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ON ALMOST-HERMITE-FEJÉR-INTERPOLATION: POINTWISE ESTIMATES

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We give a brief survey of the results obtained by numerous authors in so-called almost-Hermite-Fejér-interpolation and deal mainly with new quantitative assertions.

These are based upon more general theorems for certain continuous linear operators which yield estimates involving different types of moduli of continuity.

Our paper shows that in the case of almost-Hermite-Fejér-interpolation the underlying general technique can be used to treat three essentially different cases: sequences of positive operators, which converge uniformly for every continuous function on $[-1, 1]$, sequences of non-positive operators doing the same, and sequences of operators which converge on proper subspaces of $C[-1, 1]$ only.

1. Introduction

The image of a function $f \in C[-1, 1]$ under an (r, s)-Hermite-Fejér interpolation operator

$$F_{r,s;n} : C[-1, 1] \to \pi_{2n+r+s-1}$$

is the uniquely determined algebraic polynomial satisfying for a given

Received 4 January 1982. The author gratefully acknowledges H.B. Knoop for providing him with a preliminary form of his work cited under [9]. This paper treats - mainly from a non-quantitative point of view - the movement of the uniform convergence hexagon in 2-space as r and s vary.

sequence of nodes

$$1 = x_0 > x_1 > \dots > x_n > x_{n+1} = -1$$

the $2n + r + s$ conditions

$$F_{r,s;n}(f, x_k) = f(x_k) \,, \qquad (F_{r,s;n}f)'(x_k) = 0 \,, \quad 1 \le k \le n \,;$$

$$F_{r,s;n}(f, 1) = f(1) \quad \text{for } r \ge 1 \,, \quad (F_{r,s;n}f)^{(\rho)}(1) = 0 \,, \quad 1 \le \rho \le r-1 \,;$$

$$F_{r,s;n}(f, -1) = f(-1) \quad \text{for } s \ge 1 \,, \quad (F_{r,s;n}f)^{(\sigma)}(-1) = 0 \,, \quad 1 \le \sigma \le s-1 \,.$$

Because of the position of the nodes $x_1, x_2, \dots, x_n \in (-1, 1)$ it is natural to investigate (r, s)-Hermite-Fejér processes based on the two endpoints ± 1 and the roots $x_1, \dots, x_n$ of the Jacobi polynomials $P_n^{(\alpha,\beta)}$, $\alpha, \beta > -1$. This has been done by numerous authors (see, for example, Vértesi [20], Knoop [9], the references therein, and the bibliography in Mills' paper [11]). The corresponding operators are denoted by $F_{r,s;n}^{(\alpha,\beta)}$.

In the present paper we treat one particular case, namely $(r, s) = (1, 0)$, from a quantitative point of view. The corresponding operators $F_{1,0;n}^{(\alpha,\beta)}$ are referred to as so-called almost-Hermite-Fejér-interpolation operators and given by the formula

$$F_{1,0;n}^{(\alpha,\beta)}(f, x) = f(1) \cdot \frac{w(x)^2}{w(1)^2} + \sum_{k=1}^{n} f(x_k) \cdot \frac{1-x}{1-x_k} \left[1+c_k^*(x-x_k)\right] \cdot l_k(x)^2 \,,$$

where l_k denotes the kth Lagrange fundamental polynomial and

$$w(x) = \prod_{k=1}^{n} (x-x_k) \,.$$

Moreover,

$$c_k^* = \frac{1}{1-x_k} - \frac{w''(x_k)}{w'(x_k)} \,.$$

If $1 + c_k^*(x-x_k) \ge 0$ for all $x \in [-1, 1]$, then obviously the above

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operator is positive and linear. If $x_1, \ldots, x_n$ are the zeros of a Jacobi polynomial $P_n^{(\alpha,\beta)}$, then this is the case for all n if and only if $(\alpha, \beta) \in [0, 1] \times (-1, 0]$.

In order to give a brief survey of known results concerning almost-Hermite-Fejer-interpolation operators we choose to represent them in Diagram 1 (p. 408). The resulting pentagon encloses those values of (α, β) for which $F_{1,0;n}^{(\alpha,\beta)}f$ converges uniformly to f for all $f \in C[-1, 1]$. For values of (α, β) , with $\alpha, \beta > -1$, not enclosed in the pentagon or lying on the dotted lines there is no uniform convergence in general which is abbreviated by 'Divergence' in the diagram.

The known quantitative theorems concerning the differences $\left|F_{1,0;n}^{(\alpha,\beta)}(f, x)-f(x)\right|$ cover the approximation of arbitrary continuous functions only, using the first order modulus of continuity of the function under consideration.

It is our aim to prove - with the aid of much more general estimates in terms of a certain K-functional Ω - pointwise improvements and modifications of these results. These will involve for instance the first order modulus of continuity of the first derivative denoted by $\omega_1(f', \cdot)$ or the second order modulus of continuity $\omega_2(f, \cdot)$.

The proofs will be given for $(\alpha, \beta) = (\frac{1}{2}, -\frac{1}{2})$, $(\alpha, \beta) = (-\frac{1}{2}, -\frac{1}{2})$, and $(\alpha, \beta) = (0, 0)$ only, since these are the three essentially different cases. For $(\alpha, \beta) = (\frac{1}{2}, -\frac{1}{2})$ the corresponding operators are positive and converge uniformly for every $f \in C[-1, 1]$; for $(\alpha, \beta) = (-\frac{1}{2}, -\frac{1}{2})$ the operators are not positive but nevertheless there is uniform convergence on the whole space $C[-1, 1]$. Setting $(\alpha, \beta) = (0, 0)$ leads to a sequence of positive linear operators which converges only on a proper subspace. This subspace fails to contain the testfunction e_1 with $e_1(x) = x$; thus in this case it would be useless to apply a quantitative version of Korovkin's well-known result in order to arrive at a quantitative assertion.

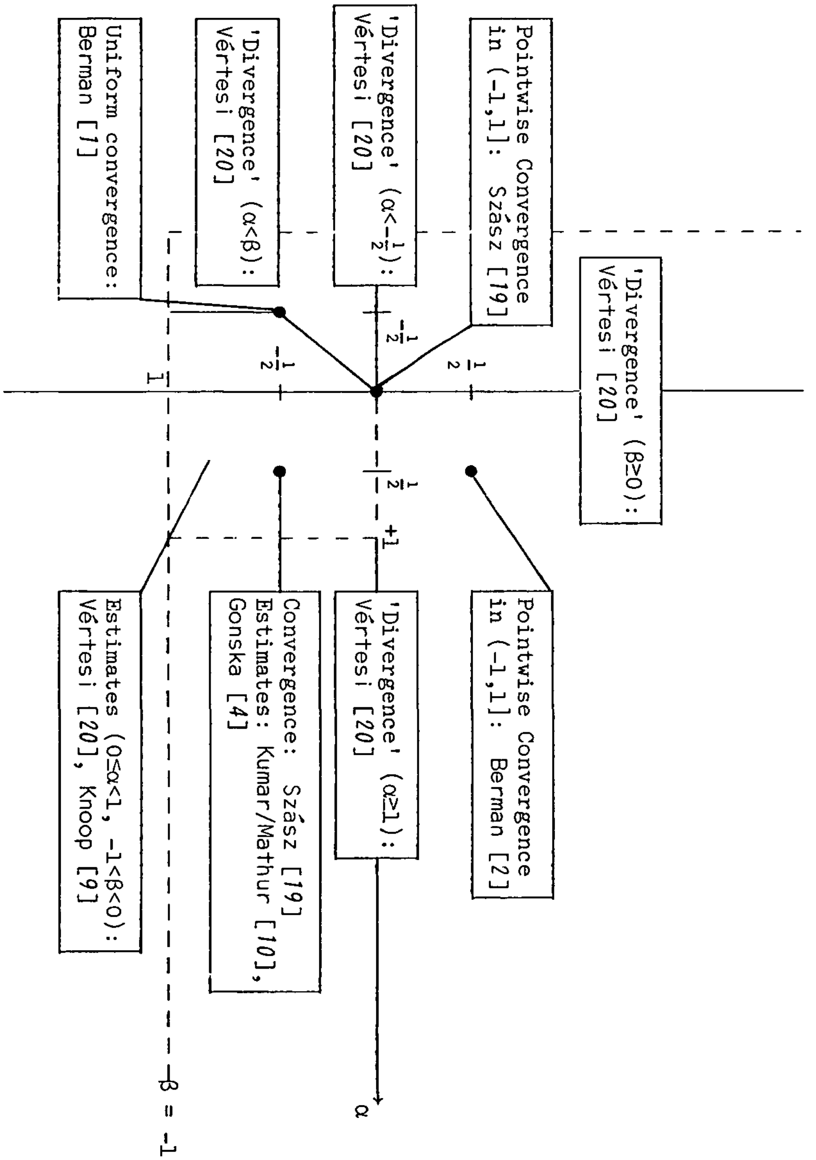
'Divergence' (β≥0): Vértesi [20]
Pointwise Convergence in (-1,1]: Szász [19]
'Divergence' (α<-½): Vértesi [20]
'Divergence' (α<β): Vértesi [20]
Uniform convergence: Berman [1]
Pointwise Convergence in (-1,1]: Berman [2]
'Divergence' (α≥1): Vértesi [20]
Convergence: Szász [19]
Estimates: Kumar/Mathur [10], Gonska [4]
Estimates (0≤α<1, -1<β<0): Vértesi [20], Knoop [9]
α
β = -1

DIAGRAM 1.

2. Basic estimates

We first sketch how to construct the functional $\Omega : C[a, b] \times \mathbb{R}_+^3 \to \mathbb{R}_+$ mentioned above. Let E denote a real vector space, U a subspace of E, and p and $\bar{p}$ seminorms on E and U, respectively. We define $\tilde{K} : \mathbb{R}_+^2 \times E \to \mathbb{R}_+$ by

$$\tilde{K}(t_1, t_2, f; (E, p), (U, \bar{p})) := \inf\{p(f-g)+t_1 p(g)+t_2\bar{p}(g) : g \in U\}$$

and $K : \mathbb{R}_+ \times E \to \mathbb{R}_+$ by

$$K(t, f; (E, p), (U, \bar{p})) := \inf\{p(f-g)+t\cdot\bar{p}(g) : g \in U\} .$$

We write for simplification $\tilde{K}(t_1, t_2, f)$ and $K(t, f)$, respectively, if it is clear what (E, p) and $(U, \bar{p})$ are. It is readily verified that for fixed (t_1, t_2) the functional $\tilde{K}(t_1, t_2, \cdot)$ is a seminorm on E; thus it makes sense to use it as the seminorm $\bar{p}$ when defining a functional $K(t, \cdot)$.

We now consider the spaces $C^i[a, b]$, $i \in \{1, 2\}$, of i-times continuously differentiable functions defined on the finite interval $[a, b]$ with the seminorms $\|\cdot^{(i)}\|_\infty$, where $\cdot^{(i)}$ denotes i-fold differentiation. Then for fixed t_1, t_2 the seminorm

$$\tilde{K}\left(t_1, t_2, f; \left(C^1[a, b], \|\cdot^{(1)}\|_\infty\right), \left(C^2[a, b], \|\cdot^{(2)}\|_\infty\right)\right)$$

may be used in the definition of

$$\Omega(f; t, t_1, t_2) := K\left(t, f; (C[a, b], \|\cdot\|_\infty), \left(C^1[a, b], \tilde{K}(t_1, t_2, \cdot)\right)\right), \quad t \geq 0, \; f \in C[a, b] .$$

We cite three theorems concerning Ω which will be used below. The proofs can be found in [4], [5], and [6].

THEOREM 2.1. *Let Ω be defined as above. Then the following inequalities are true for any* $(f; t, t_1, t_2) \in C[a, b] \times \mathbb{R}_+^3$, $0 < h \leq b-a$:

$$\Omega(f;\, t,\, t_1,\, t_2) \le \begin{cases} \left(1 + \frac{t}{h}\right) \cdot \omega_1(f,\, h)\ , & \text{(i)} \\ t \cdot t_1 \cdot \|f'\|_\infty + \left(t + \frac{t \cdot t_2}{h}\right) \cdot \omega_1(f',\, h)\ , & \text{(ii)} \\ \left(\frac{3}{2} + \frac{2tt_2}{h^2}\right) \cdot \omega_2(f,\, h) + \frac{2tt_1}{h} \cdot \omega_1(f,\, h)\ , & \text{(iii)} \\ \frac{1}{2} \cdot \omega_1^*(f,\, 2t)\ , & \text{(iv)} \\ t \cdot t_1 \cdot \|f'\|_\infty + t \cdot \omega_1^*(f',\, 2t_2)\ , & \text{(v)} \end{cases}$$

where, for $k = 1, 2$, $\omega_k(f, \cdot)$ *denotes the* kth *order modulus of continuity of* f , $\omega_1^*(f, \cdot)$ *is the least concave majorant of* $\omega_1(f, \cdot)$ *and (ii) and (v) are valid for* $f \in C^1[a, b]$ *only.*

THEOREM 2.2. *Let* $L : C[a, b] \to C[a, b]$ *be continuous and linear, satisfying, for all* $x \in [a, b]$,

(i) $|L(f_1, x) - f_1(x)| \le \phi(x) \cdot \|f_1'\|_\infty$ *for all* $f_1 \in C^1[a, b]$ *with a function* $\phi \ge 0$, *and*

(ii) $|L(f_2, x) - f_2(x)| \le \gamma_1(x) \cdot \|f_2'\|_\infty + \gamma_2(x) \cdot \|f_2''\|_\infty$ *for all* f_2 *in* $C^2[a, b]$ *with* $\gamma_1, \gamma_2 \ge 0$.

Moreover, we suppose for all $x \in [a, b]$ *that the quotients* $\gamma_i(x)/\phi(x)$, $i = 1, 2$, *are finite. Then for every* $f \in C[a, b]$ *and every* $x \in [a, b]$ *we have the inequality*

$$|L(f,\, x) - f(x)| \le (\|L\|+1) \cdot \Omega\left(f;\, \frac{\phi(x)}{\|L\|+1},\, \frac{\gamma_1(x)}{\phi(x)},\, \frac{\gamma_2(x)}{\phi(x)}\right) .$$

THEOREM 2.3. *Let* $L : C[a, b] \to C[a, b]$ *be a positive linear operator satisfying* $L(e_0, \cdot) = e_0$, *where* e_i *denotes the* ith *monomial. Then for any* $f \in C[a, b]$ *and all* $x \in [a, b]$ *the estimate*

$$|L(f,\, x) - f(x)| \le 2 \cdot \Omega\left(f;\, \frac{L(|e_1 - x|, x)}{2},\, \frac{|L(e_1 - x, x)|}{L(|e_1 - x|, x)},\, \frac{L\left((e_1 - x)^2, x\right)}{2L(|e_1 - x|, x)}\right)$$

holds. It remains true if one or more of the three 'differences' occurring

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on the right side are replaced by majorants in a way such that the quotients remain finite.

3. Pointwise estimates for $F^{(\frac{1}{2},-\frac{1}{2})}_{1,0;n}$

For the above choice of (α, β) the operators have the form (see Szász [19])

$$F^{(\frac{1}{2},-\frac{1}{2})}_{1,0;n}(f, x) = f(1)\,\frac{w(x)^2}{w(1)^2} + \sum_{k=1}^{n} f(x_k)\cdot\frac{1-x}{1-x_k}\cdot\frac{1-xx_k}{1-x_k^2}\cdot l_k(x)^2 ,$$

where

$$w(x) = \sin\frac{2n+1}{2}\arccos x/\sin\tfrac{1}{2}\arccos x , \quad x_k = \cos\frac{2k}{2n+1}\pi , \quad 1 \le k \le n ,$$

and l_k is the kth Lagrange fundamental polynomial. As can be seen from the representation given above, the operators are positive; thus Theorem 2.3 will be the main tool for proving the following

THEOREM 3.1. *For the operators* $F^{(\frac{1}{2},-\frac{1}{2})}_{1,0;n}$ *based upon the roots of the Jacobi polynomials* $P^{(\frac{1}{2},-\frac{1}{2})}_{n}$ *and the endpoint* 1 *, we have for all* $n \ge 2$ *, all* $f \in C[-1, 1]$ *and every* $x \in [-1, 1]$ *,*

(i) $$\left|F^{(\frac{1}{2},-\frac{1}{2})}_{1,0;n}(f, x)-f(x)\right| \le 8\cdot\left\{\omega_2\left(f, (1-x)^{\frac{1}{2}}\cdot|w(x)|\cdot n^{-\frac{1}{2}}\right)+n^{-\frac{1}{2}}\cdot\omega_1\left(f, (1-x)^{\frac{1}{2}}\cdot|w(x)|\cdot n^{-\frac{1}{2}}\right)\right\}.$$

Provided that f *is in* $C^1[-1, 1]$ *we have in addition*

(ii) $$\left|F^{(\frac{1}{2},-\frac{1}{2})}_{1,0;n}(f, x)-f(x)\right| \le \frac{c}{n}\cdot\left\{(1-x)^{\frac{1}{2}}\cdot|w(x)|\cdot\|f'\|_\infty+s_n(x)\cdot\omega_1\left[f', (1-x)w(x)^2/s_n(x)\right]\right\} ,$$

where $s_n(x) := 1 + (1-x^2)^{\frac{1}{2}}\cdot\ln n$ *and* c *denotes a positive real number independent of* f, x *and* n *.*

Proof. In order to apply Theorem 2.3 we first note that $F^{(\frac{1}{2},-\frac{1}{2})}_{1,0;n}(e_0, x) = 1$. Moreover, one has to evaluate the operator for the

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functions $|e_1-x|$, $(e_1-x)^2$ and $e_1 - x$, $x \in [-1, 1]$. As stated in Remark 4 in the paper of Kumar and Mathur [10], an analogue of their Theorem 1 holds for the operators $F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}$. This implies the inequality

$$F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}(|e_1-x|, x) \le c \cdot \frac{1+(1-x^2)^{\frac{1}{2}}\cdot \ln n}{2n+1} ,$$

where c is a suitable constant.

In order to find a majorant for $F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}\left((e_1-x)^2, x\right)$ it is of advantage to make use of the operator's representation as given at the beginning of this section. As shown in [4, p. 176] one arrives at

$$F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}\left((e_1-x)^2, x\right) = \frac{2(1-x)\cdot w(x)^2}{3n} .$$

The third quantity was also investigated in [4]. We have

$$\left|F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}(e_1-x, x)\right| = \frac{1}{(2n+1)^2} \cdot |(1-x)\cdot w(x)\cdot [2(1-x^2)\cdot w'(x)+(2nx-1)w(x)]|$$

$$\le \frac{4(1-x)^{\frac{1}{2}}\cdot |w(x)|}{n} , \quad n \ge 2 .$$

Theorem 2.3 now implies a general estimate for the difference $\left|F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}(f, x)-f(x)\right|$ in terms of Ω , and statement *(iii)* in Theorem 2.1 yields

$$\left|F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}(f, x)-f(x)\right|$$

$$\le 2 \cdot \left\{\left(\frac{3}{2} + \frac{(1-x)\cdot w(x)^2}{h^2\cdot 3n}\right) \cdot \omega_2(f, h) + \frac{4(1-x)^{\frac{1}{2}}\cdot |w(x)|}{h\cdot n} \cdot \omega_1(f, h)\right\} ,$$

$$0 < h \le 2 .$$

The choice $h = (1-x)^{\frac{1}{2}} \cdot |w(x)| \cdot n^{-\frac{1}{2}}$ now leads to the first statement in Theorem 3.1 if the latter quantity is greater than 0 . Obviously the assertion is also true if $(1-x)^{\frac{1}{2}} \cdot |w(x)| \cdot n^{-\frac{1}{2}} = 0$.

In order to obtain the estimate for continuously differentiable functions we use statement *(ii)* in Theorem 2.1 $(0 < h \le 2, n \ge 2)$. This gives

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$$\left|F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}(f, x)-f(x)\right|$$
$$\le \left|F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}(e_1-x, x)\right| \cdot \|f'\|_\infty$$
$$+ \left[F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}(|e_1-x|, x) + \frac{1}{2h} \cdot F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}\left((e_1-x)^2, x\right)\right] \cdot \omega_1(f', h)$$
$$\le \frac{4(1-x)^{\frac{1}{2}} \cdot |w(x)|}{n} \cdot \|f'\|_\infty$$
$$+ \left[c \cdot \frac{1+(1-x^2)^{\frac{1}{2}} \cdot \ln n}{2n+1} + \frac{(1-x)\cdot w(x)^2}{h \cdot 3n}\right] \cdot \omega_1(f', h) \ .$$

Putting $h = \left((1-x)\cdot w(x)^2\right)/\left(1+(1-x^2)^{\frac{1}{2}} \cdot \ln n\right)$ we arrive at the second estimate in Theorem 3.1 provided that $h > 0$; the estimate remains true if $(1-x) \cdot w(x)^2 = 0$.

As an immediate consequence of Theorem 3.1 we obtain the following uniform estimates.

COROLLARY 3.2. *For the operators* $F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}$ *the following inequalities are valid* $(f \in C[-1, 1]$ *and* $C^1[-1, 1]$, *respectively,* $n \ge 2$)*:*

(i) $$\left\|F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}f-f\right\|_\infty \le 8 \cdot \left\{\omega_2\left(f, (2/n)^{\frac{1}{2}}\right)+n^{-\frac{1}{2}}\cdot\omega_1\left(f, (2/n)^{\frac{1}{2}}\right)\right\} ;$$

(ii) $$\left\|F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})}f-f\right\|_\infty \le \frac{c}{n} \left\{\|f'\|_\infty + \ln n \cdot \omega_1\left(f', 1/(\ln n)\right)\right\} .$$

Proof. Inequality *(i)* is obtained by observing that $(1-x)^{\frac{1}{2}} \cdot |w(x)| \le 2^{\frac{1}{2}}$. The second estimate is obtained by using $(1-x)w(x)^2 \le 2$ and putting $h = 1/(\ln n)$ in the proof of Theorem 3.1.

The most interesting consequence of Theorem 3.1 is contained in its following corollary; it shows how smoothness properties of f' accelerate the order of approximation $O(\ln n/n)$, $n \to \infty$, which we have for any $f \in C^1[-1, 1]$.

COROLLARY 3.3. *If* $f \in C^1[-1, 1]$ *such that* $f' \in \mathrm{Lip}\,\alpha$, $0 < \alpha \le 1$, *then*

$$\left\| F_{1,0;n}^{(\frac{1}{2},-\frac{1}{2})} f - f \right\|_\infty = O\left((\ln n)^{1-\alpha}/n\right) \ , \quad n \to \infty \ .$$

REMARK 3.4. It would also have been possible to give estimates involving $\omega_1(f, \cdot)$, $\omega_1^*(f, \cdot)$ or $\omega_1^*(f', \cdot)$ by using Theorem 2.1. This was omitted since estimates using $\omega_1(f, \cdot)$ are known (see, for example, Vértesi [20]) and since estimates using the least concave majorants are only interesting when determining optimal constants (see, for example, [7]).

4. Pointwise estimates for $F_{1,0;n}^{(-\frac{1}{2},-\frac{1}{2})}$

Since $F_{1,0;n}^{(\alpha,\beta)}$ is a positive operator for all n if and only if $(\alpha, \beta) \in [0, 1] \times (-1, 0]$, Theorem 2.3 is not applicable for the choice $(\alpha, \beta) = (-\frac{1}{2}, -\frac{1}{2})$. In this case, however, the analogue of Theorem 3.1 can be derived from Theorem 2.2.

THEOREM 4.1. *The almost-Hermite-Fejér-interpolation operators* $F_{1,0;n}^{(-\frac{1}{2},-\frac{1}{2})}$ *based upon the roots of the Čebyšev polynomials* T_n *and the end-point* 1 *satisfy the following inequalities* $(n \geq 2,\ f \in C[-1, 1]$, $x \in [-1, 1])$:

$$\text{(i)} \quad \left| F_{1,0;n}^{(-\frac{1}{2},-\frac{1}{2})}(f, x) - f(x) \right|$$
$$\leq \frac{C_1}{n} \cdot T_n^2(x) \cdot \sum_{k=1}^{n} \left[\omega_1\left(f, \frac{(1-x^2)^{\frac{1}{2}}}{k}\right) + \omega_1\left(f, \frac{1}{k^2}\right) \right]$$
$$+ C_2 \left(\omega_1\left(f, \frac{|T_n(x)|}{n}\right) + T_n^2(x) \cdot \omega_1\left(f, \frac{1}{n}\right) \right) \ ;$$

$$\text{(ii)} \quad \left| F_{1,0;n}^{(-\frac{1}{2},-\frac{1}{2})}(f, x) - f(x) \right|$$
$$\leq C_3 \cdot \left\{ \omega_2\left(f, |T_n(x)| \cdot n^{-\frac{1}{2}}\right) + n^{-\frac{1}{2}} \cdot \omega_1\left(f, |T_n(x)| \cdot n^{-\frac{1}{2}}\right) \right\} \ .$$

If f *is continuously differentiable we also have*

$$\text{(iii)} \quad \left| F_{1,0;n}^{(-\frac{1}{2},-\frac{1}{2})}(f, x) - f(x) \right|$$
$$\leq \frac{C_4}{n} \cdot |T_n(x)| \cdot \left\{ \|f'\|_\infty + s_n(x) \cdot \omega_1\left(f', |T_n(x)|/s_n(x)\right) \right\} \ ,$$

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where $s_n(x) := 1 + (1-x^2)^{\frac{1}{2}} \cdot \ln n$.

$C_1, \ldots, C_4$ are constants independent of f, x and n .

Proof. In order to prove *(i)* in Theorem 4.1 we use the method employed by Berman [1] writing

$$F^{(-\frac{1}{2},-\frac{1}{2})}_{1,0;n}(f,\ x) - f(x)$$

$$= F^{(-\frac{1}{2},-\frac{1}{2})}_{0,0;n}(f,\ x) - f(x) + \left[f(1)-F^{(-\frac{1}{2},-\frac{1}{2})}_{0,0;n}(f,\ 1)\right] \cdot T_n^2(x) ,$$

where $F^{(-\frac{1}{2},-\frac{1}{2})}_{0,0;n}$ is the ordinary Hermite-Fejér operator for the Čebyšev nodes and where T_n denotes the nth Čebyšev polynomial.

Thus we can estimate $\left|F^{(-\frac{1}{2},-\frac{1}{2})}_{1,0;n}(f,\ x)-f(x)\right|$ by

$$\left|F^{(-\frac{1}{2},-\frac{1}{2})}_{0,0;n}(f,\ x)-f(x)\right| + T_n^2(x) \cdot \left|F^{(-\frac{1}{2},-\frac{1}{2})}_{0,0;n}(f,\ 1)-f(1)\right| .$$

Applying a recent result of Goodenough and Mills [8] the first quantity in the above sum is less than or equal to

$$\frac{C_9}{n} \cdot T_n^2(x) \cdot \sum_{k=1}^{n} \left[\omega_1\left(f,\ \frac{(1-x^2)^{\frac{1}{2}}}{k}\right)+\omega_1\left(f,\ \frac{1}{k^2}\right)\right] + C_{10} \cdot \omega_1\left(f,\ \frac{|T_n(x)|}{n}\right) .$$

Here C_9 and C_{10} are suitable constants, $-1 \le x \le 1$, $n \ge 2$. The second quantity is less than or equal to

$$T_n^2(x) \cdot \left\{\frac{C_9}{n} \cdot \sum_{k=1}^{n} \omega_1\left(f,\ \frac{1}{k^2}\right)+C_{10}\cdot\omega_1\left(f,\ \frac{1}{n}\right)\right\} .$$

Adding the two expressions yields

$$\frac{C_1}{n} \cdot T_n^2(x) \cdot \sum_{k=1}^{n} \left[\omega_1\left(f,\ \frac{(1-x^2)^{\frac{1}{2}}}{k}\right)+\omega_1\left(f,\ \frac{1}{k^2}\right)\right]$$

$$+ C_2 \cdot \left\{\omega_1\left(f,\ \frac{|T_n(x)|}{n}\right) + T_n^2(x)\cdot\omega_1\left(f,\ \frac{1}{n}\right)\right\} ;$$

this gives the estimate under *(i)*.

Since statement *(i)* implies in particular that the norms $\left\|F^{(-\frac{1}{2},-\frac{1}{2})}_{1,0;n}\right\|$

are uniformly bounded by a constant B we can now use Theorem 2.2 in order to prove the remaining two estimates. First of all, for $f_1 \in C^1[-1, 1]$, *(i)* implies

$$\left|F_{1,0;n^2}^{(-\frac{1}{2},-\frac{1}{2})}(f_1, x)-f_1(x)\right| \le \frac{C_3}{n} \cdot \left(1+\left(1-x^2\right)^{\frac{1}{2}} \cdot \ln n\right) \cdot |T_n(x)| \cdot \|f_1'\|_\infty =: \phi_n(x) \cdot \|f_1'\|_\infty ,$$

where C_3 is another suitable constant.

For the proof of *(ii)* in Theorem 2.2 we apply the Berman decomposition again, use the positivity of $F_{0,0;n^2}^{(-\frac{1}{2},-\frac{1}{2})}$, the fact that $F_{0,0;n^2}^{(-\frac{1}{2},-\frac{1}{2})}e_0 = e_0$ and Lemma 2 of Min'kova [12] to arrive first at $\left(f_2 \in C^2[-1, 1],\ x \in [-1, 1]\right)$:

$$\begin{aligned}&\left|F_{0,0;n^2}^{(-\frac{1}{2},-\frac{1}{2})}(f_2, x)-f_2(x)\right| \\ &\quad\le \left|F_{0,0;n^2}^{(-\frac{1}{2},-\frac{1}{2})}(e_1-x, x)\right| \cdot \|f_2'\|_\infty + \frac{1}{2} \cdot F_{0,0;n^2}^{(-\frac{1}{2},-\frac{1}{2})}\left((e_1-x)^2, x\right) \cdot \|f_2''\|_\infty \\ &\quad= \frac{1}{n}\,|T_n(x)\cdot T_{n-1}(x)| \cdot \|f_2'\|_\infty + \frac{1}{2n} \cdot T_n^2(x) \cdot \|f_2''\|_\infty\end{aligned}$$

(see [4] and Popoviciu [13], respectively). This leads to

$$\begin{aligned}\left|F_{1,0;n^2}^{(-\frac{1}{2},-\frac{1}{2})}(f_2, x)-f_2(x)\right| &\le \frac{1}{n} \cdot |T_n(x)T_{n-1}(x)| \cdot \|f_2'\|_\infty + \frac{1}{2n} \cdot T_n^2(x)\|f_2''\|_\infty \\ &\qquad + T_n^2(x) \cdot \left[\frac{1}{n} \cdot \|f_2'\|_\infty + \frac{1}{2n} \cdot \|f_2''\|_\infty\right] \\ &\le \frac{2}{n} \cdot |T_n(x)| \cdot \|f_2'\|_\infty + \frac{1}{n} \cdot T_n^2(x) \cdot \|f_2''\|_\infty \\ &=: \gamma_{1,n}(x) \cdot \|f_2'\|_\infty + \gamma_{2,n}(x) \cdot \|f_2''\|_\infty .\end{aligned}$$

Theorem 2.2 now implies

$$\begin{aligned}&\left|F_{1,0;n^2}^{(-\frac{1}{2},-\frac{1}{2})}(f, x)-f(x)\right| \\ &\quad\le \left(\left\|F_{1,0;n^2}^{(-\frac{1}{2},-\frac{1}{2})}\right\|+1\right) \cdot \Omega\left(\frac{\phi_n(x)}{\left\|F_{1,0;n}^{(-\frac{1}{2},-\frac{1}{2})}\right\|+1}, \frac{\gamma_{1,n}(x)}{\phi_n(x)}, \frac{\gamma_{2,n}(x)}{\phi_n(x)}\right) .\end{aligned}$$

The estimate containing the second order modulus of continuity is now obtained from Theorem 2.1 by putting $h = |T_n(x)| \cdot n^{-\frac{1}{2}}$ and using the

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uniform boundedness of the sequence $\left(\left\|F_{1,0;n}^{(-\frac{1}{2},-\frac{1}{2})}\right\|\right)_{n\geq 1}$.

The inequality for continuously differentiable functions follows from the choice $h = |T_n(x)|/\left(1+\left(1-x^2\right)^{\frac{1}{2}} \cdot \ln n\right)$.

REMARK 4.2. Statements analogous to the ones in Corollary 3.2, Corollary 3.3 and Remark 3.4 hold for the case $(\alpha, \beta) = (-\frac{1}{2}, -\frac{1}{2})$, as well.

5. Pointwise estimates for $F_{1,0;n}^{(0,0)}$

In this case there is no uniform convergence for all $f \in C[-1, 1]$ which was shown by Vertési [20]. However, as proved by Knoop [9, Beispiel 7.8 (d)] uniform convergence holds if $f \in C[-1, 1]$ and if the additional assumption $f(1) = f(-1)$ is fulfilled. It is the aim of this part of our paper to prove a quantitative version of Knoop's assertion. This will be done by comparing $F_{1,0;n}^{(0,0)}$ to $F_{1,1;n}^{(0,0)}$. An alternative approach could for instance consist of comparing $F_{1,0;n}^{(0,0)}$ to $F_{0,0;n}^{(0,0)}$. We do not do this since $F_{0,0;n}^{(0,0)}f$ converges uniformly to f if and only if f satisfies the equalities $f(1) = f(-1) = \frac{1}{2}\int_{-1}^{1} f(t)dt$ (see Fejér [3] and Schönhage [18]). Thus the power of our 'helping operator' would be rather poor if we used $F_{0,0;n}^{(0,0)}$.

$F_{1,1;n}^{(0,0)}$ has much nicer properties. This was only recently emphasized in a paper of Prasad and Varma [15] who proved the following

THEOREM (Prasad and Varma [15]). *For any* $f \in C[-1, 1]$ *, any* $x \in [-1, 1]$ *and all* $n \geq 1$ *the estimate*

$$\left|F_{1,1;n}^{(0,0)}(f, x)-f(x)\right| \leq \frac{c}{n} \cdot \sum_{i=1}^{n} \omega_1\left(f, \frac{(1-x^2)^{\frac{1}{2}}}{i}\right)$$

holds, where C *is independent of* f, n *and* x .

A slightly weaker form of this theorem was published by Prasad and

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Saxena in 1975 [14]. The particular advantage of its form given above is due to the fact that the right-hand side in the estimate reproduces the interpolation conditions at the endpoints of $[-1, 1]$. As will be seen below the same estimate holds for the subspace consisting of all functions satisfying the equation $f(1) = f(-1)$.

THEOREM 5.1. *The almost-Hermite-Fejér-interpolation operators* $F_{1,0;n}^{(0,0)}$ *based upon the roots of the Legendre polynomials* P_n *and the end-point* 1 *satisfy the following inequalities provided that* $f \in C[-1, 1]$ *is such that* $f(1) = f(-1)$, $n \geq 1$, $x \in [-1, 1]$:

$$\text{(i)} \qquad \left|F_{1,0;n}^{(0,0)}(f, x)-f(x)\right| \leq C \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} \omega_1\left(f, \frac{(1-x^2)^{\frac{1}{2}}}{i}\right),$$

where C *is a suitable constant;*

$$\text{(ii)} \quad \left|F_{1,0;n}^{(0,0)}(f, x)-f(x)\right| \leq 5 \cdot \omega_2\left(f, (1-x^2)^{\frac{1}{2}} \cdot |P_n(x)|\right) + 3 \cdot \omega_1\left(f, (1-x^2)^{\frac{1}{2}} \cdot |P_n(x)|\right).$$

Proof. As announced above we use the decomposition

$$F_{1,0;n}^{(0,0)}(f, x) - f(x) = F_{1,1;n}^{(0,0)}(f, x) - f(x) + P_n(x)^2 \cdot \left(\frac{1}{2} \cdot \left[f(-1)-F_{1,0;n}^{(0,0)}(f, -1)\right]x+\frac{1}{2}\left[F_{1,0;n}(f, -1)-f(-1)\right]\right),$$

which is easily obtained by comparing the polynomials $F_{1,0}^{(0,0)}f$ and $F_{1,1}^{(0,0)}f$ and holds for an arbitrary $f \in C[-1, 1]$.

Looking again at $F_{1,0;n}^{(0,0)}(f, -1)$ it is immediately clear that this quantity is equal to $f(1)$. Thus the above decomposition reduces to the equality

$$F_{1,0;n}^{(0,0)}(f, x) - f(x) = F_{1,1;n}^{(0,0)}(f, x) - f(x) + P_n(x)^2 \cdot \left(\frac{1}{2} \cdot [f(-1)-f(1)]x+\frac{1}{2}[f(1)-f(-1)]\right)$$

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and in case that $f(1) = f(-1)$ attains the simple form

$$F_{1,0;n}^{(0,0)}(f,\ x) - f(x) = F_{1,1;n}^{(0,0)}(f,\ x) - f(x)\ .$$

This proves the first statement in Theorem 5.1.

In order to arrive at *(ii)* we use again the above equality if $f(1) = f(-1)$. Since $F_{1,1}^{(0,0)}$ is a positive linear operator (as is $F_{1,0}^{(0,0)}$) satisfying $F_{1,1}^{(0,0)}e_0 = e_0$ we can use Theorems 2.3 and 2.1 to find an inequality involving the second order modulus of continuity of f . For this purpose we do not necessarily have to know what $F_{1,1}^{(0,0)}(|e_1-x|,\ x)$, $F_{1,1}^{(0,0)}(e_1-x,\ x)$ and $F_{1,1}^{(0,0)}\left((e_1-x)^2,\ x\right)$ exactly look like as will be seen below.

We have

$$F_{1,1;n}^{(0,0)}\left((e_1-x)^2,\ x\right) = \left(1-x^2\right) \cdot P_n^2(x) \cdot \left[1 + \sum_{k=1}^{n} \frac{1}{\left(1-x_k^2\right) \cdot P_n'\left(x_k\right)^2}\right] .$$

The quantity in square brackets was investigated by Fejér [3]; he showed that it equals 2 . Thus

$$F_{1,1}^{(0,0)}\left((e_1-x)^2,\ x\right) = 2\left(1-x^2\right) \cdot P_n^2(x)\ .$$

Using now Theorems 2.3 and 2.1 we find that

$$\left|F_{1,1}^{(0,0)}(f,\ x)-f(x)\right|$$

$$\leq 2 \cdot \Omega\left(f;\ \frac{F_{1,1;n}^{(0,0)}(|e_1-x|,x)}{2},\ \frac{\left|F_{1,1;n}^{(0,0)}(e_1-x,x)\right|}{F_{1,1;n}^{(0,0)}(|e_1-x|,x)},\ \frac{2\left(1-x^2\right)\cdot P_n^2(x)}{2\cdot F_{1,1;n}^{(0,0)}(|e_1-x|,x)}\right)$$

$$\leq 2 \cdot \left\{\left(\frac{3}{2} + \frac{\left(1-x^2\right)\cdot P_n(x)^2}{h^2}\right) \cdot \omega_2(f,\ h) + \frac{\left|F_{1,1;n}^{(0,0)}(e_1-x,x)\right|}{h} \cdot \omega_1(f,\ h)\right\}$$

for any $0 < h \leq 2$. The particular choice

$$h = \left(1-x^2\right)^{\frac{1}{2}} \circ \left|P_n(x)\right| \leq \left(1-x^2\right)^{\frac{1}{4}} \cdot (2/\pi)^2 \cdot n^{-\frac{1}{2}}$$

(see Schönhage [17, p. 75]) yields for the case that this number is greater than 0 the inequality

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$$\left|F_{1,1}^{(0,0)}(f,\ x)-f(x)\right|$$
$$\leq 5\cdot\omega_2\left(f,\ (1-x^2)^{\frac{1}{2}}\cdot|P_n(x)|\right)$$
$$+\ 3\ \frac{\left|F_{1,1}^{(0,0)}(e_1-x,x)\right|}{\left(F_{1,1}^{(0,0)}((e_1-x)^2,x)\right)^{\frac{1}{2}}}\cdot\omega_1\left(f,\ (1-x^2)^{\frac{1}{2}}\cdot|P_n(x)|\right)\ .$$

The quotient in front of $\omega_1(f,\ \ldots)$ is bounded by 1 since the positivity of $F_{1,1;n}^{(0,0)}$ gives

$$\left|F_{1,1;n}^{(0,0)}(e_1-x,\ x)\right|\leq F_{1,1;n}^{(0,0)}(|e_1-x|,\ x)$$
$$\leq\left(F_{1,1;n}^{(0,0)}(e_0,\ x)\right)^{\frac{1}{2}}\cdot\left(F_{1,1;n}^{(0,0)}\left((e_1-x)^2,\ x\right)\right)^{\frac{1}{2}}=\left(F_{1,1;n}^{(0,0)}\left((e_1-x)^2,\ x\right)\right)^{\frac{1}{2}}\ .$$

The inequality remains true if x is a zero of $(1-x^2)\cdot P_n(x)^2$ since for these points the operator $F_{1,1;n}^{(0,0)}$ interpolates.

This proves statement *(ii)*.

REMARK 5.2. (i) It is also possible for the case $\alpha=\beta=0$ to prove uniform estimates or to give such involving $\omega_1^*(f,\ \cdot)$ or $\omega_1^*(f',\ \cdot)$.

(ii) Theorem 5.1 is not as informative as Theorems 3.1 and 4.1 in the sense that it does not provide us with information on how smoothness properties of f' accelerate the order of approximation of f . This goal could be achieved if we knew more on the behavior of the difference $F_{1,1;n}^{(0,0)}(f,\ x)-f(x)$, $f\in C^1[-1,\ 1]$, as can be seen from the equation

$$F_{1,0;n}^{(0,0)}(f,\ x)-f(x)=F_{1,1;n}^{(0,0)}(f,\ x)-f(x)\quad\text{if}\quad f(1)=f(-1)\ .$$

The real problem is that we do not have a 'good' pointwise estimate for $F_{1,1;n}^{(0,0)}(e_1-x,\ x)$. Such an estimate would provide us with the necessary tools to take full advantage of Berman's idea and arrive at inequalities for differentiable functions as well.

6. Concluding remarks

There is a case symmetric to that considered in this paper, namely the one for $(r, s) = (0, 1)$. Here the historical diagram has the form shown in Diagram 2. Again the pentagon encloses those values of (α, β) for which $F_{0,1;n}^{(\alpha,\beta)}f$ converges uniformly to f for all $f \in C[-1, 1]$; for other values of (α, β), $\alpha, \beta > -1$, there is no uniform convergence in general.

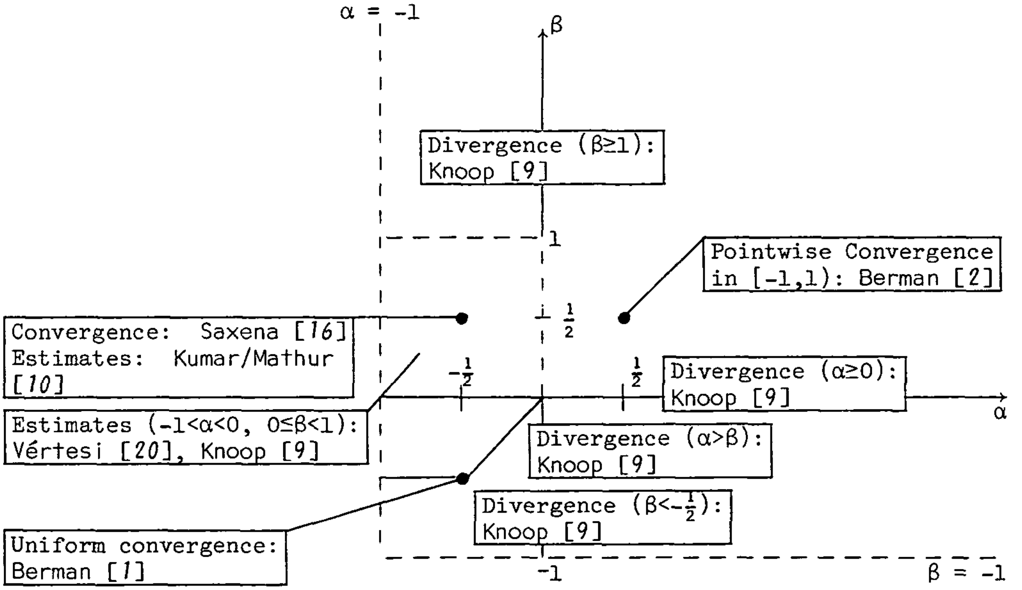

DIAGRAM 2

In this symmetric case it is possible to obtain estimates similar to the ones proved in this paper. There are again three essentially different situations: positive linear operators which converge uniformly on the whole space $C[-1, 1]$, non-positive linear operators having this property, and positive linear operators converging uniformly on proper subspaces of $C[-1, 1]$ only.

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