Canad. Math. Bull. Vol. **54** (2), 2011 pp. 311–315
doi:10.4153/CMB-2011-005-0

Some Remarks Concerning the Topological Characterization of Limit Sets for Surface Flows

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Abstract. We give some extension to theorems of Jiménez López and Soler López concerning the topological characterization for limit sets of continuous flows on closed orientable surfaces.

1 Introduction

Let M be a closed orientable connected surface. A continuous flow ϕ on M will be a continuous map $\phi\colon \mathbb{R}\times M \to M$ with the properties:
(i) $\phi\big((t+s),x\big) = \phi\big(t,\phi(s,x)\big)$ for every $t,s\in\mathbb{R}$ and $x\in M$,
(ii) $\phi(0,x) = x$ for every $x\in M$.
Given a point $x\in M$, we define the map $\phi_x\colon \mathbb{R}\to M$ by $\phi_x(t) = \phi(t,x)$, $t\in\mathbb{R}$. We call $L_x = \phi_x(\mathbb{R})$ the orbit of x. We say that x is a singular point (or a singularity) of ϕ if ϕ_x is constant. Denote by:

- $\operatorname{sing}(\phi)$ the set of singular points of ϕ.
- $M^\star = M\setminus \operatorname{sing}(\phi)$.
- M_1 the union of all orbits of ϕ which are closed in $M^\star$.
- $U_1 = M^\star\setminus M_1$.

A subset A of M is called *invariant* if $\phi(\mathbb{R}\times A) = A$, that is, A is a union of orbits. For every orbit L of ϕ and $x\in L$, we call

$$L_x^+ = \{\phi_x(t); t\in\mathbb{R}_+\} \quad (\text{resp. } L_x^- = \{\phi_x(t); t\in\mathbb{R}_-\})$$

the *positive* (resp. *negative*) *semi-orbit* of x. The set

$$\Omega_L = \bigcap_{x\in L}\overline{L_x^+} \quad (\text{resp. } A_L = \bigcap_{x\in L}\overline{L_x^-}\ )$$

is called the ω-*limit* (resp. α-*limit*) set of L, and $\lim L = \Omega_L\cup A_L$ is called the *limit set* of L. A point $y\in\Omega_L$ (resp. $y\in A_L$) means that there exists a sequence $t_n\mapsto +\infty$ (resp. $t_n\mapsto -\infty$) such that $\lim_{n\to\infty}\phi(t_n,x) = y$. We have $\lim L = \overline{L}\setminus L$ if L is a non-periodic orbit. If L is a periodic orbit, $\Omega_L = A_L = L$. The set Ω_L (resp. A_L) is closed, connected, invariant, and non-empty.

Received by the editors November 21, 2007; revised July 22, 2008.
Published electronically January 26, 2011.
AMS subject classification: **37B20**, 37E35.
Keywords: flows on surfaces, orbits, class of an orbit, singularities, minimal set, limit set, regular cylinder.

We say that an orbit L of ϕ is *proper* if $\overline{L} \setminus L$ is closed in M. For example, a closed orbit in $M^\star$, is proper. In particular, if L is a periodic orbit, it is called *trivial recurrent*. A non-proper orbit L is called *non-trivial recurrent*, that is, either *locally dense* if $\overline{L}$ has non-empty interior, or *exceptional* if L is nowhere dense. For non-trivial recurrent orbits L, we have one of the following types: ω-recurrent if $\overline{L} = \Omega_L$; α-recurrent if $\overline{L} = A_L$; or both ω-recurrent and α-recurrent if $\Omega_L = A_L = \overline{L}$. If L is proper and non-periodic, then $\lim L = \overline{L} \setminus L$. Otherwise, $\lim L = \overline{L}$. In particular, if L is a non-proper orbit then $\overline{L} = \Omega_L$ or $\overline{L} = A_L$.

A subset E of M is called *a minimal set* of ϕ if E is closed in M, non-empty, ϕ-invariant and has no proper subset with these properties. This is equivalent to saying that for every orbit L contained in E, we have $\overline{L} = E$. In particular, a closed orbit in M is a minimal set of ϕ.

We call the *class of an orbit* L of ϕ the union $\mathrm{cl}(L)$ of orbits G of ϕ such that $\overline{G} = \overline{L}$. We note that orbits which are in the same class are either all proper or locally dense or exceptional. In particular, if L is proper, $\mathrm{cl}(L) = L$.

We call the *lower structure* of an orbit L of ϕ the subset $\mathrm{SI}(L) = \overline{L} \setminus \mathrm{cl}(L)$. In the case where L is proper, $\mathrm{SI}(L) = \overline{L} \setminus L$ is always closed in M.

We call the *higher structure* of an orbit L of ϕ the union $\mathrm{SS}(L)$ of orbits G of ϕ such that $L \subset \overline{G}$ with $\overline{G} \neq \overline{L}$.

For a subset A of M, $\mathrm{int}(A)$, $bd(A) = \overline{A} \setminus \mathrm{int}(A)$ will denote respectively, the interior and the boundary of A.

We call a *regular cylinder* an open connected set of M which is homeomorphic to an open annulus and its boundary has two connected components.

In [6], the problem of the topological characterization of limit sets of flows on closed surfaces has partially answered by Jiménez López and Soler López.

For ω (resp. α)-limit sets which has empty interior, their result can be paraphrased as follows:

Theorem 1.1 (Jiménez López and Soler López [6]) *Let ϕ be a continuous flow on a closed orientable surface M and let L be an orbit of ϕ. Assume that L is proper or that* $\mathrm{int}(\Omega_L) = \varnothing$ *and $M \setminus \Omega_L$ has a* finite *number of components. Then Ω_L is a boundary component of a regular cylinder in M. Conversely, if Ω is a boundary component of a regular cylinder in M then there are a smooth flow on M and an orbit of this flow such that $\Omega = \Omega_L$.*

For ω (resp. α)-limit sets with non-empty interior, there were also characterized:

Theorem 1.2 (Jiménez López and Soler López [5]) *Let ϕ be a continuous flow on a closed orientable surface M and let L be an orbit of ϕ. Assume that* $\mathrm{int}(\Omega_L) \neq \varnothing$. *Then $\Omega_L = \overline{O}$ where $O =$* $\mathrm{int}(\Omega_L)$ *and M is not homeomorphic to the sphere S^2.*

Conversely, if $O \subset M$ is an open set not homeomorphic to a subset of the sphere S^2 then there are a smooth flow Φ on M and an orbit L of Φ such that $\Omega_L = \overline{O}$.

The problem remains open in the case of an ω (resp. α)-limit set which, simultaneously, is the limit set of one of its orbits, has empty interior, and whose complement has an *infinite* number of components.

The aim of this note is to extend the above works by characterizing ω (resp. α)-limit sets of surface flow orbits; we show that the assumption: "$M \setminus \Omega_L$ has a finite number of components" in Theorem 1.1 is unnecessary provided the flow has a finite number of singularities, or that $\overline{L}$ is a non-trivial minimal set. Moreover, for non-empty interior limit set, we give a precise description of its topological characterization.

Our main results can be stated as follows.

Theorem 1.3 *Let ϕ be a continuous flow on a closed orientable surface M and let L be an orbit of ϕ such that $\operatorname{int}(\Omega_L) = \varnothing$. Then Ω_L is a boundary component of a regular cylinder in M if one of the following conditions hold:*

(i) *ϕ has finitely many singularities.*

(ii) *L is an exceptional orbit with $\overline{L}$ does not contain singular points.*

In the case of flows with finitely many singularities, the set U_1 is open in M [4, Theorem, p. 386] and we have precisely the following.

Proposition 1.4 *Let ϕ be a continuous flow with finitely many singularities on a closed orientable surface M and let L be an orbit of ϕ such that $\operatorname{int}(\Omega_L) \neq \varnothing$. Then*

(i) *Ω_L is the closure of the connected component V of U_1 containing L;*

(ii) *the flow $\phi_{|V}$ is minimal (every orbit of $\phi_{|V}$ is dense in V).*

Notice that we have the same statements as for Theorem 1.3 and Proposition 1.4 for α-limit sets.

Remark (1) In condition (i) of Theorem 1.3, a flow without singularities or with finitely many singularities may admit ω-limit sets with empty interior and having infinitely many components in its complementary. Informally speaking the construction of such flow consists of making the suspension of an homeomorphism $f\colon S^1 \to S^1$ obtained by blowing up infinitely many orbits of the irrational rotation on S^1. This is the procedure usually used to obtain the Denjoy flow, but for that case only one orbit is blowing up.

Following [1, Theorem 2.5, p. 208] there exist a homeomorphism $f\colon S^1 \to S^1$ and a continuous increasing surjective map $h\colon S^1 \to S^1$ semi-conjugating f to the irrational rotation R_α of angle α. Moreover there exist countable many orbits of R_α $(R_\alpha^n(x_i)_{n\in\mathbb{Z}})_{i\in\mathbb{N}}$ such that $\chi = \bigcup_{n\in\mathbb{Z}}\{R_\alpha^n(x_i) : i \in \mathbb{N}\}$ is the set of point $x \in S^1$ for which $h^{-1}(x)$ contains more than one point. Now it suffices of defining the flow Φ as the suspension of f, see *e.g.*, [1, p. 16]. It is easy to verify that Φ satisfies the desired property and the following ones:

(i) There exists a set Ω such that $\operatorname{int}(\Omega) = \varnothing$ and $\Omega = \Omega_L = A_L$ for every orbit L.

(ii) Φ admits proper and non-proper orbits.

(iii) Φ has no singularities nor periodic orbits.

(2) Flows having infinitely many singularities and admitting an ω-limit set with empty interior and with infinitely many components in its complementary also exist. Moreover, this ω-limit set is not boundary of any regular cylinder, see [7].

2 Some Results

In the following, we give some properties for the dynamics of recurrent orbits.

Proposition 2.1 ([8, Proposition 2.1]) *Let ϕ be a continuous flow on a closed orientable surface M and suppose that* $\text{sing}(\phi)$ *is a compact totally disconnected set on M. Then, if L is a non-proper orbit of ϕ then every orbit contained in* $\text{SI}(L)$ *is closed in $M^\star$.*

Proposition 2.2 *Under the hypothesis of Proposition 2.1, if L is an orbit of ϕ such that $\overline{L}$ contains a periodic orbit γ then L is proper.*

Proof Suppose that L is a non-proper orbit, then $\overline{L}$ is one of its limit set, say $\overline{L} = \Omega_L$. By [3, Proposition 7.11], we will have $\Omega_L = \gamma$ thus $\gamma = \overline{L}$, which is impossible. ∎

Proposition 2.3 ([9, Theorem 2.2]) *Let ϕ be a continuous flow on a closed orientable surface M with finitely many singularities. If L is an exceptional orbit then $V = \text{SS}(L) \cup \text{cl}(L)$ is open in M.*

Proposition 2.4 ([2, Theorem 1.1]) *Let ϕ be a continuous flow on an orientable surface M. Let $E \subset M$ be a non-trivial compact minimal set. Then, there exists a connected, open, ϕ-invariant neighborhood U of E with the following property:*

- *if $L \subset U$ is an orbit, then $\Omega_L \cup A_L \subset E \cup bd(U)$ and $\Omega_L = E$ or $A_L = E$.*

In particular, Proposition 2.4 holds for $E = \overline{L}$ if $\overline{L}$ does not contain singular point and L is an exceptional orbit (since in this case $\overline{L}$ is non-trivial compact minimal set).

3 Proof of Theorem 1.3 and Proposition 1.4

3.1 Proof of Theorem 1.3

Lemma 3.1 *Let ϕ be a continuous flow on a closed orientable surface M and let L be an exceptional orbit. Suppose that ϕ has finitely many singularities or that $\overline{L}$ does not contain singular point. If $(W_j)_{j\in J}$ are the connected components of $M \setminus \overline{L}$ then there exists $m \in J$ such that $\overline{L} = bd(W_m)$.*

Proof The inclusion $bd(W_m) \subset \overline{L}$ is clear since $bd(W_m)$ is closed in $M \setminus \overline{L}$. To prove the other inclusion $\overline{L} \subset bd(W_m)$, suppose the contrary; that is for every $j \in J$, we have $L \subset M \setminus \overline{W_j}$.

If ϕ has finitely many singularities then by Proposition 2.3, the set $V = \text{SS}(L) \cup \text{cl}(L)$ is open in M. Then (since $M \setminus \overline{W_j}$ is ϕ-invariant), for every orbit $G \subset V$, we have $G \subset M \setminus \overline{W_j}$. Therefore, $V \subset M \setminus \overline{W_j}$ for every $j \in J$. It follows that $M \setminus \overline{L} \subset M \setminus V$ and then $\text{int}(\overline{L}) \neq \varnothing$, a contradiction.

If $\overline{L}$ does not contain singular point then $E = \overline{L}$ is an exceptional compact minimal set. Hence, by Proposition 2.4, there exists a connected neighborhood U of E such that for every orbit $G \subset U$, we have $E \subset \overline{G}$ so $G \subset M \setminus \overline{W_j}$. Therefore, $U \subset M \setminus \overline{W_j}$ for every $j \in J$. It follows that $M \setminus \overline{L} \subset M \setminus U$ and then $\text{int}(\overline{L}) \neq \varnothing$, a contradiction. ∎

Proof of Theorem 1.3 Let L be an orbit of ϕ such that $\text{int}(\Omega_L) = \varnothing$. If L is periodic, obviously L is a boundary of a regular cylinder. Now, suppose that L is non-periodic. We distinguish two cases.

https://doi.org/10.4153/CMB-2011-005-0 Published online by Cambridge University Press

If $L \subset M \setminus \Omega_L$ then decompose $U = M \setminus \Omega_L$ into its connected components by $U = \cup_{j \in J} W_j$ and define W_m the component containing L. It is easy to check that $bd(W_m) = \Omega_L$ for some m.

If $L \subset \Omega_L$ that is $\overline{L} = \Omega_L$; then L is an exceptional orbit. By Lemma 3.1, there exists $m \in J$ such that $\Omega_L = \overline{L} = bd(W_m)$.

The reminder of the proof, that is W_m is homeomorphic to a regular cylinder, is similar to that of the proof of Lemma 3.3 in [7]. ■

3.2 Proof of Proposition 1.4

Lemma 3.2 *Let ϕ be a continuous flow with finitely many singularities on a closed orientable surface M. If L is a non-proper orbit of ϕ then* $\mathrm{cl}(L) = \overline{L} \cap U_1$. *In particular, if L is locally dense then* $\mathrm{cl}(L)$ *is the connected component of U_1 containing L.*

Proof Let L be a non-proper orbit. If $G \subset \overline{L} \cap U_1$ is an orbit of ϕ then G is non-closed in $M^\star$. From Proposition 2.1, we have $\overline{G} = \overline{L}$. So, $G \subset \mathrm{cl}(L)$ and $\mathrm{cl}(L) = \overline{L} \cap U_1$. Now, let V be the connected component of U_1 containing L. We also have $\overline{L} \cap V = \mathrm{cl}(L)$. Suppose that L is locally dense; that is $\mathrm{int}(\overline{L}) \neq \varnothing$. We have $L \subset \mathrm{int}(\overline{L})$ and therefore $\mathrm{cl}(L) \subset \mathrm{int}(\overline{L})$. It follows that $\mathrm{cl}(L) = \overline{L} \cap V = \mathrm{int}(\overline{L}) \cap V$ thus, $\mathrm{cl}(L)$ is open and closed in V. As V is connected, we have $\mathrm{cl}(L) = V$. ■

Proof of Proposition 1.4 Let L be an orbit such that $\mathrm{int}(\Omega_L) \neq \varnothing$. Then L is locally dense and we have $\overline{L} = \Omega_L$. By Lemma 3.2, if V is the connected component of U_1 containing L then $\mathrm{cl}(L) = V$. Thus, $\overline{L} = \overline{V} = \Omega_L$ and assertion (i) follows. Assertion (ii) is clear since $\mathrm{cl}(L) = V$. ■

Acknowledgments The author would like to thank Gabriel Soler López for useful comments.

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