Brilliance or steadiness? A suggestion of an alternative model to Hardy's model concerning golf (1945)

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Introduction

There have been numerous attempts to model the governing dynamics between the two ostensibly competing concepts of brilliance and mechanical steadiness. One interesting study is given by the English mathematician G. H. Hardy in his model [1] describing two characteristically different golfers playing a match against each other. The model challenges the apparently accepted doctrine of the 'Brilliant player' having the advantage over the 'Steady player' in a long series of golf matches by holes. Hardy defines 'brilliance' as the capacity to produce ingenious results as well as the capacity to make mistakes, compared to 'steadiness' being completely mechanical producing the same average result all the time. The two players in his model are equal in performance on average, only the brilliant player has a higher standard deviation whereas the consistent player has a standard deviation of 0.

Hardy's argument overthrowing the 'commonly accepted doctrine' using his mathematical analysis can be simply encapsulated in the notion that the brilliant player has more room for error. His model categorises stroke quality into three divisions, good shots, ordinary shots, and bad shots, where the player gains a stroke, gains no stroke, and loses one stroke respectively. In a game of golf, a birdie and a bogey means completing the hole one stroke ahead of par and one stroke behind par respectively. Then to win with a birdie at a par 4 hole, you need a good shot in your first 3 shots, whereas to lose with a bogey means a bad shot in one of your first 4 shots. Therefore, with a balance of $4x - 3x = x$, where x is the probability of producing a good/bad shot, there is scope for loss for the brilliant player. Hardy's model is very simple in that all strokes are independent from each other and the probability of producing a good shot is equal to the probability of producing a bad shot. In this paper, a new approach using an alternative model will be discussed taking into account the dependency of shots, different probabilities for each outcome, and exploring the 'likely flaw in the model'. The 'flaw', as Hardy put it, is 'To play a sub-shot is to give yourself an opportunity of a super-shot … thus the chance of a super-shot is to some extent automatically increased'. Hardy himself questions whether this modification might resolve the paradox of his mathematical analysis which goes against the widely accepted doctrine, yet does not venture into an 'unpleasantly complex' model. Here he is probably keeping true to his belief as expressed in his book *A Mathematician's Apology* [2], stating 'Beauty is the first test: there is no permanent place in the world for ugly mathematics'.

https://doi.org/10.1017/mag.2017.64 Published online by Cambridge University Press

Previous attempts on further analysis of Hardy's model

Several attempts have been made to evaluate and develop the ideas of Hardy's model in recent years. Notably, R. Minton [3, 4] and G. L. Cohen [5] have both published interesting papers regarding the subject. Yet Minton's papers are limited to Hardy's model, with only extensions that evaluate Hardy's model in terms of the expected number of strokes for the 'erratic player' for a par 4 hole, and discuss the plausibility of the results depending on the type of match. He argues that the 'standard wisdom' is true for matches scored by holes won, as the steady player has a greater advantage than in matches scored by the number of strokes played. When considering the possible ways for the brilliant player to win a par 4 hole (an eagle or a birdie) against a mechanical player, one can evaluate the probability of the 'wins' (8 possible combinations of shots) and by subtracting all the probabilities of a 'win' and a 'par' (20 possible combinations of shots) from 1, one can find the probability of 'loss'. From this point onwards, x denotes the probability of hitting a 'super-shot' (a good shot), y denotes the probability of hitting a 'sub-shot' (a bad shot) and the difference in the two functions, 'Win' and 'Loss', will be referred to as the balance function ($x = y$ in Hardy/Minton's case). Hardy and Minton's balance function $f(x) = \text{Loss}(x) - \text{Win}(x)$ is as follows

$$f(x) = x - 9x^2 + 30x^3 - 35x^4.$$

Using a similar approach, Minton reaches the conclusion that the mean score for a 'brilliant player' is

$$\mu = 4 + x\left(1 - \frac{x^4}{(1 - x)^4}\right).$$

This is close to $4 + x$ for small x, reinforcing the observation that the steady player is advantageous in stroke play.

Cohen takes another step forward and includes the possibility that the probability of good shots and bad shots can differ. With such analysis he shows his balance function to be

$$f(x, y) = \left(4y - 6y^2 + 4y^3 - y^4\right) + \left(-3 - 6y + 21y^2 - 12y^3\right)$$
$$+ x^2\left(3 + 6y - 18y^2\right) + x^3\left(-1 - 4y\right).$$

He then reaches the conclusion that when $x > y$, the brilliant player is more likely to be victorious, but for $x = y$, it is the other way around.

Most interestingly, he calculates the expected number of strokes for an 'erratic golfer' on par 3, 4 and 5 holes using a simple Markov chain analysis, and finds the final function for the mean score for the 'erratic player' for a par 4 hole to be

$$\mu = \frac{4}{1 - y} - \frac{3x}{(1 - y)^2} + \frac{2x^2}{(1 - y)^3} - \frac{x^3}{(1 - y)^4}.$$

Proposed alternative model

When we consider the models presented so far and the possible flaw identified by Hardy in his model, it seems that the assumption of the quality of the strokes being independent cannot be justified. Therefore a new type of model is needed. A review of certain papers on emotions and sports/golf performance, makes it clear that 'performance was significantly greater in the anger condition compared with the happiness and emotion-neutral condition' [6]. Several pieces of evidence also support the claim that 'optimal emotional states would increase the probability of improved golf performance' [7]. With those issues in mind, and other common notions of psychology and performance such as being 'on tilt', which Wikipedia describes nicely as 'a state of mental or emotional confusion or frustration in which a player adopts a less than optimal strategy, usually resulting in the player becoming over-aggressive' [8], the following alternative model is proposed, which I claim balances these additional psychological effects as a 'bonus' to the pre-existing probabilities. The state diagrams in Figures 1 and 2, adapted from a template by Gareth Jones, describe the difference between the suggested new model and Hardy/Minton's idea.

The psychological effects must also mean that the previous shot will affect the golf player's current shot. For example, if the previous shot was bad, a sense of guilt or anger can adversely affect the player, or in other cases motivate them to play the current shot better. This dependency suggests a Markov chain approach. The 'memorylessness' quality is satisfied, as one would naturally presume only the previous shot would affect the current shot. The reason for this is that the seed of players is divided at random into smaller groups, and the grouping changes through several rounds in a tournament. Therefore, the player's knowledge of each other's performance is limited. Furthermore, there is a big variation in how players are affected by the performance of opposing players. Thus the psychological effect of the opposing players' performance is deemed negligible, compared to the importance of personal performance. So only considering the previous shot in the evaluation of the current shot is a natural assumption. Accordingly, I have set out the transition matrices for the different models in Table 1.

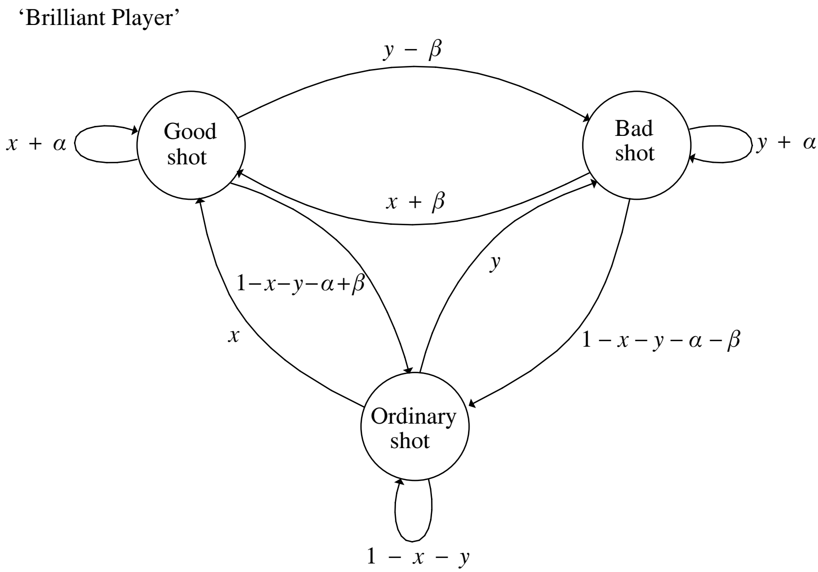

FIGURE 1: Markov Chain state diagram representing the respective probabilities depending on the current quality of the stroke.

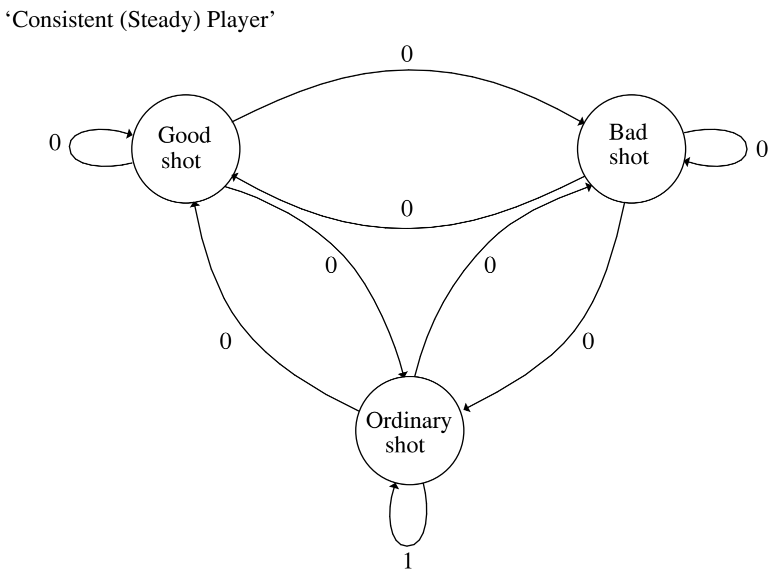

FIGURE 2: Hardy's idea of a 'Steady player'. As a purely mechanical being, he/she only plays par.

https://doi.org/10.1017/mag.2017.64 Published online by Cambridge University Press

$$\text{Transition Matrix}_{Hardy/Minton} = \begin{array}{c c} & \begin{array}{ccc} \textit{Ordinary} & \textit{Good} & \textit{Bad} \end{array} \\ \begin{array}{c} \textit{Ordinary} \\ \textit{Good} \\ \textit{Bad} \end{array} & \begin{bmatrix} 1-2x & x & x \\ 1-2x & x & x \\ 1-2x & x & x \end{bmatrix} \end{array}$$

$$\text{Transition Matrix}_{Cohen} = \begin{array}{c c} & \begin{array}{ccc} \textit{Ordinary} & \textit{Good} & \textit{Bad} \end{array} \\ \begin{array}{c} \textit{Ordinary} \\ \textit{Good} \\ \textit{Bad} \end{array} & \begin{bmatrix} 1-x-y & x & y \\ 1-x-y & x & y \\ 1-x-y & x & y \end{bmatrix} \end{array}$$

$$\text{Transition Matrix}_{Alternative} = \begin{array}{c c} & \begin{array}{ccc} \textit{Ordinary} & \textit{Good} & \textit{Bad} \end{array} \\ \begin{array}{c} \textit{Ordinary} \\ \textit{Good} \\ \textit{Bad} \end{array} & \begin{bmatrix} 1-x-y & x & y \\ 1-x-y-\alpha+\beta & x+\alpha & y-\beta \\ 1-x-y-\alpha-\beta & x+\beta & y+\alpha \end{bmatrix} \end{array}$$

TABLE 1: The Ordinary/Good/Bad columns show the probabilities for the quality of the next stroke after a player has just played a shot of the quality indicated to the left of the matrix.

α quantifies the bonus one gets through natural responses. When a good shot has been played, at the simplest level, it is natural for the player to try harder to keep up the good work, and likewise when a bad shot has been played, anger and guilt will increase the chance of another bad shot. β on the other hand, quantifies the bonus the player gets through controlling and moderating their behaviour. A previous good shot will result in a lower likelihood of a bad shot as long as the player does not get overly aggressive or lured into a state of false security. Similarly, playing a previous bad shot gives the player an opportunity to redeem him/herself and, as Hardy's paper notes, will be more 'keyed up to take' a good shot. Therefore, both α and β should be set high enough to affect the outcome for the player.

To make the values realistic, it seems plausible to take $\beta > \alpha$ in most cases to ensure that controlled moderation has a bigger effect than natural responses. By ensuring β overpowers A from a certain point, Hardy's idea of 'keying up' and balancing the good player's probabilities of hitting a good shot and a bad shot can be incorporated. A greater β compared to α will reflect the fact that the 'bonus' does not overpower the probabilities, yet it will only represent the fact that after a good shot the player will be less likely to make a mistake rather than to play a good shot again. This model therefore includes 'human control' of performance. Ideally, α and β should be functions of x and y, taking into account the relative ability to control one's emotion and the quality of one's shot based on the player's consistency.

As α describes a natural bonus that is applied when the current shot is in the same category as the previous shot, it seems sensible to suggest the bonus is related to the player's consistency. The quantity $(1 - x - y)$ which describes the probability of an ordinary shot, seems appropriate to describe a player's consistency. Therefore, I propose that α is proportional to $(1 - x - y)$, with the constant of proportionality chosen for this essay as 1/8 to satisfy the constraints described below for x and y.

Similarly, as β is a moderating parameter relating to control and psychology, the notion of overcoming a predicament or a precarious situation in a game suggests that β should be related to x, the probability of hitting a good shot. β is set to be proportional to the square root of x multiplied by $\ln(1 + x)$. (It will therefore follow closely to $y = x$, starting slowly at first then overtaking $y = x$.) The constant of proportionality is set to 12/5 to satisfy the constraints for x and y.

Hence in this paper, $\alpha = \dfrac{(1 - x - y)}{8}$, and $\beta = \dfrac{12}{5}\sqrt{x}\ln(1 + x)$ are proposed to incorporate the effects of the change based on Hardy's thought of being 'keyed up' to play a better shot. With the constants of proportionality set as above, we need to impose some constraints on the values of x and y. Consideration of the first entry of the 'bad' row in the transition matrix gives the constraint that the sum of the entry in the 'bad' column and the 'good' column should be less than or equal to 1. This is because for large x and y, the sum of $x + \beta$ and $y + \alpha$ may go beyond 1 which is out of our required range. Additionally, consideration of the 'good' row in the transition matrix gives the constraint that $\beta \leqslant y$. If we combine all the constraints, we arrive at:

$$x + \frac{12}{5}\sqrt{x}\ln(1 + x) + y + \frac{(1 - x - y)}{8} \leqslant 1, \tag{1}$$

$$\frac{12}{5}\sqrt{x}\ln(1 + x) \leqslant y \leqslant 0.5. \tag{2}$$

Both are realistic constraints, as the first only affects the extreme range of x and y (x beyond 0.35). For the second constraint, y is likely to be greater than $\frac{12}{5}\sqrt{x}\ln(1 + x)$ as in real life, it is most likely that $y > x$ and $\frac{12}{5}\sqrt{x}\ln(1 + x) < x$ for the majority of values of x between 0 and 0.5.

Calculations and results

Evaluating the Markov process stage-wise, as inspired by Cohen's paper, I have categorised 4 stages, E_0, E_1, E_2, E_3 as the expected number of strokes from points 0, 1, 2, 3 respectively to 4 points (= par). Using the transition matrix defined above, this resulted in 10 equations:

(A Good shot = 2 points, an Ordinary shot = 1 point, and a Bad shot = 0 points)

$$E_0 = (1 - x - y)(E_1O) + x(E_2G) + y(E_0B) + 1$$

$$E_0B = (1 - x - y - \alpha - \beta)(E_1O) + (x + \beta)(E_2G) + (y + \alpha)(E_0B) + 1$$

$$E_1O = (1 - x - y)(E_2O) + x(E_3G) + y(E_1B) + 1$$

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$$E_1B = (1 - x - y - \alpha - \beta)\left(E_2O\right) + (x + \beta)\left(E_3G\right) + (y + \alpha)\left(E_1B\right) + 1$$

$$E_2O = (1 - x - y)\left(E_3O\right) + y\left(E_2B\right) + 1$$

$$E_2G = (1 - x - y - \alpha + \beta)\left(E_3O\right) + (y - \beta)\left(E_2B\right) + 1$$

$$E_2B = (1 - x - y - \alpha - \beta)\left(E_3O\right) + (y + \alpha)\left(E_2B\right) + 1$$

$$E_3O = y\left(E_3B\right) + 1$$

$$E_3G = (y - \beta)\left(E_3B\right) + 1$$

$$E_3B = (y + \alpha)\left(E_3B\right) + 1$$

These ten equations are interpreted as follows: E_nQ denotes the expected number of additional strokes required to reach 4 points for a player who currently has n points and whose last shot was of quality Q. For example, E_2G means the previous shot was Good, reaching 2 points (therefore only coming from E_0). The probabilities used can be read easily off the 'Good' row of the transition matrix. E_0G and E_1G are not displayed as these states can never arise (a previous shot with quality G gives 2 points, so a 0 or 1 is impossible). Any occurrence of E_4 that arises is also immediately replaced with 0 as the player has finished the game reaching 4 points. We are only interested in E_0, the expected number of strokes from point 0, the start of a hole. Rearranging the last equation, we get

$$E_3B = \frac{1}{1 - y - \alpha}.$$

Further substitutions of E_3B into the other occurrences of E_3 and thereafter of E_2, of E_1, and of E_0 result in

$$\begin{aligned} E_0 = \mu = 4 + \beta y - ax - 3x + 2xy + 2x^2 - x^3 \\ - \left(\frac{\beta^2 x + 2\beta y^2 + 3xy^2 + \left(-3x^2 + 4x - 4\right)\beta y + 2(x-1)\beta x + \left(-4x^2 + 6x^2 - 8x + 4\right)y}{\alpha + y - 1}\right) \\ + \left(\frac{y(\beta + x)\left(\beta^2 + 2\beta(2x - 1) + y^2 + 2y\left(-3x^2 + 3x - 2\right) + (2 - 3x)\beta y\right)}{(\alpha + y - 1)^2}\right) \\ - \left(\frac{y^2(\beta + x)^2(y(2 - \beta - 4x) + 2\beta)}{(\alpha + y - 1)^3}\right) \\ - \left(\frac{y^4(\beta + x)^3}{(\alpha + y - 1)^4}\right). \end{aligned} \tag{3}$$

Example 1: The mean number of strokes from stage 0 (0 points) to stage 4 (par 4).

Letting $\alpha, \beta \to 0$ we can confirm it reproduces the same result as Cohen's balance function and if we set further $y \to x$, it correctly produces

https://doi.org/10.1017/mag.2017.64 Published online by Cambridge University Press

the same result as Minton's model. Taking $\alpha = \frac{(1 - x - y)}{8}$, and $\beta = \frac{12}{5}\sqrt{x}\ln(1 + x)$ and cutting the graph with the given constraints as shown above, the following surface is generated which describes the expected number of strokes for each x and v combination.

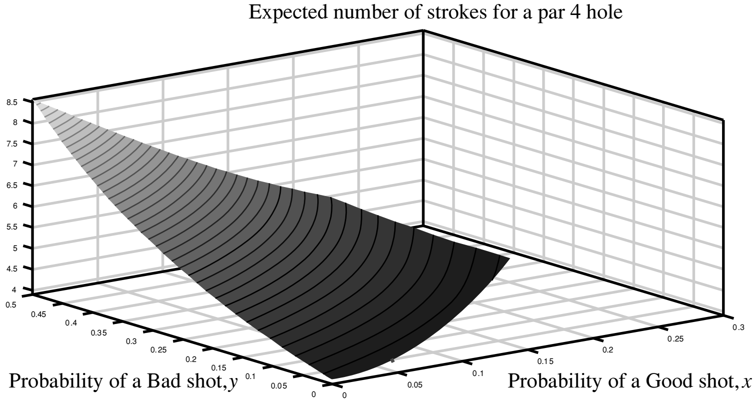

FIGURE 3: Surface plot of the Expected number of strokes for a par 4 hole. $\alpha = (1 - x - y)/8$ and $\beta = \frac{12}{5}\sqrt{x}\ln(1 + x)$.

When $x = y = 0$, describing a 'Steady player', $E_0 = 4$ as expected. We can see that generally as y increases, the expected number of strokes increase, whereas as x increases, the expected number of strokes decreases.

Unlike Cohen's model, depending on the parameters of α, β and the values of x and y, one can still have 'erratic players' winning against 'steady players' when $y > x$:

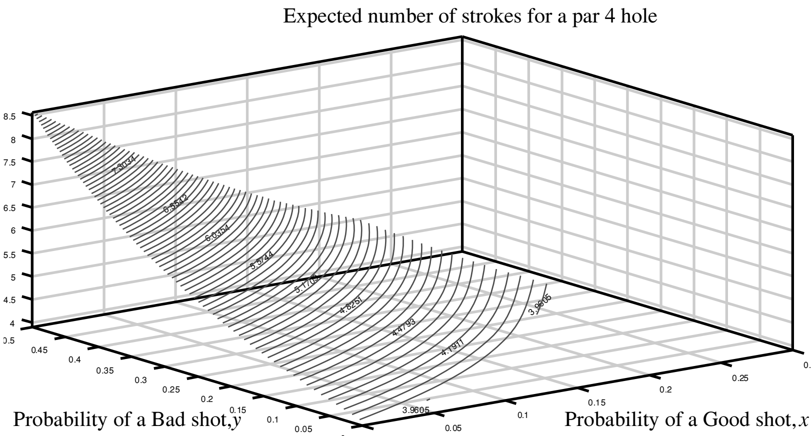

FIGURE 4: Contour plot of the Expected number of strokes for a par 4 hole with labels, illustrating that the mean number of strokes can be less than 4 even if $y(0.23) > x(0.22)$. $\alpha = (1 - x - y)/8$ and $\beta = \frac{12}{5}\sqrt{x}\ln(1 + x)$.

https://doi.org/10.1017/mag.2017.64 Published online by Cambridge University Press

Table 2 summarises the expected numbers of stroke for a par 4 hole for several x, y combinations generated by the proposed model

$\alpha = (1 - x - y)/8$		*Probability of a sub-shot*, y							
$\beta = \frac{12}{5}\sqrt{x}\ln(1 + x)$		0	0.05	0. 1	0.15	0.2	0.25	0.3	0.35
	0	4	4.241	4.508	4.807	5.143	5.524	5.959	6.462
Probability	0.05	–	4.062	4.300	4.563	4.856	5.185	5.557	5.979
of a	0.1	–	–	4.089	4.315	4.564	4.839	5.145	5.488
super-shot,	0.15	–	–	–	4.069	4.275	4.500	4.747	5.018
x	0.2	–	–	–	–	3.998	4.179	4.375	4.587
	0.24	–	–	–	–	–	–	4.106	4.281

TABLE 2: The expected number of strokes at a par 4 hole, for selected values of x and y, where '-' represents omitted values due to the constraints.

The big difference we can see from this table is that, with Hardy/Minton's and Cohen's model, the expected number of shots when $x = y$ is close to $4 + x$, yet my table clearly shows that for $x = 0$ to $x = 0.2$ (where the constraints allow $x = y$), the expected number of shots is under $4 + x$ (except when $x = 0.05$). In particular, at $x = y = 0.2$, the expected number of shots is under 4. More significantly, looking at the diagonal entries of the table, the expected number of strokes seems to reduce as you go down the diagonal from a certain point (around $x = 0.1$). This is contrary to the constant increase in the expected number of strokes of the previous models. Additionally, a key finding would be that, even if $y > x$, as it would be in most realistic situations, the alternative model reduces the number of strokes for a par 4 compared to the previous models, suggesting a more balanced game between a 'Brilliant' player and a 'Steady' player. For reference, the following are some comparisons between Cohen's calculation and the table above: (C=Cohen, A= Alternative) ($x = 0.1$, $y = 0.2 \rightarrow$ C = 4.6, A = 4.56), ($x = 0.2$, $y = 0.2 \rightarrow$ C = 4.2, A = 4.00), ($x = 0.2$, $y = 0.3 \rightarrow$ C = 4.7, A = 4.38).

Further evaluation of the model and examination of the probabilities shows that the balance function is:

Balance Function $= \text{Loss}(x, y) - \text{Win}(x, y)$

$\text{Loss}(x, y) = 1 - \text{Par}(x, y) - \text{Win}(x, y)$

$\text{Win}(x, y)$ = Sum of the probabilities of the 4 different stroke sequences resulting in Eagle/Birdie

$\text{Par}(x, y)$ = Sum of the probabilities of the 20 different 4 stroke sequenceS of reaching Par.

$$f(x, y) = x^3(-1 - 4y) + x^2\left(3 + 6y - 18y^2 + 3y(\beta - 2\alpha) + 2\beta(1 + \alpha)\right)$$
$$+ x\left(-3 - 6y + 21y^2 - 12y^3 + \beta(1 + \alpha)(\beta - 2) + 2y\left(-\alpha^2 + \alpha\beta + 2\alpha + 3\beta^2\right) - 3y^2(3\alpha + 2\beta)\right)$$
$$- y^4 + 4y^3 - 6y^2 + 4y - \beta y\left(3y^2 + y(2\alpha - \beta - 6) + \alpha(1 + \alpha) - \beta(\beta - 2) + 3\right)$$

Example 2: Balance function of the new model, where the rectangles represent Cohen's balance function referenced again. (α and β are arbitrary constants/functions of x and y.)

For $x = y$, $\alpha = \frac{1-x-y}{8}$ and $\beta = \frac{12}{5}\sqrt{x}\ln(1+x)$ the following graph explains the result

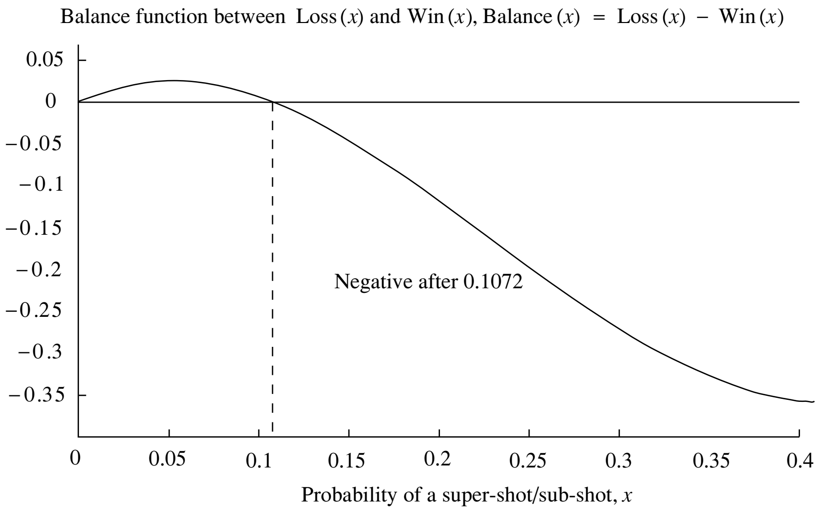

FIGURE 5: Plot of the Balance function (Loss (x) $-$ Win (x)) with the given α and β values. The function returns a negative value after $x \approx 0.1072$ suggesting 'Win' is more probable after that point.

As can be seen from the graph, for relatively realistic values ($x \approx 0.2$), the alternative model seems to show that 'brilliant players' have a greater chance of winning. Near $x = 0.1$, which Hardy, Minton, and Cohen would all agree as probable values for a professional player, the new balance function indicates the 'steady' player does not have a distinct advantage over the 'brilliant' player.

Conclusions and Thoughts

If the analysis is done purely by the balance function as Hardy did in his paper, this alternative model suggests the consideration of the apparent flaw in Hardy's model does indeed resolve the paradox. Yet with the expected value analysis and the uncertainty of α and β values, this model cannot concretely conclude that it resolves Hardy's flaw. Nonetheless, as a whole, the alternative model proposed seems to take into account the psychological effects linking shot performance with the quality of the stroke before. Overall it produced a lower expected value than Hardy/Minton's or Cohen's, AND the distribution shape is slightly different. It is possible to conclude from the attempt above that the alternative model eliminates the notion of an 'advantage' resulting from the style of play. This allows the reader to have a comforting feeling that golf can be a fair game between different styles of players, making it the interesting sport it is.

One can definitely modify the values/functions of α and β to change the results in favour of the 'Brilliant' player, yet our discussion was a demonstration using what the writer thinks are realistic bonuses given to the player in a game. An interesting study would be to investigate the effect different functions for α and β will have on the alternative model, or to suggest a 'better' model that describes the phenomenon discussed above 'better'.

This attempt to explore the 'flaw' of Hardy's model, prompts one to wonder whether Hardy would have considered it as 'resolving the paradox' or an 'ugly piece of physical application of mathematics'. The eternal question of brilliance versus steadiness is not solved through this model. It merely displays just a little exploration made with what Hardy would call 'ugly mathematics', complicating his results. The model suggests a sense of equality between the erratic player and the consistent player. Rather than trying to find the winner of the two, it is with great hope that this study points to the optimal observation of the game golf, as a 1935 *Glasgow Herald* newspaper [9] described a golf match, 'Brilliant Steadiness' or 'Steady Brilliance'.

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10.1017/mag.2017.64

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