(k+1)}(\mu_{k+1,s})) - \phi_0(\mu_{0,s})\Big) + \mathcal{O}(1) + \mathcal{O}\Big(\max \ln\ln \frac{1}{\langle\mu_{k,e}\rangle_h}\Big)\Big),$$

where the maximum is taken over all k with $\gamma(I_k) \cap B_e \neq \varnothing$, so if γ never meets B_e, we only have the remainder $\mathcal{O}(1)$.

Proof Notice that the first term of the right-hand side of (10.1) can also be written

$$\operatorname{Re} \frac{1}{2\pi h} \sum_{k=0}^{M} \big(\phi_{\nu(k)}(\mu_{k,e}) - \phi_{\nu(k)}(\mu_{k,s})\big).$$

Consider an interval J_k. If we first assume $\nu(k) \neq 4^{\pm}$ (if we are in Case 1), then for $t \in J_k$:

$$G(\gamma(t)) = b_{\nu(k)}(\gamma(t)) a_{\nu(k)}(\gamma(t)),\ |b_{\nu(k)}(\gamma(t)) - 1| < \tfrac{1}{2}. \tag{10.2}$$

Let $\widetilde{\mu}_{k,s}$ and $\widetilde{\mu}_{k,e}$ be the start and the end points of $\gamma_{|J_k}$, so that

$$\widetilde{\mu}_{k,s} = \mu_{k,s} + \mathcal{O}\Big(h\frac{\ln\ln}{\ln}\Big(\frac{1}{\langle\mu_{k,s}\rangle_h}\Big)\Big), \quad \widetilde{\mu}_{k,e} = \mu_{k,e} + \mathcal{O}\Big(h\frac{\ln\ln}{\ln}\Big(\frac{1}{\langle\mu_{k,e}\rangle_h}\Big)\Big),$$

with the $\frac{\ln\ln}{\ln}$ improving to $\frac{1}{\ln}$ if the corresponding neighboring interval I_k does not meet B_e. Using (10.2), we see that

$$\frac{1}{2\pi i}\int_{\gamma_{|J_k}}\frac{G'}{G}\,d\mu = \mathcal{O}(1)+\phi_{\nu(k)}(\widetilde{\mu}_{k,e})-\phi_{\nu(k)}(\widetilde{\mu}_{k,s}). \tag{10.3}$$

In the case $\nu(k)=4^{\pm}$, we know that a_4 is dominating along $\gamma_{|J_k}$ and (10.3) holds with $\phi_{\nu(k)}$ replaced by $\frac{h}{i}\ln a_4$. Now we also know that along $\gamma_{|J_k}$, we have

$$a_4(\gamma(t)) = c(\gamma(t))a_{\nu(k)}(\gamma(t)),$$

with

$$1\Big/\Big(C\ln\frac{1}{\langle\mu\rangle_h}\Big)\leq|c(\gamma(t))|\leq C,$$

and with $\arg c(\gamma(t))$ of bounded variation. It follows that the real part of the equation (10.3) still holds in this case.

We next estimate the integral along $\gamma|_{I_k}$, and let us consider the worst case, when $\gamma(I_k)\cap B_e\neq\varnothing$. Let

$$r=\mathcal{O}\Big(h\frac{\ln\ln}{\ln}\Big(\frac{1}{\langle\mu\rangle_h}\Big)\Big)$$

be such that $\gamma(I_k)\subset D\big(\widetilde{\mu}_{k-1,e},\frac{r}{2}\big)$. On this disc, we write

$$G(\mu;h)=a_{\nu(k-1)}(\mu)b_k(\mu), \tag{10.4}$$

where b_k is holomorphic, and

$$C\geq|b_k(\widetilde{\mu}_{k-1,e})|\geq 1\Big/\Big(C\ln\frac{1}{\langle\widetilde{\mu}_{k-1,e}\rangle_h}\Big), \tag{10.5}$$

$$|b_k(\mu)\leq\exp\mathcal{O}(1)\Big[\frac{\ln\frac{1}{\langle\mu\rangle_h}}{h}h\frac{\ln\ln}{\ln}\Big(\frac{1}{\langle\mu\rangle_h}\Big)\Big]=\exp\mathcal{O}\Big(\ln\ln\frac{1}{\langle\mu\rangle_h}\Big).$$

Using (10.5) and the elementary arguments recalled in the second part of Section 7, we get

$$\frac{1}{2\pi i}\int_{\gamma_{|I_k}}\frac{b_k'}{b_k}d\mu=\mathcal{O}(1)\ln\ln\frac{1}{\langle\mu\rangle_h}. \tag{10.6}$$

On the other hand,

$$\frac{1}{2\pi i}\int_{\gamma|_{I_k}}\frac{a'_{\nu(k-1)}}{a_{\nu(k-1)}}\,d\mu=\frac{1}{2\pi h}(\phi_{\nu(k-1)}(\widetilde{\mu}_{k,s})-\phi_{\nu(k-1)}(\widetilde{\mu}_{k-1,e})). \tag{10.7}$$

Combining the real parts of (10.3), (10.4), (10.6), and (10.7), we get

$$\operatorname{Re}\frac{1}{2\pi i}\int_{\gamma}\frac{G'}{G}d\mu = \operatorname{Re}\Big(\phi_{\nu(M)}(\widetilde{\mu}_{M,e}) + \sum_{k=0}^{M-1}\big(\phi_{\nu(k)}(\widetilde{\mu}_{k+1,s}) - \phi_{\nu(k+1)}(\widetilde{\mu}_{k+1,s})\big) - \phi_{\nu(0)}(\widetilde{\mu}_{0,s})\Big) + \mathcal{O}(1)\ln\ln\frac{1}{\langle\mu\rangle_h},$$

with the remainder improving to $\mathcal{O}(1)$ if we do not encounter B_e. Now, $\widetilde{\mu}_{M,e} = \mu_{M,e}$, $\widetilde{\mu}_{0,s} = \mu_{0,s}$, and

$$\mu_{k,e} = \mu_{k+1,s} + \mathcal{O}\Big(h\frac{\ln\ln}{\ln}\Big(\frac{1}{\langle\mu\rangle_h}\Big)\Big),$$

with the last remainder improving to $\mathcal{O}\big(h/\ln\frac{1}{\langle\mu\rangle_h}\big)$, if we avoid B_e, and (10.1) follows. ∎

We end this section by some rough estimates on the location of the skeleton and the corresponding distribution of eigenvalues for the reduced operators constructed in Sections 2 and 3. Our starting point is the reduced symbol in (2.5) and the corresponding 1-dimensional symbol

$$(10.8)\quad Q\Big(\tau, x, \xi, \epsilon, \frac{h^2}{\epsilon}; h\Big) = \langle q\rangle(\tau, x, \xi) + \mathcal{O}(\epsilon) + \frac{h^2}{\epsilon}p_2(\tau, x, \xi) + h\widetilde{p}_1 + h^2\widetilde{p}_2 + \cdots.$$

Here we shall take τ real (and eventually of the form $h\big(k - \frac{k_0}{4}\big) - \frac{S_0}{2\pi}$). If z is the original spectral parameter, we introduce the new spectral parameter w, by

$$z = g(\tau) + i\epsilon w,$$

and we will work under the assumption $h^2 \ll \epsilon \ll h^{1/2}$.

Recall the microlocal normal form for Q near the branch point, given by Proposition 6.2 and in particular (6.6):

$$U^{-1}QU = K_{\epsilon,h^2/\epsilon}(\tau, I; h) + \mathcal{O}(e^{-\frac{1}{Ch}}),\quad I = \tfrac{1}{2}(x\circ hD_x + hD_x\circ x),$$

where the leading symbol in $K_{\epsilon,h^2/\epsilon}$ is $k_{\epsilon,h^2/\epsilon}(\tau,\iota)$ with $\iota = x\xi$, given in Proposition 6.1. Correspondingly, we replace w by the new spectral parameter μ, given by

$$(10.9)\qquad K_{\epsilon,h^2/\epsilon}(\tau, \mu; h) = w.$$

We next estimate the location of the skeleton in the μ-plane, and start with the case $|\mu| \geq Ch$. Assume for simplicity that we are in Case 1: $\operatorname{Im}\mu \geq -C|\operatorname{Re}\mu|$. We will only be concerned with the horizontal part S' of the skeleton. When $\operatorname{Re}\mu \geq 0$, it is given by the curves $\Gamma_{3,4^+} = \Gamma_{1,2}$, $\Gamma_{2,4^+} = \Gamma_{1,3}$ in (8.5), where

$$X(\mu) = Y(\mu) + \frac{\pi}{2}\operatorname{Re}\mu,\quad Y(\mu) = (\operatorname{Re}\mu)\arg\Big(\frac{\mu}{i}\Big) - \operatorname{Im}\mu + h\operatorname{Re}\mathcal{O}_-\Big(\frac{h}{\mu}\Big).$$

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Clearly $X(\mu)$ is uniformly Lipschitz continuous and for $\mu > 0$, we get

$$X(\mu) = h \operatorname{Re} \mathcal{O}_-\Big(\frac{h}{\mu}\Big).$$

According to (5.5), we have $\operatorname{Re} \mathcal{O}_-\big(\frac{h}{\mu}\big) = \mathcal{O}(e^{-2|\mu|/h})$.

When $\epsilon = 0$, $h^2/\epsilon = 0$, we know that the leading part of Q in (10.8) is real-valued (assuming that $\langle q \rangle$ is real for simplicity), so it follows in this case that when μ is real, then $\operatorname{Im} S_{j,k} = \mathcal{O}(h)$. Since $S_{j,k}$ depends holomorphically on μ, we conclude that in general $\operatorname{Im} S_{j,k}(\mu) = \mathcal{O}(\epsilon + h^2/\epsilon)$, $\mu \in \mathbf{R}$. Now combine this with (8.5), the estimate $X(\mu) = \mathcal{O}(he^{-2\pi|\mu|/h})$ and Proposition B.1 to conclude that in the region $|\mu| \geq Ch$, $\operatorname{Re}\mu \geq 0$, the horizontal part S' of the spectrum is given by the union of two curves of the form $\operatorname{Im}\mu = f(\operatorname{Re}\mu)$, with f' satisfying (B.17), and further,

$$|f(x)| \leq C\Big(\epsilon + \frac{h^2}{\epsilon}\Big) \max\Big(\frac{1}{\ln\frac{1}{|x|}}, \frac{1}{\ln\frac{1}{(\epsilon+h^2/\epsilon)}}\Big). \tag{10.10}$$

In the left half-plane, we recall that S' has a more complicated structure. Assume, to fix the ideas, that $\operatorname{Re}\mu_A \leq 0$. Then S' is the union of the curves (defined in (8.6)):

$$\Gamma_{1,3}: \quad \operatorname{Im}\mu = \gamma_{1,3}(\operatorname{Re}\mu) \Leftrightarrow (\operatorname{Im}\mu)\ln\frac{1}{|\mu|} = \operatorname{Im} S_{1,2} + (X - \pi\operatorname{Re}\mu) + \pi\operatorname{Re}\mu$$

$$\Gamma_{3,4-}: \quad \operatorname{Im}\mu = \gamma_{3,4-}(\operatorname{Re}\mu) \Leftrightarrow (\operatorname{Im}\mu)\ln\frac{1}{|\mu|} = \operatorname{Im} S_{3,4} + (X - \pi\operatorname{Re}\mu) - \pi\operatorname{Re}\mu,$$

in the region $\operatorname{Re}\mu_A \leq \operatorname{Re}\mu \leq 0$. Here $\gamma_{3,4-}(\operatorname{Re}\mu) \leq \gamma_{1,3}(\operatorname{Re}\mu)$ and the two curves cross at μ_A. In the region $\operatorname{Re}\mu \leq \operatorname{Re}\mu_A$ S' is given by

$$\Gamma_{1,4-}: \quad \operatorname{Im}\mu = \gamma_{1,4-}(\operatorname{Re}\mu) \Leftrightarrow (\operatorname{Im}\mu)\ln\frac{1}{|\mu|} = \frac{1}{2}(\operatorname{Im} S_{1,2} + \operatorname{Im} S_{3,4}) + (X - \pi\operatorname{Re}\mu),$$

and this curve also contains μ_A.

When $\mu < 0$, we have

$$X - \pi\operatorname{Re}\mu = h\operatorname{Re}\mathcal{O}_-\Big(\frac{h}{\mu}\Big) = h\mathcal{O}(e^{-2\pi|\mu|/h}),$$

so again $f := \gamma_{1,4-}$ satisfies (10.10), while

$$\begin{aligned}\widetilde{\gamma}_{3,4-}(\operatorname{Re}\mu) := \operatorname{Im} S_{3,4} + (X - \pi\operatorname{Re}\mu) &\leq \gamma_{3,4-}(\operatorname{Re}\mu) \leq \gamma_{1,3}(\operatorname{Re}\mu)\\ &\leq \operatorname{Im} S_{1,2} + (X - \pi\operatorname{Re}\mu) =: \widetilde{\gamma}_{1,3}(\operatorname{Re}\mu),\end{aligned}$$

for $\operatorname{Re}\mu_A \leq \operatorname{Re}\mu \leq 0$, where $f = \widetilde{\gamma}_{1,3}, \widetilde{\gamma}_{3,4-}$ satisfy (10.10).

In the region $|\mu| \leq Ch$ the horizontal part of the spectrum is a union of curves $\Gamma_{j,k}$ given in (9.2) and (9.3). Here the new function X is uniformly Lipschitz and $\mathcal{O}(h)$, so the skeleton is here contained in a region

$$|\operatorname{Im}\mu| \leq \mathcal{O}(1)\frac{(\epsilon + h^2/\epsilon)}{\ln\frac{1}{h}}.$$

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The *overall conclusion* is that the skeleton is contained in a region

$$(10.11)\qquad |\operatorname{Im}\mu| \le \mathcal{O}(1)\Big(\epsilon + \frac{h^2}{\epsilon}\Big) \max\Big(\frac{1}{\ln\frac{1}{\langle \operatorname{Re}\mu\rangle_h}}, \frac{1}{\ln\frac{1}{\epsilon + h^2/\epsilon}}\Big),$$

where we recall that $\langle \operatorname{Re}\mu\rangle_h = (h^2 + (\operatorname{Re}\mu)^2)^{1/2}$.

We end this section by establishing a simplified statement to be used in Theorem 1.1. We shall simply remove a small rectangle around $\mu = 0$ where we have seen that the description of the spectrum is more intricate.

Start by recalling the definition of μ_A, μ_B prior to (8.14). For instance μ_A is the intersection of the curves

$$A: \quad -2\pi \operatorname{Re}\mu = \operatorname{Im} S_{1,2} - \operatorname{Im} S_{3,4},$$
$$\Gamma_{1,4-}: \quad \operatorname{Im}\mu = \gamma_{1,4-}(\operatorname{Re}\mu),$$

where $f = \gamma_{1,4-}$ satisfies (10.10). Using that $\operatorname{Im} S_{1,2}$ and $\operatorname{Im} S_{3,4}$ are $\mathcal{O}(\epsilon + h^2/\epsilon + |\operatorname{Im}\mu|)$, we get

$$\operatorname{Re}\mu_A = \mathcal{O}\Big(\epsilon + \frac{h^2}{\epsilon} + |\operatorname{Im}\mu_A|\Big),$$
$$\operatorname{Im}\mu_A = \mathcal{O}\Big(\epsilon + \frac{h^2}{\epsilon}\Big) \max\Big(\frac{1}{|\ln|\operatorname{Re}\mu_A||}, \frac{1}{|\ln(\epsilon + \frac{h^2}{\epsilon})|}\Big),$$

implying,

$$\operatorname{Re}\mu_A = \mathcal{O}\Big(\epsilon + \frac{h^2}{\epsilon}\Big), \quad \operatorname{Im}\mu_A = \frac{\mathcal{O}(\epsilon + \frac{h^2}{\epsilon})}{|\ln(\epsilon + \frac{h^2}{\epsilon})|}.$$

We have of course the same estimates for μ_B.

Choose $C > 0$ sufficiently large so that the "black box"

$$\mathcal{B} = [-a, a] + i[-b, b], \quad \text{with } a = C\Big(\epsilon + \frac{h^2}{\epsilon}\Big),\ b = C\frac{(\epsilon + \frac{h^2}{\epsilon})}{|\ln(\epsilon + \frac{h^2}{\epsilon})|},$$

contains μ_A, μ_B. Then we have the following.

Proposition 10.2 *The number of eigenvalues in $\mathcal{B}$ is $\mathcal{O}(\frac{\epsilon}{h} + \frac{h}{\epsilon})|\ln(\epsilon + \frac{h^2}{\epsilon})|$. The eigenvalues outside $\mathcal{B}$ are exponentially close to $\Gamma_{1,4-} \cup \Gamma_{1,2} \cup \Gamma_{1,3}$. More precisely introduce*

$$E_{1,4-} = \{\mu \in \Gamma_{1,4-} \setminus \mathcal{B}\,;\, a_1 + a_{4-} = 0,\ \operatorname{Re}\mu < 0\},$$
$$E_{1,2} = \{\mu \in \Gamma_{1,2} \setminus \mathcal{B}\,;\, a_1 + a_2 = 0,\ \operatorname{Re}\mu > 0\},$$
$$E_{1,3} = \{\mu \in \Gamma_{1,3} \setminus \mathcal{B}\,;\, a_1 + a_3 = 0,\ \operatorname{Re}\mu > 0\}.$$

Then there is a bijection b (possibly after a slight modification of $\mathcal{B}$) between the set of eigenvalues outside $\mathcal{B}$ and $E_{1,4-} \cup E_{1,2} \cup E_{1,3}$, such that

$$b(\mu) - \mu = \mathcal{O}(e^{-\pi|\operatorname{Re}\mu|/h} h/|\ln|\mu||).$$

Proof We may first notice that we can replace the index 4^- by 4 without changing the validity of the statement of the proposition, since $a_4 - a_{4^-} = \mathcal{O}(e^{-2\pi|\operatorname{Re}\mu|/h})$, $\operatorname{Re}\mu \ll -h$. In view of (10.11), we know that there are no eigenvalues outside $\mathcal{B}$ with $|\operatorname{Re}\mu| \leq a$ and the discussion in Section 8 then shows that the eigenvalues outside $\mathcal{B}$ have to be exponentially close to $\Gamma_{1,4} \cup \Gamma_{1,2} \cup \Gamma_{1,3}$ and that there is a bijection b as stated. To estimate the number of eigenvalues inside $\mathcal{B}$, we simply apply Theorem 10.1. with γ a rectangular contour containing $\mathcal{B}$ but contained in $2\mathcal{B}$ and working directly with a_4 instead of $a_{4^\pm}$. ∎

Consider $E_{1,4^-}$ of the preceding proposition. In view of (8.3), it is given by the quantization condition

$$S_{1,2} + S_{3,4} + 2\mu(\ln(-\mu) - 1) + \frac{\pi h}{2} + 2hi\mathcal{O}_-\Big(\frac{h}{\mu}\Big) = 2\pi\Big(k + \frac{1}{2}\Big)h, \quad k \in \mathbf{Z}. \tag{10.12}$$

Here we recall from the beginning of Section 5, that the term $\mathcal{O}_-(\frac{h}{\mu})$ is $\sim C_1\frac{h}{\mu} + C_2(\frac{h}{\mu})^2 + \cdots$, as $\frac{h}{\mu} \to 0$. We also know that if $\alpha = (\epsilon, h^2/\epsilon)$ denote the small additional parameters in the problem, then $S_{j,k} \sim \sum_0^\infty S_{j,k}^\nu(\mu,\alpha)h^\nu$, for $(j,k) = (1,2),(3,4)$, where $S_{j,k}^\nu$ are smooth in α and analytic in τ. Hence the condition (10.12) takes the form $b_{1,4^-}(\mu,\alpha;h) = 2\pi(k+\frac{1}{2})h$, where

$$b_{1,4^-}(\mu,\alpha;h) \sim \sum_{\nu=0}^\infty b_{1,4^-}^\nu(\mu,\alpha)h^\nu, \tag{10.13}$$

in the space of bounded holomorphic functions defined in the truncated sector:

$$\operatorname{Re}\mu \leq -Ch, \quad |\operatorname{Im}\mu| \leq \frac{1}{C}(-\operatorname{Re}\mu),$$

with

$$\begin{array}{l} b_{1,4^-}^0(\mu,\alpha) - 2\mu\ln(-\mu),\ b_{1,4^-}^1 \text{ holomorphic} \\ \text{in a full neighborhood of } \mu = 0,\ \alpha = 0, \end{array} \tag{10.14}$$

and

$$b_{1,4^-}^\nu(\mu,\alpha) = \mathcal{O}(\mu^{1-\nu}),\ \nu \geq 2. \tag{10.15}$$

Notice that the singularity structure (10.13), (10.14), and (10.15) of $b_{1,4^-}$ is essentially unchanged if we replace μ by $\widetilde{\mu} = a(\mu,\alpha;h)(\mu + hd(\mu,\alpha;h))$, where a, d are classical symbols of order 0 in h with coefficients that are analytic near $\mu = 0, \alpha = 0$ and with a elliptic, $\operatorname{Re} a > 0$, $|\operatorname{Im} a| \ll \operatorname{Re} a$.

On the other hand, in the region $\operatorname{Re}\mu < -1/C$, $C \gg 0$, we know (and that was done for instance in [14, §4]), that the eigenvalues sit on a curve and are given by a Bohr–Sommerfeld condition

$$\widetilde{b}(\mu,\alpha;h) = 2\pi\Big(k + \frac{1}{2}\Big)h, \quad k \in \mathbf{Z},$$

where $\widetilde{b}$ is a classical analytic symbol of order 0: $\widetilde{b} \sim \sum_0^\infty \widetilde{b}^\nu(\mu,\alpha)h^\nu$, and where

$$\widetilde{b}^0(\mu,\alpha) = \int_{\gamma_{\mathrm{ext}}(\mu,\alpha)} \xi\, dx.$$

Here $\gamma_{\mathrm{ext}}(\mu,\alpha)$ denotes a closed loop in the energy surface $Q^0(\mu,\alpha,x,\xi) = w$, with w and μ related by (10.9), that can be obtained from the real energy curve we get by taking μ real and putting $\alpha = 0$. Clearly $\widetilde{b} = b_{1,4-}$, so our discussion gives detailed description about how one can push the standard WKB-construction to the limit $|\mu| \gg h$ in the region $\operatorname{Re}\mu < 0$.

The same discussion applies to $E_{1,2}$, $E_{1,3}$. We get the conditions

$$b_{1,2}(\mu,\alpha;h) = 2\pi kh \quad \text{and} \quad b_{1,3}(\mu,\alpha;h) = 2\pi kh,$$

respectively, where $b_{j,k}$, $(j,k) = (1,2),(1,3)$ are defined in the truncated sector $\operatorname{Re}\mu \geq Ch$, $|\operatorname{Im}\mu| \leq \operatorname{Re}\mu/C$, and $b^\nu_{j,k}$ have the analogous properties to those in (10.14), (10.15), for $\nu \geq 1$, while the first part of (10.14) should be replaced by the condition that

$$b^0_{j,k}(\mu,\alpha) - \mu\ln\mu \text{ is holomorphic near } \mu = 0,\ \alpha = 0.$$

Then $b^0_{1,2}$ is the action along a closed loop inside the appropriate complex energy curve, that can be obtained by deformation from the case $\mu > 0$, $\alpha = 0$ where we take the left real component, close to the left loop in the ∞-shaped set $K_{0,0}$. For $b^0_{3,4}$ we deform from the right real component.

11 Justification by Means of a Global Grushin Problem

11.1 One Dimensional Grushin Problems

We may assume here without loss of generality, that $\langle q\rangle$ is real-valued. Then we know that f in (6.7) satisfies $f(w;h) \sim \sum_0^\infty f_k(w;h)$, where f_0 is real-valued when $\epsilon, h^2/\epsilon = 0$, $\tau \in \mathbf{R}$. Recall that f and f_k depend analytically on τ and smoothly on $\epsilon, h^2/\epsilon$.

Also recall that the spectrum of Q is localized to the region (6.4):

$$|\operatorname{Im} w| = \mathcal{O}(h + \epsilon + h^2/\epsilon)$$

(as follows from the more refined estimate (10.11)), and in view of (6.8): $f(w;h) = -\mu$, it follows that μ is localized to a domain of the same type.

We shall now introduce three different Grushin problems for $Q - w$ in the spirit of [13, 31]. Let $\chi \in C_0^\infty((\operatorname{neigh}(0,0),\mathbf{R}^2))$ be equal to one near $(0,0)$. We realize χ as an h-pseudodifferential operator, that we also denote by χ, using a Gaussian resolution of the identity (see [17, §3] and [27]), so that our calculus errors will be exponentially small rather than just $\mathcal{O}(h^\infty)$. Assume, in order to fix the ideas, that the support of $\nabla\chi$ is a thin annulus around $(0,0)$, containing the points α_j, $j = 1,\dots,4$

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(see Section 6). Recall the definition of g_j in (6.10). Let g_j^* be the corresponding functions defined for $Q^* - \overline{w}$, depending anti-holomorphically on w. Define

$$R_+^j u = (-1)^j (u|\frac{i}{h}[Q^*, \chi]g_j^*)_{W_j}.$$

Here W_j is a small neighborhood of α_j and $(u|v)_{W_j} = (\chi_j u|v)$ where $\chi_j \in C_0^\infty(W_j)$ is equal to 1 near α_j, and we also let χ_j denote the corresponding Gaussian quantization. We may normalize the choice of g_j^* so that $R_+^j g_j = 1$. Expanding the commutator, we see that the definition of $R_+^j u$ (up to an exponentially small error) is independent of the choice of χ, provided that u is a microlocal null-solution of $Q - w$.

Assume for simplicity that χ is real-valued and that the corresponding quantization is selfadjoint. Put $R_-^j u_- = (-1)^j u_- i/h[Q, \chi]g_j$.

Our first Grushin problem is directly adapted to the derivation of the quantization condition (6.18) in Section 6. It is the form

$$(Q - w)u + R_- u_- = v, \quad R_+ u = v_+, \tag{11.1}$$

with $R_+ = R_+^4 \colon L^2_{\theta'} \to \mathbf{C}$, $R_- = R_-^1 \colon \mathbf{C} \to L^2_{\theta'}$. Using Section 6 we see as in [13] and [31, §5] that it is well posed for w in some fixed complex neighborhood of 0 with a solution of the form

$$u = Ev + E_+ v_+, \quad u_- = E_- v + E_{-+} v_+. \tag{11.2}$$

Here we get

$$E_{-+} v_+ = ihv_+ (c_{2,3} e^{2\pi i(\theta_1+\theta_2) + \frac{i}{h}(S_{3,4}+S_{1,2})} + c_{2,4} e^{2\pi i\theta_2 + \frac{i}{h} S_{1,2}} - c_{1,3} e^{2\pi i \theta_1 + \frac{i}{h} S_{3,4}} - c_{1,4}),$$

where the parenthesis is the same as in the quantization condition (6.18). As usual, we read off the approximate eigenvalues as the zeros of E_{-+}.

The drawback with this first Grushin problem is that the solution operator (11.2) will grow exponentially when $\mu > 0$. This can be seen either directly from the explicit formulae for $a_{j,k}$ and the slightly less explicit expression for $c_{j,k}$, or from the fact that for $\mu > 0$, we have approximately a double well problem and with R_+ we prescribe the solution u in (11.1) in one of the wells, and hence u will in general be exponentially large in the other well. Of course, we will have to accept some exponential growth with a rate $\mathcal{O}(|\operatorname{Im} w| + \epsilon + h^2/\epsilon)$ but certainly would like to avoid exponential growth with a fixed rate when μ is real.

It seems impossible to cover a full neighborhood of $w = 0$ with a single Grushin problem whose solution operator does not exhibit exponential growth in some region, so we shall use two Grushin problems where one will be nice roughly in the upper half-plane and the other in the lower half-plane.

The second Grushin problem is designed to cover the region $\operatorname{Im} \mu \geq 0$ with some margin. It is of the form (11.1) with

$$R_+ u = (R_+^2 u, R_+^4 u), \quad R_- u_- = R_-^1 u_-^1 + R_-^3 u_-^3, \tag{11.3}$$

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so that $R_+ = L^2_{\theta'} \to \mathbf{C}^2$, $R_- : \mathbf{C}^2 \to L^2_{\theta'}$. For the corresponding model problem for $P_0 - \mu$ in Section 4, we get (*cf.* (4.6) and (4.8)):

$$\begin{pmatrix} u_1 \\ u_3 \end{pmatrix} = U \begin{pmatrix} u_2 \\ u_4 \end{pmatrix}, \tag{11.4}$$

$$U = \begin{pmatrix} u_{1,2} & u_{1,4} \\ u_{3,2} & u_{3,4} \end{pmatrix} = \Gamma\Big(\frac{1}{2} - i\frac{\mu}{h}\Big) h^{-i\frac{\mu}{h}} \sqrt{2\pi} \begin{pmatrix} e^{\frac{\pi}{2}\frac{\mu}{h}+i\frac{\pi}{4}} & e^{-\frac{\pi}{2}\frac{\mu}{h}-i\frac{\pi}{4}} \\ e^{-\frac{\pi}{2}\frac{\mu}{h}-i\frac{\pi}{4}} & e^{\frac{\pi}{2}\frac{\mu}{h}+i\pi 4} \end{pmatrix}. \tag{11.5}$$

This is basically the approach of [13, §4] and as there, we see by direct calculation or by a more general normalization argument that $U(\mu)$ is unitary when μ is real. We also see that U is uniformly bounded in any disc $D(0, Ch)$ for

$$\operatorname{Im}\mu \geq -\frac{Ch}{\ln\frac{1}{h}}.$$

For μ outside an angle around $-i[0, +\infty[$ with $|\mu| \gg h$, we apply Stirling's formula (5.2) and get

$$u_{j,k} = \exp\Big(\frac{i\mu}{h} - \frac{i\mu}{h}\ln(-i\mu) \pm \Big(\frac{\pi}{2}\frac{\mu}{h} + i\frac{\pi}{4}\Big) + \mathcal{O}_-\Big(\frac{h}{\mu}\Big)\Big)$$

with the + sign valid for $u_{1,2}$, $u_{3,4}$, and the $-$ sign for $u_{1,4}$, $u_{3,2}$. As in Section 8, we get

$$|u_{j,k}| = \exp\Big(-\frac{1}{h}\Big(1 + \ln\frac{1}{|\mu|}\Big)\operatorname{Im}\mu + \frac{\operatorname{Re}\mu}{h}\Big(\arg\Big(\frac{\mu}{i}\Big) \pm \frac{\pi}{2}\Big) + \operatorname{Re}\mathcal{O}_-\Big(\frac{h}{\mu}\Big)\Big).$$

We now also assume that $|\mu| \leq C^{-1} \ll 1$, so that $\ln(|\mu|^{-1}) \gg 1$. We shall estimate the exponent from above. When $\operatorname{Re}\mu \geq 0$, the worst exponent is the one with $+\pi/2$ in the middle term and we approximate

$$\frac{1}{h}\operatorname{Re}\mu\Big(\arg\Big(\frac{\mu}{i}\Big) + \frac{\pi}{2}\Big) = \frac{1}{h}(\operatorname{Re}\mu)(\arg\mu) \sim \frac{1}{h}\frac{(\operatorname{Re}\mu)(\operatorname{Im}\mu)}{|\mu|},$$

which is dominated by the first term and hence

$$|u_{j,k}| \leq \exp\Big(-\frac{1}{h}\Big(\ln\frac{1}{|\mu|} + \mathcal{O}(1)\Big)\operatorname{Im}\mu\Big), \tag{11.6}$$

when $\operatorname{Re}\mu \geq 0$. When $\operatorname{Re}\mu \leq 0$, the worst case is the one with $-\pi/2$ in the middle term and we approximate

$$\frac{1}{h}\operatorname{Re}\mu\Big(\arg\Big(\frac{\mu}{i}\Big) - \frac{\pi}{2}\Big) = \frac{-\operatorname{Re}\mu}{h}(\pi - \arg\mu) \sim \frac{1}{h}\frac{|\operatorname{Re}\mu|(\operatorname{Im}\mu)}{|\mu|},$$

leading to (11.6) also in this case. We conclude that U is bounded in a domain of the form

$$\Big\{\mu\,;\operatorname{Im}\mu \geq -\frac{Ch}{\ln\frac{1}{h+|\mu|}}\Big\}. \tag{11.7}$$

As a consequence, we get the following.

Proposition 11.1 *The problem* (11.1) *with* $R_\pm$ *given by* (11.3) *is microlocally well posed (with errors* $\mathcal{O}(e^{-1/(Ch)})$*) for* $|\mu|$ *small with* μ *in* $D(0, \frac{h}{4})$ *or away from a small conic neighborhood of* $i\mathbf{R}_-$*. If we write the solution as in* (11.2)*, then for* μ *as in* (11.7)*, we have*

$$h\|E\|,\ \|E_\pm\|,\ |E_{-+}| = \mathcal{O}(1)\exp\frac{C}{h}\Big(|\operatorname{Im}\mu| + \epsilon + \frac{h^2}{\epsilon}\Big). \tag{11.8}$$

Let us also compute E_{-+}. Near the branching point, we recall that we have the relation (6.11) for null-solutions to $P - z$, equal to $u_j g_j$ near α_j. To determine E_{-+}, we consider (11.1) with $v = 0$, so that $u_j = v_j^+$ for $j = 2, 4$. We then want to express u_j, $j = 1, 3$ in terms of u_2, u_4. We can do this using (11.4), (11.5) above and redo some of the work in Section 6, but it is easier to to use the work already done and "solve" (6.11). We get

$$\begin{pmatrix} u_1 \\ u_3 \end{pmatrix} = \begin{pmatrix} \frac{c_{1,3}}{c_{2,3}} & c_{1,4} - \frac{c_{1,3}c_{2,4}}{c_{2,3}} \\ \frac{1}{c_{2,3}} & -\frac{c_{2,4}}{c_{2,3}} \end{pmatrix} \begin{pmatrix} u_2 \\ u_4 \end{pmatrix}.$$

Notice that u_1^-, u_3^- in our Grushin problem (with $v = 0$) are the discontinuities we obtain at α_1, α_3 when trying to extend a null-solution near $(0, 0)$ with prescribed $u = v_j^+ g_j$ near α_j, $j = 2, 4$, to a global null-solution near $K_{0,0}$. We get

$$\begin{pmatrix} u_1^- \\ u_3^- \end{pmatrix} = E_{-+}\begin{pmatrix} v_2^+ \\ v_4^+ \end{pmatrix} = \begin{pmatrix} e^{2\pi i\widetilde{\theta}_2} - \frac{c_{1,3}}{c_{2,3}} & \frac{c_{1,3}c_{2,4}}{c_{2,3}} - c_{1,4} \\ -\frac{1}{c_{2,3}} & e^{2\pi i\widetilde{\theta}_1} + \frac{c_{2,4}}{c_{2,3}} \end{pmatrix} \begin{pmatrix} v_2^+ \\ v_4^+ \end{pmatrix},$$

where

$$2\pi\widetilde{\theta}_1 = 2\pi\theta_1 + \tfrac{1}{h}S_{3,4},\ 2\pi\widetilde{\theta}_2 = 2\pi\theta_2 + \tfrac{1}{h}S_{1,2}. \tag{11.9}$$

It follows that

$$\begin{aligned} \det E_{-+} &= \frac{1}{c_{2,3}}(e^{2\pi i(\widetilde{\theta}_1+\widetilde{\theta}_2)}c_{2,3} + c_{2,4}e^{2\pi i\widetilde{\theta}_2} - c_{1,3}e^{2\pi i\widetilde{\theta}_1} - c_{1,4}) \\ &= e^{2\pi i(\widetilde{\theta}_1+\widetilde{\theta}_2)} + \frac{c_{2,4}}{c_{2,3}}e^{2\pi i\widetilde{\theta}_2} - \frac{c_{1,3}}{c_{2,3}}e^{2\pi i\widetilde{\theta}_1} - \frac{c_{1,4}}{c_{2,3}}. \end{aligned} \tag{11.10}$$

From the middle expression, we see that this determinant is equal to the expression in the quantization condition (6.18) times a non-vanishing factor.

The third problem is designed to cover the region $\operatorname{Im}\mu \leq 0$. It is of the form (11.1) with

$$R_+u = (R_+^1 u, R_+^3 u), \quad R_-u_- = R_-^2 u_-^2 + R_-^4 u_-^4. \tag{11.11}$$

For the corresponding model problem for $P_0 - \mu$ in Section 4 we get (*cf.* (4.6), (4.8)):

$$\begin{pmatrix} u_2 \\ u_4 \end{pmatrix} = V\begin{pmatrix} u_1 \\ u_3 \end{pmatrix} = U^{-1}\begin{pmatrix} u_1 \\ u_3 \end{pmatrix},$$

with U as in (11.4), unitary for real μ, so that $V(\mu) = U(\overline{\mu})^*$. Thus from (11.6) we see that the matrix elements $v_{j,k}$ satisfy

$$|v_{j,k}| \leq \exp \frac{1}{h}\Big(\ln \frac{1}{|\mu|} + \mathcal{O}(1)\Big) \operatorname{Im} \mu,$$

and V is bounded in a domain of the form

$$\Big\{\mu\, ; \operatorname{Im} \mu \leq \frac{Ch}{\ln \frac{1}{h+|\mu|}}\Big\}. \tag{11.12}$$

Proposition 11.2 *The problem* (11.1) *with* $R_\pm$ *given by* (11.11) *is microlocally well posed (with errors* $\mathcal{O}(e^{-1/(Ch)})$*) for* $|\mu|$ *small with* μ *in* $D(0, h/4)$ *or away from a small conic neighborhood of* $i\mathbf{R}_+$. *If we write the solution as in* (11.2)*, then for* μ *as in* (11.12)*, we have the estimate* (11.8).

We now compute the corresponding E_{-+}, so we put $v = 0$ in (11.1) and repeat the arguments above. Near the branching point u is a null-solution, and is $u_j g_j$ near α_j, now with $v_j^+ = u_j$, $j = 1, 3$. By "solving" (6.11), we express u_2, u_4 in terms of u_1, u_3 and find

$$\begin{pmatrix} u_2 \\ u_4 \end{pmatrix} = \begin{pmatrix} \frac{c_{2,4}}{c_{1,4}} & c_{2,3} - \frac{c_{1,3}c_{2,4}}{c_{1,4}} \\ \frac{1}{c_{1,4}} & -\frac{c_{1,3}}{c_{1,4}} \end{pmatrix} \begin{pmatrix} v_1^+ \\ v_3^+ \end{pmatrix}.$$

We then get

$$\begin{pmatrix} u_2 \\ u_4 \end{pmatrix} = E_{-+} \begin{pmatrix} v_1^+ \\ v_3^+ \end{pmatrix} = \begin{pmatrix} \frac{c_{2,4}}{c_{1,4}} - e^{-2\pi i\widetilde{\theta}_2} & c_{2,3} - \frac{c_{1,3}c_{2,4}}{c_{1,4}} \\ -\frac{1}{c_{1,4}} & \frac{c_{1,3}}{c_{2,4}} + e^{-2\pi i\widetilde{\theta}_1} \end{pmatrix} \begin{pmatrix} v_1^+ \\ v_3^+ \end{pmatrix},$$

and

$$\begin{aligned} \det E_{-+} &= \frac{e^{-2\pi i(\widetilde{\theta}_1+\widetilde{\theta}_2)}}{c_{1,4}} (c_{2,3}e^{2\pi i(\widetilde{\theta}_1+\widetilde{\theta}_2)} + c_{2,4}e^{2\pi i\widetilde{\theta}_2} - c_{1,3}e^{2\pi i\widetilde{\theta}_1} - c_{1,4}) \\ &= \frac{c_{2,3}}{c_{1,4}} + \frac{c_{2,4}}{c_{1,4}}e^{-2\pi i\widetilde{\theta}_1} - \frac{c_{1,3}}{c_{1,4}}e^{-2\pi i\widetilde{\theta}_2} - e^{-2\pi i(\widetilde{\theta}_1+\widetilde{\theta}_2)}. \end{aligned} \tag{11.13}$$

11.2 The Global Grushin Problem

We first explain what the natural range will be for ϵ. Our global Grushin problem will be built from a direct sum of problems for the operators $\widehat{P}_\epsilon - z$ in (2.3). These operators can be written

$$g\Big(h\Big(k - \frac{k_0}{4}\Big) - \frac{S_0}{2\pi}\Big) + i\epsilon Q\Big(h\Big(k - \frac{k_0}{4}\Big), x, hD_x, \epsilon, \frac{h^2}{\epsilon}; h\Big), \tag{11.14}$$

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where Q is the operator appearing earlier in this section and in Section 6.

Clearly, we add conditions $R_\pm$ only for such k for which

$$w_k = \frac{z - g(h(k - \frac{k_0}{4}) - \frac{S_0}{h})}{i\epsilon}$$

is close to the spectrum of Q, *i.e.,* for which

$$|\operatorname{Im} w_k| \le \mathcal{O}\Big(\epsilon + \frac{h^2}{\epsilon}\Big), \tag{11.15}$$

(*cf.* (6.4)), assuming for simplicity that $\langle q\rangle$ is real-valued. Then according to (11.8) we can expect that our Grushin problem will have an inverse $\mathcal{E}$ with norm $\|\mathcal{E}\| = \mathcal{O}(1)e^{C(\frac{\epsilon}{h}+\frac{h}{\epsilon})}$.

Now we cannot expect to have a complete decomposition into a direct sum of operators (11.14), but Proposition 3.3 shows that it is possible up to an error

$$\mathcal{O}(1)\exp(-1/(C(\epsilon + h))).$$

Thus in order to absorb the error by a standard perturbation argument, we need $e^{C(\frac{\epsilon}{h}+\frac{h}{\epsilon})-\frac{1}{C(\epsilon+h)}} \ll 1$, which would follow from

$$\frac{1}{\epsilon + h} \gg \frac{\epsilon}{h} + \frac{h}{\epsilon}, \tag{11.16}$$

or equivalently $\epsilon^3 + h^3 \ll \epsilon h$. Here we already know that $h^3 \ll \epsilon h$, since we assume $h^2/\epsilon \ll 1$, so the new constraint is $\epsilon^2 \ll h$. From now on we work in the range

$$h^2 \ll \epsilon \ll h^{1/2}.$$

Since g is real-valued with $g' \neq 0$, it follows from (6.4) (or (6.5) in the general case, when $\langle q\rangle$ is not assumed to be real) that the operators (11.14) have disjoint spectra. When (11.15) is never satisfied for any k, it is straightforward to see that z is not in the spectrum of our original operator P_ϵ. Assume now that (11.15) holds for (at most) one $k = \widetilde{k}$. Let $R_\pm^{(\widetilde{k})}$ be the corresponding operators $R_\pm$ defined earlier in this section. Using the notation of Proposition 2.1, we define

$$R_+u = R_+^{(\widetilde{k})}((e^{\frac{i}{h}A}U^{-1}e^{-\frac{\epsilon}{h}G}u|e_{\widetilde{k}-\frac{k_0}{4}-\frac{S_0}{2\pi h}})_{L^2(S^1)}),$$

$$R_-u_- = e^{\frac{\epsilon}{h}G}Ue^{-\frac{i}{h}A}(e_{\widetilde{k}-\frac{k_0}{4}-\frac{S_0}{2\pi h}} \otimes R_-^{(\widetilde{k})}u_-),$$

for $u \in L^2(M)$, $u_- \in \mathbf{C}$. Here $e_k(t) = e^{ikt}$.

Repeating the arguments from [14, §6], (see also [15]), we see that we get a well posed problem with

$$E_{-+}(z) = E_{-+}^{(\widetilde{k})}(z) + \mathcal{O}(e^{\frac{-1}{C(\epsilon+h)}}). \tag{11.17}$$

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Here $E_{-+}^{(\widetilde{k})}$ is the E_{-+} of the approximate 1-dimensional Grushin problem treated in either Proposition 11.1 or 11.2, depending on the sign of the corresponding parameter $\operatorname{Im}\mu$.

In the expressions (11.10) and (11.13), we have the term $e^{\pm 2\pi i(\widetilde{\theta}_1+\widetilde{\theta}_2)}$, where $\widetilde{\theta}_j$ are given by (11.9). We see that $\operatorname{Im}\widetilde{\theta}_j = \mathcal{O}(\frac{\epsilon h}{+}\frac{h}{\epsilon})$, so $e^{\pm 2\pi i(\widetilde{\theta}_1+\widetilde{\theta}_2)} = e^{\mathcal{O}(\frac{\epsilon}{h}+\frac{h}{\epsilon})}$. We conclude from this and (11.16) that the remainder in (11.17) is $\mathcal{O}(e^{-\frac{1}{C(\epsilon+h)}})$ times the dominating term in the expression for $E_{-+}^{(\widetilde{k})}$ in (11.10), or (11.13) respectively. This implies that if we pass to the μ-variable (for $k = \widetilde{k}$) and define the skeleton as in Sections 8, 9 and the corresponding body B as in the beginning of Section 10, then the zeros of $\det E_{-+}$ are confined to B and Theorem 10.1 still applies to give the number of eigenvalues (in the μ-plane) inside an admissible curve.

12 Improved Parameter Range for Barrier Top Resonances in the Resonant Case

We start the discussion in this section with the following general observation. Let P_ϵ be a smooth family of operators, satisfying all the assumptions of the introduction, and in particular (1.9). In Proposition 2.1 we have seen that the operator $P_{\epsilon=0}$ can be reduced by successive averaging procedures to

$$\widehat{P}_{\epsilon=0} = g(hD_t) + hp_1(hD_t, x, hD_x) + h^2 p_2(hD_t, x, hD_x) + \cdots, \quad t \in S^1,\ \tau, x, \xi \approx 0. \tag{12.1}$$

Proposition 12.1 *Assume that the subprincipal symbol of $P_{\epsilon=0}$ vanishes and that the spectrum of $P_{\epsilon=0}$ clusters into bands of size $\le \mathcal{O}(1)h^{N_0}$, for some integer $N_0 \ge 2$. Then $p_j(\tau, x, \xi) = p_j(\tau)$ are independent of (x, ξ) for $1 \le j \le N_0 - 1$ in* (12.1).

Proof Since the subprincipal symbol vanishes, we already know that $p_1 = 0$. Suppose that the conclusion of the proposition does not hold and let $N_1 \in \{2, 3, \dots, N_0 - 1\}$ be the smallest N with $p_N(\tau_0, x, \xi)$ non-constant for some $\tau_0 \approx 0$. Take a family of Gaussian quasimodes $e_\alpha(x)$, $\alpha = (\alpha_x, \alpha_\xi) \in \operatorname{neigh}(0, \mathbf{R}^2)$ with

$$\|e_\alpha\| = 1, \quad p_{N_1}^w(\tau_0, x, hD_x)e_\alpha = p_{N_1}(\tau_0, \alpha)(\alpha)e_\alpha + \mathcal{O}(h^{1/2}) \text{ in } L^2.$$

See [5] for the standard construction of such quasimodes. Then put

$$f_{\alpha,h} = (2\pi)^{-1/2} e^{\frac{i}{h}(h(k-\frac{k_0}{4})-\frac{S_0}{2\pi})t} e_\alpha(x),$$

with $k = k(h)$ such that $h(k(h) - k_0/4) - S_0/2\pi \to \tau_0$, so that

$$\widetilde{P}_0^w f_{\alpha,k} = g\Big(h\Big(k-\frac{k_0}{4}\Big) - \frac{S_0}{2\pi}\Big) + h^{N_1} p_{N_1}\Big(h\Big(k-\frac{k_0}{4}\Big) - \frac{S_0}{2\pi}, \alpha\Big) f_{\alpha,k} + \mathcal{O}(h^{N_1+\frac{1}{2}}) \text{ in } L^2.$$

Hence, since we are dealing with selfadjoint operators,

$$\operatorname{dist}\Big(g\Big(h\Big(k-\frac{k_0}{4}\Big) - \frac{S_0}{2\pi}\Big) + h^{N_1} p_{N_1}\Big(h\Big(k-\frac{k_0}{4}\Big) - \frac{S_0}{2\pi}, \alpha\Big), \sigma(P_0)\Big) \le \mathcal{O}(h^{N_1+\frac{1}{2}}),$$

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and varying α, so that the values $p_{N_1}(h(k-\frac{k_0}{4})-\frac{S_0}{2\pi},\alpha)$ fill up a whole interval, we get a contradiction to the clustering assumption. ■

Remark Proposition 12.1 remains to hold in the case described in [14, §4], where the operator $P_{\epsilon=0}$ is conjugated into a normal form in a neighborhood of a Lagrangian torus, rather than near a closed H_p-trajectory.

From now on we shall assume that $P_{\epsilon=0}$ satisfies the assumptions of Proposition 12.1. Let us now switch on ϵ. An application of Proposition 2.1 together with Proposition 12.1 then shows that microlocally, near a closed H_p-trajectory, P_ϵ can be reduced to the form

$$\widehat{P}_\epsilon = g(hD_t) + \epsilon\Big(i\langle q\rangle(hD_t, x, hD_x) + \mathcal{O}(\epsilon) + \mathcal{O}\Big(\frac{h^{N_0}}{\epsilon}\Big) + h\widetilde{p}_1 + \cdots\Big). \tag{12.2}$$

It follows therefore that in the results of [14, Theorems 6.4, 6.7] we can replace the exponent 2 by the exponent N_0 in the parameter range for ϵ. Thus for the study of the spectrum of P_ϵ in a region where $|\mathrm{Re}\, z| < 1/\mathcal{O}(1)$ and $|\mathrm{Im}\, z/\epsilon - F_0| \leq 1/\mathcal{O}(1)$, when F_0 is a non-critical value or a non-degenerate maximum or minimum of $\mathrm{Re}\langle q\rangle$ along $p^{-1}(0)$, it suffices to assume that

$$h^{N_0} \ll \epsilon \leq h^\delta, \tag{12.3}$$

for some $\delta > 0$. In the case when F_0 is a saddle point value of $\mathrm{Re}\langle q\rangle$, from Theorem 1.1 we get the condition

$$h^{N_0} \ll \epsilon \ll h^{1/2}. \tag{12.4}$$

Indeed, in the latter case we still have Proposition 3.3 and the decoupling condition analogous to (11.16) becomes $\epsilon^3 + h^{N_0+1} \ll \epsilon h$, which is fulfilled by (12.4).

We shall now apply these observations to improve the result of [14, Proposition 7.1], giving a description of the individual barrier top resonances of the semiclassical Schrödinger operator in the resonant case. Before doing so, and also for the future use in Section 13, we shall first briefly recall the general setup in [14, §7], as well as in [15, §5].

As in [14, 15], let us consider

$$P = -h^2\Delta + V(x),\ P(x,\xi) = \xi^2 + V(x),\ (x,\xi) \in T^*\mathbf{R}^2, \tag{12.5}$$

where V satisfies the general assumptions of [14, §7], allowing us to define the resonances of P in the lower half-plane inside some fixed neighborhood of $E_0 > 0$, where $V(0) = E_0, V'(0) = 0, V''(0) < 0$. As in [14, 15], we assume that $\{(0,0)\}$ is the only trapped H_P-trajectory in $P^{-1}(E_0)$. After a linear symplectic change of coordinates, we may write

$$P(x,\xi) - E_0 = \sum_{j=1}^{2} \frac{\lambda_j}{2}(\xi_j^2 - x_j^2) + p_3(x) + p_4(x) + \cdots, \quad (x,\xi) \to 0, \tag{12.6}$$

where $\lambda_j > 0$ and $p_j(x)$ is a homogeneous polynomial of degree $j \geq 3$. Recall further from [14] that the study of resonances of P near E_0 can be reduced to an eigenvalue problem for P after applying some variant of the method of complex scaling, and that near $x = 0$ this simply amounts to working in the new real coordinates $(\widetilde{x}, \widetilde{\xi})$, given by $x = e^{i\pi/4}\widetilde{x}$, $\xi = e^{-i\pi/4}\widetilde{\xi}$.

Performing the scaling and dropping the tildes from the notation, we see that the problem reduces to studying the eigenvalues close to 0 of the operator $i(P - E_0)$, now elliptic outside a small neighborhood of $(0, 0)$, with symbol

$$P(x,\xi) = p(x,\xi) + ie^{\frac{3i\pi}{4}} p_3(x) + ie^{i\pi} p_4(x) + ie^{\frac{5i\pi}{4}} p_5(x) + \cdots, \tag{12.7}$$

where

$$p(x,\xi) = \sum_{j=1}^{2} \frac{\lambda_j}{2}(x_j^2 + \xi_j^2). \tag{12.8}$$

Here we continue to write P to denote the scaled operator.

We assume that $\lambda_j > 0$ in (12.7) are rationally dependent,

$$\exists k^0 = (k_1^0, k_2^0) \in \mathbf{Z}^2 \setminus \{0\},\ k_1^0\lambda_1 + k_2^0\lambda_2 = 0, \tag{12.9}$$

which implies that the H_p-flow is periodic.

As in [14, 15], we are interested in eigenvalues E of P with $|E| \sim \epsilon^2$, $0 < \epsilon \ll 1$. After a rescaling $x = \epsilon\tilde{x}$ and dropping the tildes over the new variables, we get an operator $P_\epsilon = \frac{1}{\epsilon^2}P$ that we view as an $\widetilde{h}$-pseudodifferential operator with the symbol

$$P_\epsilon(x,\xi) = \tfrac{1}{\epsilon^2}P(\epsilon(x,\xi)) = p(x,\xi) + i\epsilon e^{3\pi i/4} p_3(x) - i\epsilon^2 p_4(x) + \mathcal{O}(\epsilon^3).$$

Here $\widetilde{h} = h/\epsilon^2$. Now the spectrum of $P_{\epsilon=0}$ is that of the harmonic oscillator, and hence it clusters into sets of diameter 0 and separation of order h. An application of Proposition 12.1 shows that all the p_j in (12.1) are constant. Moreover, since in this case all the eigenvalues depend linearly on h, we see from the proof that the p_j have to vanish. It follows from (12.3) that in the zone corresponding to non-critical values F_0 or non-degenerate maxima or minima, the range of energies that we get is

$$h^{2N_0/(1+2N_0)} \ll |E - E_0| \leq h^{2\delta/(1+2\delta)},$$

for all $N_0 = 2, 3, \ldots$ and all $\delta > 0$. When F_0 corresponds to a branching level, we get from (12.4)

$$h^{2N_0/(1+2N_0)} \leq |E - E_0| \leq h^{1/2}.$$

We summarize the discussion above in the following proposition, which is an improvement of [14, Proposition 7.3]. Clearly, in a similar fashion, we also obtain an improvement of [15, Theorem 5.1].

Proposition 12.2 *Assume that the principal symbol $P(x,\xi)$ in (12.5) has an asymptotic expansion (12.6), and assume that (12.9) holds. Assume furthermore that the function $\langle p_3\rangle$, defined as the average of p_3 along the Hamilton flow of p in (12.8) does not vanish identically. Then the resonances of the operator P in the domain*

$$\{z \in \mathbf{C}; h^{2N_0/(1+2N_0)} \ll |z - E_0| \le h^{\delta}\} \setminus \bigcup\left\{ z \in \mathbf{C}\,;\, \left|\operatorname{Re} z - E_0 - A\,|\operatorname{Im} z|^{3/2}\right| < \eta\,|\operatorname{Im} z|^{3/2}\right\}, \tag{12.10}$$

where $\eta > 0$, $\delta > 0$, and $N_0 = 2, 3, \ldots$ are arbitrary but fixed, are given by

$$\sim E_0 - i\Big(h(k_1 - \alpha/4) + \epsilon^3 \sum_{j=0}^{\infty} h^j \epsilon^{-2j} r_j\Big(\frac{h}{\epsilon^2}\Big(k - \frac{k_0}{4}\Big) - \frac{S}{2\pi}, \epsilon, \frac{h^{N_0}}{\epsilon^{1+2N_0}}\Big)\Big),$$

with

$$r_0 = ie^{3\pi i/4}\langle p_3\rangle(\xi) + \mathcal{O}\Big(\epsilon + \frac{h^{N_0}}{\epsilon^{1+2N_0}}\Big), \quad r_j = \mathcal{O}(1),\ j \ge 1.$$

We have $k = (k_1, k_2) \in \mathbf{Z}^2$, $S = (S_1, S_2)$ with $S_1 = 2\pi$, and $\alpha = (\alpha_1, \alpha_2) \in \mathbf{Z}^2$ is fixed, and we choose $\epsilon > 0$ with $|E - E_0| \sim \epsilon^2$. The union in (12.10) is taken over the set of critical values of $\langle p_3\rangle$, restricted to $p^{-1}(1)$, with A varying over this set.

Remark If $\langle p_3\rangle$ restricted to $p^{-1}(1)$ has precisely one non-degenerate saddle point with the critical value A, then the results of the present paper apply and give a description of the individual resonances in a half-cubic neighborhood of the curve $\operatorname{Re} z = E_0 + A\,|\operatorname{Im} z|^{3/2}$. In the following section, we shall consider explicit examples of homogeneous polynomials for which the assumptions of Theorem 1.1 are satisfied.

13 Examples in the Barrier Top Case

This section is devoted to a study of examples of potentials of the Schrödinger operator (12.5) to which Theorem 1.1 is applicable.

Let us recall from (12.7) that we are interested in eigenvalues close to 0 of the operator P, elliptic outside a small neighborhood of $(0,0)$ with symbol

$$P(x,\xi) = p(x,\xi) + ie^{\frac{3\pi i}{4}} p_3(x) + ie^{i\pi} p_4(x) + \cdots, \tag{13.1}$$

where the harmonic oscillator $p(x,\xi)$ has been defined in (12.8). As before, we make the resonant assumption (12.9).

Consider first a general perturbation of p of the form of a linear combination of terms $x^\alpha \xi^\beta$ with $|\alpha| + |\beta| = m$, for some $m \in \{3,4,5,\ldots\}$. Recall from [14] how to compute the corresponding trajectory average $\langle x^\alpha \xi^\beta\rangle$: basically we use action-angle coordinates, but to start with, we can do things a little more easily by introducing

$$z_j = x_j + i\xi_j \in \mathbf{C},$$

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and noticing that along an H_p-trajectory we get in the z_1, z_2 coordinates:

$$z_j(t) = e^{-i\lambda_j t} z_j(0).$$

Then write $x_j(t) = \operatorname{Re} z_j(t)$, $\xi_j(t) = \operatorname{Im} z_j(t)$, so that

(13.2)

$$\begin{aligned} x(t)^\alpha \xi(t)^\beta &= \prod_{j=1}^{2} ((\operatorname{Re} z_j(t))^{\alpha_j} (\operatorname{Im} z_j(t))^{\beta_j}) \\ &= \frac{1}{2^{|\alpha|+|\beta|} i^{|\beta|}} \prod_{j=1}^{2} ((z_j(0)e^{-i\lambda_j t} + \overline{z_j}(0)e^{i\lambda_j t})^{\alpha_j} (z_j(0)e^{-i\lambda_j t} - \overline{z_j}(0)e^{i\lambda_j t})^{\beta_j}). \end{aligned}$$

Then expand the product by means of the binomial theorem. The time average is equal to the time-independent term and since this average is constant along each trajectory we shall replace the symbols $z_j(0)$ simply by z_j.

In this section we consider the case when $\lambda_1 = \lambda_2 = 1$, $m = 4$, $\beta = 0$. (In [14] we noticed that in this case the average will vanish when $m = 3$ and in [15] we made a more refined study of that case taking into account one more term in the perturbative expansion.) This means that we take $p_3(x) = 0$, and for simplicity we also assume that $p_m = 0$ for all odd m in (13.1), so that we can concentrate on the perturbation $-ip_4$ in (13.1). Performing a rescaling as described in the previous section, with ϵ replaced by $\epsilon^{1/2}$, (*i.e.*, setting $x = \epsilon^{1/2}\tilde{x}$ rather than $x = \epsilon\tilde{x}$), we get

$$p(x,\xi) - i\epsilon p_4(x) - \epsilon^2 p_6(x) + i\epsilon^3 p_8(x) + \epsilon^4 p_{10}(x) + \cdots, \tag{13.3}$$

and here, as before, we choose ϵ of the same order of magnitude as the modulus of the eigenvalues for the operator $P(x, hD)$ which we want to study.

Now we continue the calculations of trajectory averages using (13.2). We have

$$\begin{aligned} \langle x_1^4 \rangle &= \frac{1}{2^4}\langle (z_1 + \overline{z}_1)^4 \rangle = \frac{1}{2^4}\langle z_1^4 + 4z_1^3\overline{z}_1 + 6z_1^2\overline{z}_1^2 + 4z_1\overline{z}_1^3 + \overline{z}_1^4 \rangle \\ &= \frac{6}{16}\langle z_1^2\overline{z}_1^2 \rangle = \frac{3}{8}|z_1|^4 \quad (= \frac{3}{8}(x_1^2 + \xi_1^2)^2). \end{aligned}$$

In the same way, we get $\langle x_2^4 \rangle = \frac{3}{8}|z_2|^4$.

Next look at the averages of mixed terms:

$$\begin{aligned} \langle x_1^3 x_2 \rangle &= \frac{1}{2^4}\langle (z_1 + \overline{z}_1)^3 (z_2 + \overline{z}_2) \rangle = \frac{1}{2^4}\langle (z_1^3 + 3z_1^2\overline{z}_1 + 3z_1\overline{z}_1^2 + \overline{z}_1^3)(z_2 + \overline{z}_2) \rangle \\ &= \frac{3}{16}(|z_1|^2\overline{z}_1 z_2 + |z_1|^2 z_1\overline{z}_2) = \frac{3}{8}|z_1|^2 \operatorname{Re}(z_1\overline{z}_2), \\ \langle x_1 x_2^3 \rangle &= \frac{3}{8}|z_2|^2 \operatorname{Re}(z_2\overline{z}_1), \end{aligned}$$

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$$\begin{aligned}\langle x_1^2 x_2^2\rangle &= \frac{1}{2^4}\langle (z_1^2 + 2z_1\bar{z}_1 + \bar{z}_1^2)(z_2^2 + 2z_2\bar{z}_2 + \bar{z}_2^2)\rangle \\ &= \frac{1}{2^4}(z_1^2\bar{z}_2^2 + 4|z_1|^2|z_2|^2 + \bar{z}_1^2 z_2^2) \\ &= \frac{1}{8}\operatorname{Re}(z_1^2\bar{z}_2^2) + \frac{1}{4}|z_1|^2|z_2|^2.\end{aligned}$$

Notice that our averages are invariant under the anti-symplectic involution

$$j\colon (x,\xi) \mapsto (x,-\xi).$$

This is necessarily the case since we stay in the framework of ordinary Schrödinger operators (without magnetic fields) whose symbols have this invariance.

Now write our results in the action angle variables (ρ_j, θ_j), given by

$$z_j = \sqrt{2\rho_j}\, e^{-i\theta_j},$$

so that $\frac{1}{2}|z_j|^2 = \frac{1}{2}(x_j^2 + \xi_j^2) = \rho_j$:

$$\langle x_1^4\rangle = \frac{3}{2}\rho_1^2, \quad \langle x_2^4\rangle = \frac{3}{2}\rho_2^2,$$

$$\langle x_1^3 x_2\rangle = \frac{3}{2}\rho_1^{3/2}\rho_2^{1/2}\cos(\theta_1-\theta_2), \quad \langle x_1 x_2^3\rangle = \frac{3}{2}\rho_1^{1/2}\rho_2^{3/2}\cos(\theta_2-\theta_1),$$

$$\langle x_1^2 x_2^2\rangle = \rho_1\rho_2 + \frac{1}{2}\rho_1\rho_2\cos 2(\theta_1-\theta_2).$$

It follows from the Hamilton equations that ρ_j and $\theta := \theta_1 - \theta_2$ are constant along every H_p-trajectory. The involution j can also be described as $(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)$, and hence in action-angle variables as $(\rho_1, \rho_2, \theta_1, \theta_2) \mapsto (\rho_1, \rho_2, -\theta_1, -\theta_2)$.

We shall study our averages as functions on the abstract symplectic manifold

$$\Sigma = p^{-1}(1)/\exp \mathbf{R}H_p.$$

Using the (z_1, z_2)-coordinates, we have $p^{-1}(1) : \frac{1}{2}(|z_1|^2 + |z_2|^2) = 1$, and the equivalence relation induced by the H_p-flow is: $(z_1, z_2) \sim (w_1, w_2)$ if and only if

$$(w_1, w_2) = (e^{it}z_1, e^{it}z_2)$$

for some $t \in \mathbf{R}$. Thus we see that Σ can be identified with the complex projective space $P(\mathbf{C}^2)$. It is well known that this space is diffeomorphic to $\mathbf{S}^2$. Indeed, $P(\mathbf{C}^2)$ can be identified with the 1-point compactification $\mathbf{C}\cup\{\infty\}$ via the map $(z_1, z_2) \mapsto z_1/z_2$ and the one point compactification can be identified with the Riemann sphere.

Thus Σ can be parametrized by (ρ_1, ρ_2, θ) with $\rho_j \geq 0$, $\rho_1 + \rho_2 = 1$, $\theta \in \mathbf{R}/2\pi\mathbf{Z}$, with the convention that all the $(1, 0, \theta)$ denote the same point and similarly for $(0, 1, \theta)$. The involution j induces the anti-symplectic involution

$$j\colon \Sigma \ni (\rho_1, \rho_2, \theta) \mapsto (\rho_1, \rho_2, -\theta).$$

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Notice that the set of fixed points of j is given by all points with $\theta = 0$ or $\theta = \pi$. These points form a (great) circle on Σ and can also be described as the set of trajectories in $p^{-1}(1)$ whose x-space projections hit the boundary of the potential well $\{x \in \mathbf{R}^2 ;\ |x| = 1\}$.

We consider perturbations of the form

$$q(x) = \tfrac{2}{3}a(x_1^4 + x_2^4) + bx_1^2x_2^2 + \tfrac{2}{3}c(x_1^3x_2 + x_1x_2^3). \tag{13.4}$$

Then on Σ we get with $\rho = \rho_1$, so that $\rho_2 = 1 - \rho$:

$$\begin{aligned}\langle q\rangle &= a(\rho_1^2 + \rho_2^2) + b\rho_1\rho_2 + \tfrac{b}{2}\rho_1\rho_2\cos(2\theta) + c(\rho_1^{\frac{1}{2}}\rho_2^{\frac{3}{2}} + \rho_1^{\frac{3}{2}}\rho_2^{\frac{1}{2}})\cos\theta\\ &= a(\rho^2 + (1-\rho)^2) + b\rho(1-\rho)(1 + \tfrac{1}{2}\cos(2\theta))\\ &\qquad + c(\rho^{\frac{1}{2}}(1-\rho)^{\frac{3}{2}} + \rho^{\frac{3}{2}}(1-\rho)^{\frac{1}{2}})\cos\theta\\ &= a + (b - 2a)\rho(1-\rho) + \tfrac{b}{2}\rho(1-\rho)\cos(2\theta) + c\rho^1/2(1-\rho)^1/2\cos\theta\\ &= a + (\tfrac{b}{2} - 2a)\rho(1-\rho) + b\rho(1-\rho)\cos^2\theta + c\rho^{\frac{1}{2}}(1-\rho)^{\frac{1}{2}}\cos\theta,\end{aligned}$$

where we used that $\rho^{1/2}(1-\rho)^{3/2} + \rho^{3/2}(1-\rho)^{1/2} = \rho^{1/2}(1-\rho)^{1/2}$.

We are interested in the critical points of this function on Σ, and the values $\rho = 0, 1$ will have to be treated separately. In particular we are interested in the number of saddle points. If we have only one saddle point we will be able to apply the results of this paper. This is still the case if there are two saddle points provided that the corresponding critical values are different. We will also encounter the case of two saddle points S_1, S_2 away from the equator and then necessarily with $j(S_1) = S_2$. In that case the critical vales will be equal and the results of this paper will not apply directly. We plan to return to that case in a future paper, where the role of symmetries will be studied.

Put $d = \frac{b}{2} - 2a$, $g = \rho^{1/2}(1-\rho)^{\frac{1}{2}}$, $y = \cos\theta$. Then, $\langle q\rangle = a + dg^2 + bg^2y^2 + cgy$. Notice that $y = \cos\theta$ is critical precisely when $\theta = 0, \pi$ and that $y \in [-1, 1]$. When $y \neq \pm 1$, we may treat g as an independent variable. The same observation is valid for $g(\rho) \in]0, \frac{1}{2}]$. It is non-critical in $[0, \frac{1}{2}[$ *i.e.,* for $\rho \neq \frac{1}{2}$. (As already mentioned, the value $g = 0$, corresponding to $\rho = 0, 1$, will require a different treatment.)

In order to avoid various degenerations, we shall assume $d \neq 0$, When $c \neq 0$, we have $b \neq 0$, $b + d \neq 0$.

Case 1 *Critical points with*

$$\theta \neq 0, \pi, \quad \rho \neq 0, \tfrac{1}{2}, 1. \tag{13.5}$$

Here both y and g can be treated as independent variables and the critical points are determined by $2bg^2y + cg = 0$ and $2dg + 2bgy^2 + cy = 0$. This can also be written $g(2bgy + c) = 0$ and $2dg + y(2bgy + c) = 0$. Under the assumption (13.5) we have $g \neq 0$, so we get $2bgy + c = 0$ and $2dg = 0$. This is in contradiction with the assumption that $d \neq 0$, so we conclude that there are no critical points away from the union of the vertical circle given by $\theta \in \{0, \pi\}$ and the horizontal circle: $\rho = 1/2$.

Case 2 *Critical points on the horizontal circle away from the vertical one:*

$$\theta \neq 0, \pi, \quad \rho = \tfrac{1}{2}. \tag{13.6}$$

Then $g = 1/2$ and this is a critical value, so we only have to look for critical points with respect to y, leading to

$$2b(\tfrac{1}{2})^2 y + c\tfrac{1}{2} = 0, \quad y = -\tfrac{c}{b}.$$

Recall that $|y| < 1$ under the assumption (13.6), so we reach the conclusion that if $\left|\frac{c}{b}\right| < 1$, then there are two distinct critical points in the region (13.6), given by $\rho = \frac{1}{2}$, $\cos\theta = -\frac{c}{b}$, and otherwise there are no such points. (In the remaining degenerate case $b = c = 0$ the whole horizontal circle is critical.)

We also study the nature of the critical points, by computing the hessian of $\langle q\rangle$ with respect to ρ, y. Using that $g'(\frac{1}{2}) = 0$, $g''(\frac{1}{2}) = -2$, we get at both points

$$\partial_y^2\langle q\rangle = \tfrac{b}{2}, \quad \partial_y\partial_\rho\langle q\rangle = 0, \quad \partial_\rho^2\langle q\rangle = -2d.$$

So both critical points are of signature $(b, -d)$ where the first component corresponds to the horizontal (θ) direction. (We use the convention that a signature described by (α, β) is given by $(\operatorname{sign}(\alpha), \operatorname{sign}(\beta))$.)

Case 3 *Critical points on the vertical circle away from the horizontal one and from the poles $\rho = 0, 1$:*

$$\theta \in \{0, \pi\}, \quad \rho \notin \{0, \tfrac{1}{2}, 1\}.$$

For $\theta = 0$, we have $y = 1$ and we look for critical points of $g \mapsto (d+b)g^2 + cg$, leading to $g = -\frac{c}{2(b+d)}$. Hence we get two critical points in this region if $-1 < \frac{c}{b+d} < 0$. and otherwise no point on this half of the vertical circle. (In the degenerate case $b+d = 0$, $c = 0$ the whole vertical circle is critical.)

For $\theta = \pi$, we have $y = -1$ and we look for critical points of $g \mapsto (d+b)g^2 - cg$, leading to $g = \frac{c}{2(b+d)}$, so we get two critical points in this case if $0 < \frac{c}{b+d} < 1$, and otherwise no critical points on this half of the vertical circle. We will see shortly that we have critical points at the poles when $c = 0$.

In both subcases, we get by a straight forward calculation:

$$\langle q\rangle''_{gg} = 2(b+d), \quad \langle q\rangle''_{g\theta} = 0, \quad \langle q\rangle''_{\theta\theta} = \frac{c^2}{2(b+d)^2}d,$$

so the signature is $(d+b, d)$ where the first component refers to the direction of the vertical circle through the critical point.

Case 4 *The two points of intersection of the two circles:* Here $\rho = \frac{1}{2}$ and $\theta \in \{0, \pi\}$. Here both g and y are critical, so our intersection points are both critical. By straight forward calculation, we get for $C_f : \theta = 0, \rho = \frac{1}{2}$

$$\langle q\rangle''_{\theta\theta} = -\tfrac{1}{2}(b+c), \quad \langle q\rangle''_{\theta\rho} = 0, \quad \tfrac{1}{2}\langle q\rangle''_{\rho\rho} = -c - b - d,$$

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and for $C_b : \theta = \pi$, $\rho = \frac{1}{2}$

$$\langle q \rangle''_{\theta\theta} = \tfrac{1}{2}(-b + c), \quad \langle q \rangle''_{\theta\rho} = 0, \quad \tfrac{1}{2}\langle q \rangle''_{\rho\rho} = c - b - d.$$

In particular, the signature is

$$\begin{cases} (-c - b - d, -b - c) & \text{when } \theta = 0, \\ (c - b - d, c - b) & \text{when } \theta = \pi. \end{cases}$$

Also notice that

$$\langle q \rangle(C_f) = a + \frac{d+b}{4} + \frac{c}{2}, \quad \langle q \rangle(C_b) = a + \frac{d+b}{4} - \frac{c}{2}. \tag{13.7}$$

Case 5 *It remains to study the "poles", given by* $\rho = 0, 1$. Here the ρ, θ coordinates degenerate and we return to the z-coordinates. Using that $\operatorname{Re}(z_1^2 \overline{z}_2^2) = 2(\operatorname{Re} z_1 \overline{z}_2)^2 - |z_1|^2 |z_2|^2$, we get

$$\langle q \rangle = \tfrac{a}{4}(|z_1|^2 + |z_2|^2)^2 + \left(\tfrac{b}{8} - \tfrac{a}{2}\right) |z_1|^2 |z_2|^2 + \tfrac{b}{4}(\operatorname{Re} z_1 \overline{z}_2)^2 + \tfrac{c}{4}(|z_1|^2 + |z_2|^2) \operatorname{Re}(z_1 \overline{z}_2).$$

Make the change of variables $\zeta_j = z_j/\sqrt{2}$ and restrict to the energy surface $p^{-1}(1)$, which now becomes $|\zeta_1|^2 + |\zeta_2|^2 = 1$. Then we get

$$\langle q \rangle = a + d|\zeta_1|^2 |\zeta_2|^2 + b(\operatorname{Re}(\zeta_1 \overline{\zeta}_2))^2 + c \operatorname{Re}(\zeta_1 \overline{\zeta}_2),$$

again with $d = \frac{b}{2} - 2a$.

Recall that we work on the projective space, described as the 3-sphere $|\zeta_1|^2 + |\zeta_2|^2 = 1$ modulo the action of the rotations $t \mapsto (e^{it}\zeta_1, e^{it}\zeta_2)$. Consider the case $\rho = 0$. Correspondingly, we can choose the point $(\zeta_1^0, \zeta_2^0) = (0, 1)$. The H_p-integral curve through that point is $t \mapsto (0, e^{-it})$ and locally, we can identify Σ with the transversal hypersurface H in the 3-sphere which is given by $\operatorname{Im} \zeta_2 = 0$. Thus $\zeta_2 = 1 - w$ with $w \in \operatorname{neigh}(0, \mathbf{R})$, and we get $w = 1 - (1 - |\zeta_1|^2)^{1/2} = \frac{1}{2}|\zeta_1|^2 + \mathcal{O}(|\zeta_1|^4)$. We can use the real and imaginary parts of ζ_1 as local coordinates on H. Then on H, we get the Taylor expansion $\langle q \rangle = a + d|\zeta_1|^2 + b(\operatorname{Re} \zeta_1)^2 + c \operatorname{Re} \zeta_1 + \mathcal{O}(|\zeta_1|^3)$.

We conclude that the "pole" $\rho = 0$ is a critical point if and only if $c = 0$ and when this point is critical, the signature is $(d+b, d)$, where the first component corresponds to the direction of the (vertical) circle through the pole. By symmetry in the indices 1, 2, we have the identical conclusion for the opposite pole, given by $\rho = 1$. Notice finally that this case together with Case 3 give a complete description of the critical points on the vertical circle away from the crossings with the horizontal one.

We observe that the critical points away from the intersection of the two circles are non-degenerate and keep constant signatures under small perturbations of the parameters (except in the degenerate cases $c = b = 0$ and $c = b + d = 0$). These critical points can only be killed or born by passing through one of the two crossing points. This happens in the following four cases.

Case 1 $\frac{c}{b} = -1$: The critical points on the horizontal circle coalesce into the crossing point $\theta = 0,\ \rho = \frac{1}{2}$. When c/b goes from $-1+\epsilon$ to $-1-\epsilon$, the two critical points disappear and the signature at $\theta = 0,\ \rho = \frac{1}{2}$ goes from $(-d, -b\epsilon)$ to $(-d, b\epsilon)$

Case 2 $\frac{c}{b} = 1$. The two critical points on the horizontal great circle coalesce into the crossing point $\theta = \pi, \rho = \frac{1}{2}$. When $\frac{c}{b}$ goes from $1-\epsilon$ to $1+\epsilon$, the signature of that crossing point goes from $(-d, -b\epsilon)$ to $(-d, b\epsilon)$.

Case 3 $\frac{c}{b+d} = -1$: The two critical points on the vertical circle coalesce into the crossing point $\theta = 0, \rho = \frac{1}{2}$. When $\frac{c}{b+d}$ goes from $-1+\epsilon$ to $-1-\epsilon$, the signature of that crossing point goes from $(c\epsilon, d)$ to $(-c\epsilon, d)$.

Case 4 $\frac{c}{b+d} = 1$: The two critical points on the vertical circle coalesce into the crossing point $\theta = \pi, \rho = \frac{1}{2}$. When $\frac{c}{b+d}$ goes from $1-\epsilon$ to $1+\epsilon$, the signature of that crossing point goes from $(-c\epsilon, d)$ to $(c\epsilon, d)$.

In the following, we may assume in order to fix the ideas that $d > 0$. In the b, c-plane, we define the following open sets, separated from each other by the 4 lines $c = \pm b$, $c = \pm(b+d)$, where all the critical points will be non-degenerate:

A : $b > 0, \quad -b < c < b.$
B_+ : $\max(b, -b) < c < b + d.$
B_- : $-(b+d) < c < \min(b, -b).$
C_+ : $c > \max(b+d, -b).$
C_- : $c < \min(b, -b-d).$
D : $b < 0, \quad \max(b, -b-d) < c < \min(-b, b+d).$
E_+ : $\max(b+d, -b-d) < c < -b.$
E_- : $b < c < \min(-b-d, b+d).$
F : $b < -d, \quad b+d < c < -b-d.$

Then the earlier discussion gives the location and the signature of the critical points in each of the cases. Let C_f denote the "forward" crossing point of the two circles, given by $\rho = \frac{1}{2}, \theta = 0$. Similarly let C_b denote the "backward" crossing point, given by $\rho = \frac{1}{2}, \theta = \pi$.

A : Signature at C_f: $(-,-)$
Signature at C_b: $(-,-)$
Away from the crossings:
On the horizontal circle: Two critical points with signature $(+,-)$
On the vertical circle: Two critical points with signature $(+,+)$

B_+ : Signature at C_f: $(-,-)$
Signature at C_b: $(-,+)$
Away from the crossings:
On the horizontal circle: No critical points
On the vertical circle: Two critical points with signature $(+,+)$
Here $\langle q\rangle(C_b)$ is smaller than $\langle q\rangle(C_f)$ but larger than the two other critical values.

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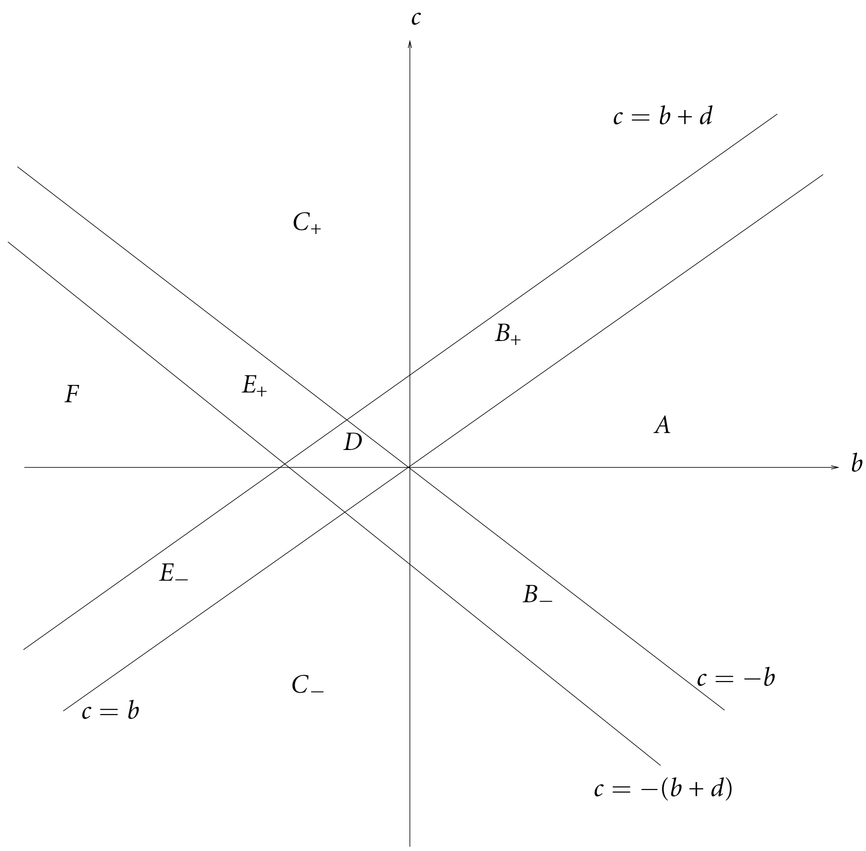

Figure 6: When the parameters are in the regions B_+, B_-, E_+, and E_-, we have precisely one saddle point, and therefore the results of the present paper apply. In the cases C_+ and C_- there are no saddle points at all, while in the case D there two saddle points. The corresponding critical values of $\langle q\rangle$ are separated provided that we assume that $c \neq 0$.

B_- : Signature at C_f: $(-,+)$
Signature at C_b: $(-,-)$
Away from the crossings:
On the horizontal circle: No critical points
On the vertical circle: Two critical points with signature $(+,+)$.
Here $\langle q\rangle(C_f)$ is smaller than $\langle q\rangle(C_b)$ but larger than the two other critical values.

C_+ : Signature at C_f: $(-,-)$
Signature at C_b: $(+,+)$
Away from the crossings:
On the horizontal circle: No critical points
On the vertical circle: No critical points

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C_- : Signature at C_f: $(+,+)$
Signature at C_b: $(-,-)$
Away from the crossings:
On the horizontal circle: No critical points
On the vertical circle: No critical points

D : Signature at C_f: $(-,+)$
Signature at C_b: $(-,+)$
Away from the crossings:
On the horizontal circle: Two critical points with signature $(-,-)$
On the vertical circle: Two critical points with signature $(+,+)$
Here $\langle q\rangle(C_f)$, $\langle q\rangle(C_b)$ are larger than the values at the critical points on the vertical circle and smaller than the values at the critical points on the horizontal circle. From (13.7) we also know that $\langle q\rangle(C_f) - \langle q\rangle(C_b) = c$.

E_+ : Signature at C_f: $(-,+)$
Signature at C_b: $(+,+)$
Away from the crossings:
On the horizontal circle: Two critical points with signature $(-,-)$.
On the vertical circle: No critical points.
In this case $\langle q\rangle(C_f)$ is larger than $\langle q\rangle(C_b)$ but smaller than the two other critical values.

E_- : Signature at C_f: $(+,+)$
Signature at C_b: $(-,+)$
Away from the crossings:
On the horizontal circle: Two critical points with signature $(-,-)$.
On the vertical circle: No critical points.
In this case $\langle q\rangle(C_b)$ is larger than $\langle q\rangle(C_f)$ but smaller than the two other critical values.

F : Signature at C_f: $(+,+)$
Signature at C_b: $(+,+)$
Away from the crossings:
On the horizontal circle: Two critical points with signature $(-,-)$.
On the vertical circle: Two critical points with signature $(-,+)$.

In cases B_+, B_-, E_+, E_- we have precisely one saddle point (necessarily) situated on the vertical circle which is the fixed point set of j. In these cases, the results of this paper apply. The results also apply in case D, provided that we assume that $c \neq 0$ in order to separate the two saddle point values. In these cases it is easy to understand the structure and the shape of the level sets $\langle q\rangle = C$. In particular, we see that when we let C be a saddle point value, we get a connected "∞" shaped set (and no "circular" components on which $\langle q\rangle$ is non-critical everywhere).

In case F, we have two saddle points situated on the vertical circle symmetrically with respect to the horizontal circle. Since we have chosen to use perturbations which are symmetric under permutation of x_1, x_2, the function $\langle q\rangle$ is invariant under the map $\rho \mapsto 1-\rho$, so the critical values are necessarily equal. Here we can break the symmetry by adding a small multiple of, for instance, x_1^4 so that we still have precisely

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two saddle points, but with different critical values. Then the results of our paper apply.

In case A, we have two saddle points on the horizontal circle. They are of course exchanged by application of j and this symmetry remains under perturbations within the class of Schrödinger operators without magnetic field. We hope to analyze this case in a future work.

We shall next compute the ϵ^2-contribution to the averaging of the principal symbol $p(x,\xi) - i\epsilon p_4(x) - \epsilon^2 p_6(x) + \mathcal{O}(\epsilon^3)$ appearing in (13.3), by applying the calculations of the end of Section 8, with $q = -p_4(x)$,$= -p_6(x)$, and $T = 2\pi$. Recall from there that we have the averaged symbol

$$p_\epsilon|_{\Lambda_{\epsilon G}} \simeq p + i\epsilon\langle q\rangle + \epsilon^2(\langle r\rangle - \tfrac{1}{2}C(q,q)) + \mathcal{O}(\epsilon^3), \tag{13.8}$$

where $C(q_1,q_2)$ and $\mathrm{Cor}(q_1,q_2)$ were defined in (8.28), (8.26).

A simple calculation gives

$$\{z^\alpha, z^\beta\},\ \{\overline{z}^\alpha, \overline{z}^\beta\} = 0,\ \{z^\alpha, \overline{z}^\beta\} = 2i\Big(\frac{\alpha_1\beta_1}{|z_1|^2} + \frac{\alpha_2\beta_2}{|z_2|^2}\Big) z^\alpha \overline{z}^\beta.$$

More generally,

$$\{z^\alpha\overline{z}^{\widetilde{\alpha}}, z^\beta\overline{z}^{\widetilde{\beta}}\} = 2i\Big(\frac{\sigma(\widetilde{\alpha}_1,\alpha_1;\widetilde{\beta}_1,\beta_1)}{|z_1|^2} + \frac{\sigma(\widetilde{\alpha}_2,\alpha_2;\widetilde{\beta}_2,\beta_2)}{|z_2|^2}\Big) z^{\alpha+\beta}\overline{z}^{\widetilde{\alpha}+\widetilde{\beta}},$$

where σ denotes the symplectic form, viewed as an alternate bilinear form on $T^*\mathbf{R}^2 \times T^*\mathbf{R}^2$. Hence

$$\mathrm{Cor}(z^\alpha\overline{z}^{\widetilde{\alpha}}, z^\beta\overline{z}^{\widetilde{\beta}}; s) = 2i\Big(\frac{\sigma(\widetilde{\alpha}_1,\alpha_1;\widetilde{\beta}_1,\beta_1)}{|z_1|^2} + \frac{\sigma(\widetilde{\alpha}_2,\alpha_2;\widetilde{\beta}_2,\beta_2)}{|z_2|^2}\Big) z^{\alpha+\beta}\overline{z}^{\widetilde{\alpha}+\widetilde{\beta}} e^{is(|\widetilde{\alpha}|-|\alpha|)},$$

when $|\widetilde{\alpha}|-|\alpha| = |\beta|-|\widetilde{\beta}|$, and $\mathrm{Cor}(z^\alpha\overline{z}^{\widetilde{\alpha}}, z^\beta\overline{z}^{\widetilde{\beta}}; s) = 0$ otherwise. If $a = (a_1,a_2)$, $b = (b_1,b_2) \in \mathbf{N}^2$ with $|a| = |b| = 4$ we get, by multinomial expansion,

(13.9)
$$\begin{aligned}
\mathrm{Cor}(x^a, x^b) &= \frac{1}{2^8}\mathrm{Cor}((z+\overline{z})^a, (z+\overline{z})^b; s)\\
&= \frac{1}{2^8}\sum_{\substack{\alpha+\widetilde{\alpha}=a\\ \beta+\widetilde{\beta}=b}} \binom{a}{\alpha}\binom{b}{\beta}\mathrm{Cor}(z^\alpha\overline{z}^{\widetilde{\alpha}}, z^\beta\overline{z}^{\widetilde{\beta}}; s)\\
&= \frac{2i}{2^8}\sum_{\substack{\alpha+\widetilde{\alpha}=a\\ \beta+\widetilde{\beta}=b\\ |\widetilde{\alpha}|-|\alpha|=|\beta|-|\widetilde{\beta}|}} \binom{a}{\alpha}\binom{b}{\beta}\Big(\frac{\sigma(\widetilde{\alpha}_1,\alpha_1;\widetilde{\beta}_1,\beta_1)}{|z_1|^2} + \frac{\sigma(\widetilde{\alpha}_2,\alpha_2;\widetilde{\beta}_2,\beta_2)}{|z_2|^2}\Big)\times\\
&\qquad\qquad \times z^{\alpha+\beta}\overline{z}^{\widetilde{\alpha}+\widetilde{\beta}} e^{is(|\widetilde{\alpha}|-|\alpha|)}.
\end{aligned}$$

When calculating this kind of expressions, it is useful to observe that the relations $|\alpha| + |\widetilde{\alpha}| = |\beta| + |\widetilde{\beta}| = 4$, $|\widetilde{\alpha}| - |\alpha| = |\beta| - |\widetilde{\beta}|$ imply $|\widetilde{\beta}| = |\alpha|$, $|\beta| = |\widetilde{\alpha}|$.

We have the Fourier series expansion

$$1_{[0,2\pi[}(s)(s-\pi) = \sum_{k\in\mathbf{Z}\setminus\{0\}} \frac{i}{k} e^{isk}.$$

Combining this with (13.9) and the Parseval identity, we get

$$\begin{aligned} &C(x^a, x^b) \\ &= \frac{1}{2\pi}\int_0^{2\pi} (s-\pi)\operatorname{Cor}(x^a, x^b; s) \\ &= \frac{2}{2^8} \sum_{\substack{\alpha+\widetilde{\alpha}=a \\ \beta+\widetilde{\beta}=b \\ |\widetilde{\alpha}|-|\alpha|=|\beta|-|\widetilde{\beta}| \\ |\widetilde{\alpha}|-|\alpha|\neq 0}} \frac{\binom{a}{\alpha}\binom{b}{\beta}}{|\widetilde{\alpha}|-|\alpha|}\Big(\frac{\sigma(\widetilde{\alpha}_1,\alpha_1;\widetilde{\beta}_1,\beta_1)}{|z_1|^2} + \frac{\sigma(\widetilde{\alpha}_2,\alpha_2;\widetilde{\beta}_2,\beta_2)}{|z_2|^2}\Big) z^{\alpha+\beta}\overline{z}^{\widetilde{\alpha}+\widetilde{\beta}} \end{aligned}$$

Using this formula we get after a few days of simple but tedious calculations:

$$\begin{aligned} C(x_1^4+x_2^4, x_1^4+x_2^4) &= -\frac{17}{16}(|z_1|^6+|z_2|^6), \\ C(x_1^4+x_2^4, x_1^2x_2^2) &= -\frac{3}{2^6}(3|z|^2(z_1^2\overline{z}_2^2+\overline{z}_1^2z_2^2)+16|z_1|^2|z_2|^2), \\ C(x_1^4+x_2^4, x_1^3x_2+x_1x_2^3) &= \frac{1}{2^7}\big(2(z_1^3\overline{z}_2^3+\overline{z}_1^3z_2^3)-(51(|z_1|^4+|z_2|^4) \\ &\qquad + 36|z_1|^2|z_2|^2)(z_1\overline{z}_2+\overline{z}_1z_2)\big), \\ C(x_1^2x_2^2, x_1^2x_2^2) &= -\frac{1}{2^6}|z|^2(9|z_1|^2|z_2|^2+8(z_1^2\overline{z}_2^2+\overline{z}_1^2z_2^2)), \\ C(x_1^2x_2^2, x_1^3x_2+x_1x_2^3) &= -\frac{1}{2^8}\big((17(|z_1|^4+|z_2|^4) \\ &\qquad + 90|z_1|^2|z_2|^2)(\overline{z}_1z_2+z_1\overline{z}_2)+12(\overline{z}_1^3z_2^3+z_1^3\overline{z}_2^3)\big), \\ C(x_1^3x_2+x_1x_2^3, x_1^3x_2+x_1x_2^3) &= -\frac{1}{2^8}\big(17(|z_1|^6+|z_2|^6)+153|z_1|^2|z_2|^2|z|^2 \\ &\qquad + 51|z|^2(z_1^2\overline{z}_2^2+\overline{z}_1^2z_2^2)\big). \end{aligned} \tag{13.10}$$

Now recall that q is given by (13.4), so that by (13.8), we have

$$\begin{aligned} &p_\epsilon|_{\Lambda_{\epsilon G}} \simeq p + i\epsilon(\langle q\rangle + i\epsilon f(r,a,b,c) + \mathcal{O}(\epsilon^2)) =: p + i\epsilon\widetilde{q}_\epsilon, \\ &f(r,a,b,c) = -\langle r\rangle + \frac{1}{2}\Big(\frac{4}{9}a^2 C(x_1^4+x_2^4, x_1^4+x_2^4) + b^2 C(x_1^2x_2^2, x_1^2x_2^2) \\ &\qquad + \frac{4}{9}c^2 C(x_1^3x_2+x_1x_2^3, x_1^3x_2+x_1x_2^3) + \frac{4ab}{3}C(x_1^4+x_2^4, x_1^2x_2^2) \\ &\qquad + \frac{8ac}{9}C(x_1^4+x_2^4, x_1^3x_2+x_1x_2^3) + \frac{4bc}{3}C(x_1^2x_2^2, x_1^3x_2+x_1x_2^3)\Big) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{13.11}$$

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According to (12.2) our reduced 1-dimensional operator has the symbol

$$Q_\epsilon = \widetilde{q}_\epsilon + \mathcal{O}\Big(h + \frac{h^{N_0}}{\epsilon}\Big).$$

Put $q_s = sQ_\epsilon + (1-s)\langle q\rangle$. If we assume that $\epsilon \gg h$, then we get according to (8.22):

$$\int_{\gamma_1(Q_\epsilon)} \xi\, dx - \int_{\gamma_1(\langle q\rangle)} \xi\, dx = -i\epsilon \int_{\gamma_1(\langle q\rangle)} [f(r,a,b,c)]_{\rho_c}^{(x(t),\xi(t))}\, dt + \mathcal{O}(\epsilon^2 + h), \tag{13.12}$$

where we recall that $\int_{\gamma_1(\langle q\rangle)} \xi\, dx$ is the (real) action along a loop in $\langle q\rangle = \text{Const} = \langle q\rangle(\rho_c)$ starting and ending at the saddle point ρ_c, and that $\int_{\gamma(Q_\epsilon)} \xi\, dx$ is the corresponding perturbed action for Q_ϵ. From (13.10), we see that $C(x_1^4 + x_2^4, x_1^4 + x_2^4)$ is minimal precisely on the horizontal circle $\rho = 1/2$. In the cases $B_\pm$, $E_\pm$, D the saddle points belong to $\{C_f, C_b\}$ situated on that circle. Since $\langle q\rangle$ is invariant under reflection in that circle, either the loop $\gamma_1(\langle q\rangle)$ is entirely in the upper or lower hemisphere intersecting the equator only at ρ_c (and this happens in the cases $B_\pm$ and for one of the saddles in case D) or $\gamma_1(\langle q\rangle)$ intersects the equator at one more point and is symmetric around the equator (and this happens in the cases $E_\pm$ and for one of the saddles in case D). In both cases we see that

$$\int_{\gamma_1(\langle q\rangle)} \big[C(x_1^4 + x_2^4, x_1^4 + x_2^4)\big]_{\rho_c}^{(x(t),\xi(t))}\, dt > 0.$$

Taking into account the form of f in (13.11), we conclude that for every r the integral in the left-hand side of (13.12) is $\neq 0$ except for (a,b,c) in a set of measure 0. For (a,b,c) outside that exceptional set, we conclude from the discussion at the end of Section 8 that the spectrum of the one dimensional localized operators has a genuinely two-dimensional structure.

A Proof of Proposition 6.2

To get a complete normal form we shall do further conjugations with analytic pseudodifferential operators of order 0 in such a way that the complete symbol also becomes a function of τ, ϵ, h^2/ϵ and $x\xi$. Moreover, we need to do so with errors that are $\mathcal{O}(e^{-1/(Ch)})$ (rather than merely $\mathcal{O}(h^\infty)$ as in [14]. Here Q is not a classical analytic symbol but it has a holomorphic realization and becomes a classical analytic symbol, if we allow some of the h-dependence to appear as an independent parameter in the coefficients of the h-asymptotic expansion. Thus, our starting point will be a symbol of the form

$$Q = Q_0(\tau, x\xi, \epsilon, h^2/\epsilon) + hQ_1(\tau, x, \xi, \epsilon, h^2/\epsilon; h), \tag{A.1}$$

where Q_1 is holomorphic and $\mathcal{O}(1)$ in some fixed complex neighborhood of $\tau = 0$, $x = \xi = 0$.

We define the ρ-quasi-norm as above, but now it is important that we work in the Weyl quantization. We then associate an analytic symbol a with the infinite order

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differential operator $A = \mathrm{Op}_a(x, \xi, D_{x,\xi}; h)$ as in (3.3). From the definition, we verify the following metaplectic invariance property. If $\kappa\colon \mathbf{C}^{2n} \to \mathbf{C}^{2n}$ is an affine linear canonical transformation and κ^*, κ_* denote the usual operations of pull-back and push-forward of functions on $\mathbf{C}^{2n}$, then $\kappa_* \mathrm{Op}_a \kappa^* = \mathrm{Op}_{\kappa_* a}$. This implies that if we define our quasi-norms with the help of a family of opens sets Ω_t which are invariant under κ, then $|||\kappa_* a|||_\rho = |||a|||_\rho$.

In the case of (A.1), we shall let Ω_t be of the form $|x|^2 + |\xi|^2 \leq r(t)$ for a suitable $r(t)$, and we observe that these balls are invariant under $\exp isH_{x\xi}$ when s is real. After applying the inverse function of $Q_0(\tau, \cdot, \epsilon, h^2/\epsilon)$ to our operator, we may assume that the principal symbol of Q is $x\xi$, so (A.1) simplifies to

$$Q = x\xi + hQ_1(\tau, x, \xi, \epsilon, h^2/\epsilon; h) \tag{A.2}$$

with a new Q_1 having the same properties as the previous one.

Using the same letters for operators and their symbols, we let $Q_0 = \frac{1}{2}(xhD + hDx)$ be the quantization of $x\xi$. Notice that $\exp(2\pi Q_0/h) = -1$, so $\exp(2\pi \,\mathrm{ad}_{Q_0}/h) = 1$. If B is an analytic h-pseudodifferential operator of order 0, we put

$$\langle B \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{tQ_0/h} B e^{-tQ_0/h}\, dt,$$

and notice that on the symbol level,

$$\langle B \rangle = \frac{1}{2\pi} \int_0^{2\pi} B \circ \exp itH_{x\xi}\, dt.$$

Also notice that $[Q_0, \langle B \rangle] = 0$. Choosing the ρ-quasi-norms as above, we further have that $|||\langle B \rangle|||_\rho \leq |||B|||_\rho$.

The equation, $\mathrm{ad}_{Q_0} A = B - \langle B \rangle$ has the solution

$$A = \frac{1}{h} \int k(t) e^{tQ_0/h} B e^{-tQ_0/h}\, dt = \frac{1}{h} \int k(t) e^{tQ_0/h} (B - \langle B \rangle) e^{-tQ_0/h}\, dt,$$

where $k(t)$ is the function with support in $[-\pi, \pi]$ which is affine on $[-\pi, 0[,]0, \pi]$ with $k(\pm\pi) = 0$, $k(\pm 0) = \mp\frac{1}{2}$. We have $|||A|||_\rho \leq C|||\frac{1}{h}(B - \langle B \rangle)|||_\rho$.

As in Section 3, we see that the map $A \mapsto \mathrm{Ad}_A(Q)$ has the differential

$$\delta A \mapsto \mathrm{ad}_{\delta A} Q_0 + \widetilde{K}(A, \delta A), \tag{A.3}$$

where

$$|||\widetilde{K}(A, \delta A)|||_\rho \leq C\rho(h + |||A|||_\rho)|||\delta A|||_\rho, \tag{A.4}$$

under the assumption that $|||A|||_\rho = \mathcal{O}(1)$.

Now return to (A.2). After a first conjugation, we may reduce ourselves to the case when $Q_1 - \langle Q_1 \rangle$ is $\mathcal{O}(h)$, so that $|||Q_1 - \langle Q_1 \rangle|||_\rho \leq Ch$, for some $C > 0$, when $\rho \leq \rho_0 > 0$. We look for A such that $\mathrm{Ad}_A Q$ commutes with Q_0, and we try $A = \sum_0^\infty A_j$

with convergence in some ρ-quasi-norm. Start by solving $[A_0, Q_0] + hQ_1 = h\langle Q_1\rangle$, with $|||A_0|||_\rho \leq C|||Q_1 - \langle Q_1\rangle|||_\rho \leq \mathcal{O}(h)$. From (A.3) and (A.4) we get $\mathrm{Ad}_{A_0} Q_0 = Q_0 + h\langle Q_1\rangle + hQ_2$,

$$\begin{aligned}|||Q_2|||_\rho &\leq h^{-1}C\rho(h + C|||Q_1 - \langle Q_1\rangle|||_\rho)|||Q_1 - \langle Q_1\rangle|||_\rho\\ &\leq C\rho(1 + C^2)|||Q_1 - \langle Q_1\rangle|||_\rho\\ &\leq \tfrac{1}{4}|||Q_1 - \langle Q_1\rangle|||_\rho,\end{aligned}$$

where the last estimate holds for $0 \leq \rho \leq \rho_0$, with $\rho_0 > 0$ small enough. Choose A_1 with

$$[A_1, Q_0] + hQ_2 = h\langle Q_2\rangle,\ |||A_1|||_\rho \leq C|||Q_2 - \langle Q_2\rangle|||_\rho \leq \frac{C}{2}|||Q_1 - \langle Q_1\rangle|||_\rho.$$

Then $\mathrm{Ad}_{A_0+A_1} Q = Q_0 + h\langle Q_1\rangle + h\langle Q_2\rangle + hQ_3$, with $|||Q_3|||_\rho \leq 2^{-2}|||Q_1 - \langle Q_1\rangle|||_\rho$. Iterating the procedure, we get A_j with $|||A_j|||_\rho \leq C2^{-j}|||Q_1 - \langle Q_1\rangle|||_\rho$, such that if $A = \sum_0^\infty A_j$, then

$$\mathrm{Ad}_A Q = Q_0 + h\langle Q_1\rangle + h\langle Q_2\rangle + \cdots,\ |||\langle Q_j\rangle|||_\rho \leq 2^{-j}|||Q_1 - \langle Q_1\rangle|||_\rho.$$

The previous discussion shows how to find U so that modulo an error $\mathcal{O}(e^{-1/(Ch)})$, $U^{-1}QU$ commutes with xhD_x. Moreover $U^{-1}QU$ is a classical analytic pseudodifferential operator (after allowing h as an independent parameter in the coefficients in the asymptotic expansions). Put $x = e^s$ and work near $x = r$ for some fixed small $r > 0$. Then $xhD_x = hD_s$ and since the class of analytic pseudodifferential operators is conserved under analytic changes of variables, we know that

$$U^{-1}QU = K_{\epsilon,h^2/\epsilon}(\tau, s, hD_s; h),$$

where K is an analytic symbol. But $[K, hD_s] = 0$, so $K = K_{\epsilon,h^2/\epsilon}(\tau, hD_s; h)$ and returning to the x-coordinates, we get the representation (6.6). ■

B Study of $\Gamma_{j,k}$

For simplicity, we restrict the attention to the right half-plane, $\mathrm{Re}\,\mu \geq 0$ and pick one of the equations in (8.5) that we write

$$(\mathrm{Im}\,\mu)\ln\frac{1}{|\mu|} = F(\mu), \tag{B.1}$$

where $F(\mu)$ is uniformly Lipschitz in a neighborhood of 0. As we have already observed,

$$\partial_{\mathrm{Im}\,\mu}\Big(\mathrm{Im}\,\mu\ln\frac{1}{|\mu|}\Big) = \ln\frac{1}{|\mu|} - \Big(\frac{\mathrm{Im}\,\mu}{|\mu|}\Big)^2 \gg 1, \tag{B.2}$$

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so (B.1) determines a curve of the form

$$\operatorname{Im}\mu = f(\operatorname{Re}\mu), \text{ where } f'(\operatorname{Re}\mu) = \mathcal{O}\Big(\frac{1}{\ln 1/|(\operatorname{Re}\mu, f(\operatorname{Re}\mu))|}\Big) \ll 1.$$

We want to express f in terms of $F(\operatorname{Re}\mu)$ up to small errors.

Let us first compare the solution μ of (B.1) with the solution $\widetilde{\mu}$ of the simplified equation

$$\operatorname{Im}\widetilde{\mu}\ln\frac{1}{|\widetilde{\mu}|} = F(\operatorname{Re}\mu), \text{ with } \operatorname{Re}\widetilde{\mu} = \operatorname{Re}\mu. \tag{B.3}$$

Using that $F(\mu) - F(\operatorname{Re}\mu) = \mathcal{O}(\operatorname{Im}\mu)$ together with (B.2), we see that

$$\operatorname{Im}\mu - \operatorname{Im}\widetilde{\mu} = \mathcal{O}\Big(\frac{\operatorname{Im}\mu}{\ln\frac{1}{|\mu|}}\Big), \text{ so } \operatorname{Im}\mu \sim \operatorname{Im}\widetilde{\mu}, \ln\frac{1}{|\mu|} \sim \ln\frac{1}{|\widetilde{\mu}|}. \tag{B.4}$$

With this estimate in mind, we now concentrate on the simplified equation (B.3), and we drop the tildes for simplicity.

Assume first that we are in the region

$$|\operatorname{Im}\mu| \leq \mathcal{O}(\operatorname{Re}\mu). \tag{B.5}$$

Then

$$\ln\frac{1}{|\mu|} = \ln\Big(\frac{1}{x}\Big)\Big(1 + \mathcal{O}\Big(\Big(\frac{y}{x}\Big)^2\frac{1}{\ln 1/x}\Big)\Big),$$

where we write $\mu = x + iy$. Thus, if $\mu = \widetilde{\mu}$ solves (B.3) and (B.5) holds, then we first see that

$$y \sim \frac{F(x)}{\ln\frac{1}{x}},$$

and then that

$$y = \frac{F(x)}{\ln\frac{1}{x}}\Big(1 + \mathcal{O}(1)\Big(\frac{F(x)}{x\ln\frac{1}{x}}\Big)^2\frac{1}{\ln\frac{1}{x}}\Big). \tag{B.6}$$

So, if we assume

$$|F(x)| \leq \mathcal{O}(1)x\ln\frac{1}{x}, \tag{B.7}$$

then we are in the region (B.5), and the solution $\mu = \widetilde{\mu} = x + iy$ of (B.3) takes the form (B.6). Combining with (B.4), we get under the assumption (B.7),

$$f(x) = \Big(1 + \frac{\mathcal{O}(1)}{\ln\frac{1}{x}}\Big)\frac{F(x)}{\ln\frac{1}{x}}.$$

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We next consider the region $x \ll |y| \ll 1$ and assume for simplicity that we have $y > 0$. Then

$$\ln \frac{1}{|\mu|} = \ln\Big(\frac{1}{y}\Big)\Big(1 + \mathcal{O}(1)\Big(\frac{x}{y}\Big)^2 \frac{1}{\ln \frac{1}{y}}\Big),$$

and (B.3) takes the form

$$y\Big(\ln \frac{1}{y}\Big)\Big(1 + \mathcal{O}(1)\Big(\frac{x}{y}\Big)^2 \frac{1}{\ln \frac{1}{y}}\Big) = F(x). \tag{B.8}$$

Consider first the simplified problem

$$y \ln \frac{1}{y} = z. \tag{B.9}$$

With $Y = \ln(1/y)$, $Z = \ln(1/z)$ (both $\gg 1$) we get

$$Y - \ln Y = Z \tag{B.10}$$

Try the approximate solution $Y_0 = Z + \ln Z$. Then by a simple calculation,

$$Y_0 - \ln Y_0 = Z + \mathcal{O}\Big(\frac{\ln Z}{Z}\Big).$$

Since the derivative of the left-hand side in (B.10) is close to 1, we see that the solution Y of that equation is of the form $Y = Y_0 + \mathcal{O}((\ln Z)/Z)$;

$$Y = Z + \Big(1 + \mathcal{O}\Big(\frac{1}{Z}\Big)\Big) \ln Z.$$

Hence the solution of (B.9) is of the form

$$y = \Big(1 + \mathcal{O}\Big(\frac{\ln \ln \frac{1}{z}}{\ln \frac{1}{z}}\Big)\Big) \frac{z}{\ln \frac{1}{z}}. \tag{B.11}$$

If we replace z by $F(x)$, we get the order of magnitude of the solution to (B.8):

$$y \sim \frac{F(x)}{\ln \frac{1}{F(x)}}, \tag{B.12}$$

and the assumption that $x \ll y$ reads:

$$\frac{F(x)}{\ln \frac{1}{F(x)}} \gg x. \tag{B.13}$$

The earlier arguments show that this condition is equivalent to

$$F(x) \gg x \ln \frac{1}{x}, \tag{B.14}$$

which indeed is complementary to (B.7). Using (B.12), we get

$$\Big(\frac{x}{y}\Big)^2\frac{1}{\ln\frac{1}{y}} \le \mathcal{O}(1)\Big(\frac{x\ln\frac{1}{F(x)}}{F(x)}\Big)^2\frac{1}{\ln\frac{\ln 1/F}{F}} \le \mathcal{O}(1)\Big(\frac{x\ln\frac{1}{F(x)}}{F(x)}\Big)^2\frac{1}{\ln\frac{1}{F(x)}},$$

where we notice that

$$\frac{x\ln\frac{1}{F}}{F} \ll 1,$$

by (B.13). Hence (B.8) gives

$$y\ln\frac{1}{y} = \Big(1+\mathcal{O}(1)\Big(\frac{x\ln 1/F(x)}{F(x)}\Big)^2\frac{1}{\ln 1/F(x)}\Big)F(x), \tag{B.15}$$

and applying (B.11) with z equal to the right-hand side of (B.15), we get

$$y = \Big(1+\mathcal{O}(1)\frac{\ln\ln 1/F}{\ln 1/F}\Big)\frac{\big(1+\mathcal{O}(1)\big(\frac{x\ln 1/F}{F}\big)^2\frac{1}{\ln 1/F}\big)F(x)}{\big(\ln\frac{1}{F}+\mathcal{O}(1)\big(\frac{x\ln 1/F}{F}\big)^2\frac{1}{\ln 1/F}\big)},$$

which simplifies to

$$y = \Big(1+\mathcal{O}(1)\frac{\ln\ln 1/F}{\ln 1/F}\Big)\frac{F(x)}{\ln 1/F(x)}. \tag{B.16}$$

Recall that here $\widetilde{\mu} = x+iy$ in the simplified equation. To get the corresponding result for (B.1), we apply (B.1) and conclude that (B.16) holds for $\mu = x+iy$ solving (B.1), under the equivalent conditions (B.13), (B.14), and assuming also $0 \le F(x) \ll 1$, $0 \le x \ll 1$.

Summing up, we have proved the following.

Proposition B.1 *Let F be a uniformly Lipschitz function with $|F| \ll 1$, defined in a neighborhood of $0 \in \mathbf{C}$. Let $\mu = x + if(x)$ be the solution of (B.1). Then for small x, we have*

$$|f'(x)| \le \mathcal{O}(1)/\ln(1/|x+if(x)|). \tag{B.17}$$

Further,

$$f(x) = \Big(1+\frac{\mathcal{O}(1)}{\ln(1/|x|)}\Big)\frac{F(x)}{\ln(1/|x|)}, \qquad \textit{when } |F(x)| \le \mathcal{O}(1)|x|\ln(1/|x|),$$

$$f(x) = \Big(1+\mathcal{O}(1)\frac{\ln\ln(1/|F(x)|)}{\ln(1/|F(x)|)}\Big)\frac{F(x)}{\ln(1/|F(x)|)}, \quad \textit{when } |F(x)| \gg |x|\ln(1/|x|),$$

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