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Ad-nilpotent Elements of Semiprime Rings with Involution

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Abstract. Let R be an $n!$-torsion free semiprime ring with involution $*$ and with extended centroid C, where $n > 1$ is a positive integer. We characterize $a \in K$, the Lie algebra of skew elements in R, satisfying $(\mathrm{ad}_a)^n = 0$ on K. This generalizes both Martindale and Miers' theorem and the theorem of Brox et al. In order to prove it we first prove that if $a, b \in R$ satisfy $(\mathrm{ad}_a)^n = \mathrm{ad}_b$ on R, where either n is even or $b = 0$, then $(a - \lambda)^{[(n+1)/2]} = 0$ for some $\lambda \in C$.

1 Results

An associative ring R is called a *prime ring* (resp. a *semiprime ring*) if, for $a, b \in R$, $aRb = 0$ implies that either $a = 0$ or $b = 0$ (resp. for $a \in R$, $aRa = 0$ implies $a = 0$). The primeness (resp. semiprimeness) of R is equivalent to saying that any product of two nonzero ideals (resp. any square of a nonzero ideal) of R is nonzero.

Throughout the paper, R always denotes a semiprime ring with center $Z(R)$ and with Martindale symmetric ring of quotients Q. The center of Q, denoted by C, is called the extended centroid of R. The center C is a commutative regular self-injective ring. Moreover, R is a prime ring if and only if C is a field. We refer the reader to [1] for details.

Let L be a Lie algebra with Lie bracket $[\cdot,\cdot]$. For $a \in L$, $\mathrm{ad}_a: L \to L$ is the adjoint map defined by $x \mapsto [a, x]$ for $x \in L$. We let $Z(L) := \{c \in L \mid [c, x] = 0\ \forall\ x \in L\}$, the center of the Lie algebra L. An element $a \in L$ is called *ad-nilpotent* if $(\mathrm{ad}_a)^k = 0$ on L for some $k \geq 1$. We let $\mathbb{Z}$ denote the ring of integers. Given a ring R, let R^- be the Lie algebra $(R, +)$ over $\mathbb{Z}$ endowed with the Lie bracket product $[x, y] := xy - yx$ for $x, y \in R$. In [18] Martindale and Miers proved the following theorem (see [18, Corollary 1]).

Theorem 1.1 (Martindale and Miers 1983) *Let R be a prime ring and let $n > 1$ be a positive integer, $a, b \in R$. Suppose that $(\mathrm{ad}_a)^n = \mathrm{ad}_b$ on R^-, where either n is even or $b = 0$. If $\mathrm{char}(R) = 0$ or a prime $p > n$, then $(a - \lambda)^{[(n+1)/2]} = 0$ for some $\lambda \in C$.*

Theorem 1.1 with $b = 0$ was first proved for simple rings by Herstein [13], and both Herstein [13] and Kovacs [16] conjectured the generalization to prime rings. We

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also refer the reader to [8, 11] for nilpotent derivations of semiprime rings. For the semiprime case with $n = 3$ and $b = 0$, Brox et al. proved the following (see [5, Theorem 3.2]).

Theorem 1.2 (Brox et al. 2016) *Let R be a 6-torsion free semiprime ring and $a \in R$. Suppose that $(\mathrm{ad}_a)^3 = 0$ on R^-. Then $(a - \lambda)^2 = 0$ for some $\lambda \in C$.*

An ad-nilpotent element a in a Lie algebra L is called a *Jordan element* if $(\mathrm{ad}_a)^3 = 0$ on L. Jordan elements in R^- play a fundamental role in the proof of Kostrikin's conjecture (see [4, 20]) and are also of great importance in the Lie inner ideal structure of associative rings (see [3]). Every Jordan element $a \in R^-$ (with $\frac{1}{2} \in R$) gives rise to a Jordan algebra $(R^-)_a$, which is called the Jordan algebra of R^- at a (see [10, Theorem 2.4]). A semiprime ring R is called *centrally closed* if $R = RC + C$. Brox et al. used Theorem 1.2 to prove that, for a 6-torsion free centrally closed semiprime ring R, the Jordan algebra of the Lie algebra R^- at a Jordan element is isomorphic to the symmetrization of a local algebra of the ring R (see [5, Lemma 5.1]). The first goal of this paper is to generalize Theorems 1.1 and 1.2 to the semiprime case from the viewpoint of orthogonal completion of semiprime rings (see [1]).

Theorem 1.3 *Let R be an $n!$-torsion free semiprime ring, where $n > 1$ is a positive integer, and $a, b \in R$. Suppose that $(\mathrm{ad}_a)^n = \mathrm{ad}_b$, where either n is even or $b = 0$. Then $(a - \lambda)^{[\frac{n+1}{2}]} = 0$ for some $\lambda \in C$.*

Let R be a semiprime ring with involution $*$ and let K denote the set of all skew elements in R; that is, $K = \{x \in R \mid x^* = -x\}$. Clearly, K forms a Lie algebra under the Lie bracket product $[x, y] = xy - yx$ for $x, y \in K$. It is known that the involution $*$ on R can be uniquely extended to an involution, denoted by $*$ also, on Q. We say that the involution $*$ is of the first kind if the restriction of $*$ to C is the identity map and it is of the second kind, otherwise. We let

$$S_m(X_1, \dots, X_m) := \sum_{\sigma \in \mathrm{Sym}(m)} (-1)^\sigma X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(m)},$$

be the standard polynomial of degree m in noncomutative indeterminates $X_1, X_2, \dots, X_m$, where $\mathrm{Sym}(m)$ denotes the permutation group on the set $\{1, 2, \dots, m\}$. By an S_m-ring R we mean that the ring R satisfies the polynomial $S_m(X_1, \dots, X_m)$. It is known that if R is a prime S_{2n}-ring, then $\dim_C RC \le n^2$ (see [21, Corollary 1] and [15, Theorem p. 17]). By [12, Corollary 8], given a prime ring R with involution $*$ and $a \in R \setminus Z(R)$, if $[a, K] = 0$, then R is an S_4-ring, *i.e.*, $\dim_C RC \le 4$. Martindale and Miers proved the following result (see [19]).

Theorem 1.4 *Let R be a prime ring with involution $*$, $\mathrm{char}(R) = 0$, or a prime $p > n$, where $n > 1$ is a positive integer, and $a \in K$. Suppose that $(\mathrm{ad}_a)^n = 0$ on K and that R is not an S_4-ring. Then $(a - \lambda)^{[(n+1)/2]+1} = 0$ for some skew element $\lambda \in C$. Moreover, if $*$ is of the first kind, then $a^{[(n+1)/2]+1} = 0$, and if $*$ is of the second kind, then $(a - \lambda)^{[(n+1)/2]} = 0$.*

Remarks (I) The theorem above was proved by Martindale and Miers with the assumption that $\operatorname{char}(R) = 0$ (see [19, Main Theorem]). Their argument is still effective when $\operatorname{char}(R) = 0$ or a prime $p > n$. We sketch its proof here for the sake of the reader. If $*$ is of the second kind, $(\operatorname{ad}_a)^n = 0$ on K implies that $(\operatorname{ad}_a)^n = 0$ on R (see Lemma 2.5). In this case, the theorem is reduced to Theorem 1.1. Thus, $*$ is assumed to be of the first kind. Let $m := n - 1$ as given in the proof of [19, Main Theorem]. It suffices to notice the following facts in [19]:

(a) On page 1049, $1 + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1} \in C \setminus \{0\}$ in Eq.(10);
(b) On page 1050, $-[\binom{m}{0} + \binom{m}{2} + \binom{m}{4} + \cdots] = -2^{m-1} \in C \setminus \{0\}$;
(c) On page 1048, let $\beta_j := (-1)^j[\binom{m}{j} - \binom{m}{j-2}] \in C$ in Eq.(8), where $0 \le j \le m+2$ and $\binom{m}{k} = 0$ if $k < 0$ or $k > m$. Indeed, let $2 \le j \le m$. We have

$$(-1)^j \beta_j = \binom{m}{j} - \binom{m}{j-2} = \frac{m!n(n-2j+1)}{j!(m-j+2)!}.$$

Note that $|n - 2j + 1| < n$. Thus, $\beta_j = 0$ in C only when $2j - 1 = n$. Clearly, $\beta_j \neq 0$ for $j = 0, 1, m+1, m+2$. We now go to the proof on page 1050 with $a^{r+1} = 0$ but $a^r \neq 0$. In this case, recall that $*$ is of the first kind. By Eq.(17), we have $\sum_{j=0}^{n+1} \beta_j a^{n+1-j} \otimes a^j = 0$, where each $\beta_j \neq 0$ except in the one case when n is odd and $j = \frac{n+1}{2}$. This implies that $a^{n+1} = 0$ and so $r \le n+1$. It follows from the proof on page 1050 that $a^{[(n+1)/2]+1} = 0$, as asserted.

(II) Suppose that $*$ is of the second kind. There exists a nonzero skew element $\nu \in C$. Since $\nu^* = -\nu \in C$ and $(\operatorname{ad}_a)^n = 0$ on K, we get $(\operatorname{ad}_{\nu a})^n = 0$ on Q (see Lemma 2.5). In view of Theorem 1.3, $(\nu a - \mu)^{[(n+1)/2]} = 0$ for some $\mu \in C$. By the primeness of R, C is a field. Therefore, $(a - \nu^{-1}\mu)^{[(n+1)/2]} = 0$. Together with fact that $(a-\lambda)^{[(n+1)/2]+1} = 0$, we see that $\lambda - \nu^{-1}\mu$ is a nilpotent element in C and so $\lambda = \nu^{-1}\mu$. Therefore, $(a - \lambda)^{[(n+1)/2]} = 0$, as asserted.

Let $\mathbb{Z}\{\widehat{X}\}$ be the free associative $\mathbb{Z}$-algebra in noncommutative indeterminates $X_1, X_2, \ldots$, where $\widehat{X} := \{X_1, X_2, \ldots\}$. Given a polynomial $f(X_1, \ldots, X_t) \in \mathbb{Z}\{\widehat{X}\}$ that has zero constant term, a semiprime ring R is called *faithful f-free* if every nonzero ideal of R does not satisfy f. The second goal of this paper is to generalize Theorem 1.4 to the semiprime case.

Theorem 1.5 *Let R be an $n!$-torsion free semiprime ring with involution $*$ and $a \in K$, where $n > 1$ is a positive integer. Suppose that $(\operatorname{ad}_a)^n = 0$ on K. Then there exist an idempotent $e = e^* \in C$ and a skew element $\lambda \in C$ such that $(ea - \lambda)^{[(n+1)/2]+1} = 0$, eR is a faithful S_4-free ring, and $(1-e)R$ is an S_4-ring. Moreover,*

$$\big(\mathrm{E}[\lambda]ea - \lambda\big)^{[\frac{n+1}{2}]} = 0 \quad \textit{and} \quad \big((1 - \mathrm{E}[\lambda])ea\big)^{[\frac{n+1}{2}]+1} = 0.$$

We refer the reader to the next section for the definition of $\mathrm{E}[\lambda]$ for $\lambda \in C$. Given a ring T with involution $*$, let $K(T)$ denote the Lie algebra of all skew elements in T. We also write K instead of $K(R)$ for simplicity. An element $s \in T$ is called *symmetric* if $s^* = s$. With Theorem 1.5 in hand, we have to characterize skew ad-nilpotent elements in a semiprime S_4-ring with involution $*$. Such a characterization is obtained as follows.

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Theorem 1.6 *Let R be a $(2n-1)!$-torsion free semiprime S_4-ring with involution $*$, where $n > 1$ is a positive integer and $a \in K$. Suppose that $(\mathrm{ad}_a)^n = 0$ on K. Then there exist orthogonal symmetric idempotents $e_1, e_2 \in C$, $e_1 + e_2 = 1$, and a skew element $\lambda \in e_2 C$ such that $e_1 a \in Z(e_1 K)$ and $(e_2 a - \lambda)^2 = 0$. In particular, $e_2 a$ is a Jordan element of the Lie algebra $(e_2 R)^-$.*

As a consequence of Theorems 1.5 and 1.6, we have the following corollary.

Corollary 1.7 *Let R be a $(2n-1)!$-torsion free semiprime ring with involution $*$ and $a \in K$, where $n > 1$ is a positive integer. Suppose that $(\mathrm{ad}_a)^n = 0$ on K. Then there exist orthogonal symmetric idempotents $e_1, \ldots, e_5 \in C$, $e_1 + \cdots + e_5 = 1$, and skew elements $\lambda_1, \lambda_2 \in C$ satisfying the following:*

(i) $(e_1 a - \lambda_1)^{[\frac{n+1}{2}]} = 0$;
(ii) $(e_2 a)^{[\frac{n+1}{2}]+1} = 0$ *and* $(e_1 + e_2)R$ *is an faithful* S_4*-free ring;*
(iii) $[e_3 a, K] = 0$;
(iv) $(e_4 a - \lambda_2)^2 = 0$, $(e_3 + e_4)R$ *is an* S_4*-ring, and* $e_5 a = 0$;
(v) RaR *is an essential ideal of* $(1 - e_5)R$.

2 Proofs

Recall that R always denotes a semiprime ring with extended centroid C. The set $\mathbf{B}$ of all idempotents of C forms a Boolean algebra with respect to the operations $e \dot{+} h := e + h - 2eh$ and $e \cdot h := eh$ for all $e, h \in \mathbf{B}$. It is complete with respect to the partial order $e \le h$ (defined by $eh = e$) in the sense that any subset S of $\mathbf{B}$ has a supremum $\bigvee S$ and an infimum $\bigwedge S$. Given a subset S of Q, we define $\mathrm{E}[S]$ to be the infimum of the subset $\{e \in \mathbf{B} \mid ex = x \ \forall x \in S\}$. If $S = \{b\}$, we write $\mathrm{E}[b]$ instead of $\mathrm{E}[S]$ for simplicity.

We call a set $\{e_\nu \in \mathbf{B} \mid \nu \in \Lambda\}$ an *orthogonal subset* of $\mathbf{B}$ if $e_\nu e_\mu = 0$ for $\nu \ne \mu$ and a *dense subset* of $\mathbf{B}$ if $\sum_{\nu\in\Lambda} e_\nu C$ is an essential ideal of C. A subset T of Q, where $0 \in T$, is called *orthogonally complete* in the following sense: given any dense orthogonal subset $\{e_\nu \mid \nu \in \Lambda\}$ of $\mathbf{B}$, there exists a one-one correspondence between T and the direct product $\prod_{\nu\in\Lambda} Te_\nu$ via the map

$$x \longmapsto \langle x e_\nu \rangle \in \prod_{\nu\in\Lambda} Te_\nu \quad \text{for } x \in T.$$

Therefore, given any subset $\{a_\nu \in T \mid \nu \in \Lambda\}$, there exists a unique $a \in T$ such that $a \mapsto \langle a_\nu e_\nu \rangle$. The element a is written as $\sum^{\perp}_{\nu\in\Lambda} a_\nu e_\nu$ and is characterized by the property that $ae_\nu = a_\nu e_\nu$ for all $\nu \in \Lambda$.

In view of [1, Proposition 3.1.10], Q is orthogonally complete. Moreover, P is a minimal prime ideal of Q if and only if $P = \mathbf{m}Q$ for some $\mathbf{m} \in \mathrm{Spec}(\mathbf{B})$, the spectrum of $\mathbf{B}$ (*i.e.*, the set of all maximal ideals of $\mathbf{B}$) (see [1, Theorem 3.2.15]). In particular, it follows from the semiprimeness of Q that $\bigcap_{\mathbf{m}\in\mathrm{Spec}(\mathbf{B})} \mathbf{m}Q = 0$. We refer the reader to [1] for details.

To begin with, we prove the following.

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Lemma 2.1 *Let R be an $n!$-torsion free semiprime ring, where n is a positive integer. Then* $\operatorname{char}(Q/\mathbf{m}Q) = 0$ *or a prime $p > n$ for any* $\mathbf{m} \in \operatorname{Spec}(\mathbf{B})$.

Proof Let $\mathbf{m} \in \operatorname{Spec}(\mathbf{B})$. Suppose on the contrary that $\operatorname{char}(Q/\mathbf{m}Q)$ is a prime $p \le n$. Then $n!(Q/\mathbf{m}Q) = 0$; that is, $n!Q \subseteq \mathbf{m}Q$. Since $n!Q$ is orthogonally complete, it follows from [1, Corollary 3.2.4] that $n!eQ = 0$ for some $e \in \mathbf{B} \setminus \mathbf{m}$. Thus, $n!e = 0$. Since R is an $n!$-torsion free semiprime ring, so is Q. This implies that $e = 0$, a contradiction. This proves that $\operatorname{char}(Q/\mathbf{m}Q) = 0$ or a prime $p > n$. ∎

We let $C[t]$ denote the polynomial ring over C in the indeterminate t.

Theorem 2.2 *Let R be a semiprime ring, $a_i \in Q$ and $g_i(t) \in C[t]$ for $1 \le i \le n$. Suppose that, given any $\mathbf{m} \in \operatorname{Spec}(\mathbf{B})$, there exists $\lambda_{\mathbf{m}} \in C$ such that $\sum_{i=1}^{n} g_i(\lambda_{\mathbf{m}})a_i \in \mathbf{m}Q$. Then $\sum_{i=1}^{n} g_i(\lambda)a_i = 0$ for some $\lambda \in C$.*

Proof Let

$$\Sigma := \Big\{ e \in \mathbf{B} \mid e\Big(\sum_{i=1}^{n} g_i(\beta)a_i \Big) = 0 \text{ for some } \beta \in C \Big\}.$$

We claim that Σ is an ideal of the complete Boolean algebra $\mathbf{B}$. Clearly, if $f \le e$ and $e \in \Sigma$, then $f \in \Sigma$. Let $e, f \in \Sigma$. We have to prove that $e + f - ef \in \Sigma$. Since $e + f - ef = e + f(1-e)$ and $e, f(1-e) \in \Sigma$, we may assume from the start that $ef = 0$. Choose $\alpha, \beta \in C$ such that

$$e\Big(\sum_{i=1}^{n} g_i(\alpha)a_i \Big) = 0 = f\Big(\sum_{i=1}^{n} g_i(\beta)a_i \Big).$$

Note that $(e + f)g_i(\alpha e + \beta f) = eg_i(\alpha) + fg_i(\beta)$, and so

$$(e + f)\Big(\sum_{i=1}^{n} g_i(\alpha e + \beta f)a_i \Big) = e\Big(\sum_{i=1}^{n} g_i(\alpha)a_i \Big) + f\Big(\sum_{i=1}^{n} g_i(\beta)a_i \Big) = 0.$$

This proves that $e + f \in \Sigma$, as asserted. If $1 \in \Sigma$, then we are done. Suppose on the contrary that $1 \notin \Sigma$. Then $\Sigma \subseteq \mathbf{m}$ for some $\mathbf{m} \in \operatorname{Spec}(\mathbf{B})$. By hypothesis, there exists $\lambda_{\mathbf{m}} \in C$ such that $\sum_{i=1}^{n} g_i(\lambda_{\mathbf{m}})a_i \in \mathbf{m}Q$. Thus, there exists $e \in \mathbf{B} \setminus \mathbf{m}$ such that $e(\sum_{i=1}^{n} g_i(\lambda_{\mathbf{m}})a_i) = 0$. This implies that $e \in \Sigma$ and so $e \in \mathbf{m}$, a contradiction. ∎

Proof of Theorem 1.3 Since R and Q satisfy the same GPIs with coefficients in Q (see [1, Theorem 6.4.1]), we have $(\operatorname{ad}_a)^n = \operatorname{ad}_b$ on Q. Let

$$q := \Big[\frac{n+1}{2} \Big] \quad \text{and} \quad g_i(t) := (-1)^{q-i}\binom{q}{i}t^{q-i} \in C[t]$$

for $0 \le i \le q$. Then

$$(a - \lambda)^q = \sum_{i=0}^{q} g_i(\lambda)a^i$$

for all $\lambda \in C$. Let $\mathbf{m} \in \operatorname{Spec}(\mathbf{B})$. By Lemma 2.1, $\operatorname{char}(Q/\mathbf{m}Q) = 0$ or a prime $p > n$. Moreover, $(\operatorname{ad}_{\overline{a}})^n = \operatorname{ad}_{\overline{b}}$ on $Q/\mathbf{m}Q$, where $\overline{z} := z + \mathbf{m}Q \in Q/\mathbf{m}Q$ for $z \in Q$. Note that $C + \mathbf{m}Q/\mathbf{m}Q$ is the extended centroid of the prime ring $Q/\mathbf{m}Q$ (see [1, Theorem 3.2.5]). In view of Theorem 1.1, there exists $\lambda_{\mathbf{m}} \in C$ such that $(\overline{a} - \overline{\lambda_{\mathbf{m}}})^q = 0$.

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That is, $\sum_{i=0}^{q} g_i(\lambda_{\mathbf{m}})a^i \in \mathbf{m}Q$. In view of Theorem 1.3, there exists $\lambda \in C$ such that $\sum_{i=0}^{q} g_i(\lambda)a^i = 0$, *i.e.*, $(a-\lambda)^q = 0$. ∎

Lemma 2.3 *Let R be an $n!$-torsion free, faithful S_4-free semiprime ring with involution $*$, $a \in K$, where $n > 1$. Suppose that $(\mathrm{ad}_a)^n = 0$ on K. Then $(a-\lambda)^{[(n+1)/2]+1} = 0$ for some skew element $\lambda \in C$.*

Proof Let $\mathbf{m} \in \mathrm{Spec}(\mathbf{B})$. By Lemma 2.1, $\mathrm{char}(Q/\mathbf{m}Q) = 0$ or a prime $p > n$. Since R is a faithful S_4-free semiprime ring, by [22, Theorem 2.3] $Q/\mathbf{m}Q$ does not satisfy S_4.

Case 1: $\mathbf{m}^* = \mathbf{m}$. Then $(\mathbf{m}Q)^* = \mathbf{m}Q$. Thus, $Q/\mathbf{m}Q$ can be endowed with an involution, denoted by $*$ also, defined by $\overline{x}^* = \overline{x^*}$ for $x \in Q$. Since $(\mathrm{ad}_a)^n(x-x^*) = 0$ for all $x \in R$, it follows from [2, Theorem 1.4.1] that $(\mathrm{ad}_a)^n(x-x^*) = 0$ for all $x \in Q$. This implies that $(\mathrm{ad}_{\overline{a}})^n(\overline{x}-\overline{x}^*) = 0$ for all $x \in Q$. Thus, $(\mathrm{ad}_{\overline{a}})^n(\overline{z}) = 0$ for all $\overline{z} \in K(Q/\mathbf{m}Q)$ as $Q/\mathbf{m}Q$ is 2-torsion free. In view of Theorem 1.4, there exists $\lambda_{\mathbf{m}} \in C$ such that $(\overline{a}-\overline{\lambda}_{\mathbf{m}})^{[(n+1)/2]+1} = 0$; that is, $(a-\lambda_{\mathbf{m}})^{[(n+1)/2]+1} \in \mathbf{m}Q$.

Case 2: $\mathbf{m}^* \neq \mathbf{m}$. As proved in Case 1, $(\mathrm{ad}_a)^n(x-x^*) = 0$ for all $x \in Q$. Then $\overline{x} = \overline{x} - \overline{x}^* \in \mathbf{m}Q + \mathbf{m}^*Q/\mathbf{m}Q$ for all $x \in \mathbf{m}^*Q$. Thus, $(\mathrm{ad}_{\overline{a}})^n(\overline{z}) = 0$ for $\overline{z} \in \mathbf{m}Q + \mathbf{m}^*Q/\mathbf{m}Q$. Note that $\mathbf{m}Q + \mathbf{m}^*Q/\mathbf{m}Q$ is a nonzero ideal of the prime ring $Q/\mathbf{m}Q$. In view of [1, Theorem 6.4.1] or [7, Theorem 2], $\mathbf{m}Q + \mathbf{m}^*Q/\mathbf{m}Q$ and $Q/\mathbf{m}Q$ satisfy the same GPIs. Therefore, $(\mathrm{ad}_{\overline{a}})^n(\overline{z}) = 0$ for $\overline{z} \in Q/\mathbf{m}Q$. In view of Theorem 1.3, there exists $\lambda_{\mathbf{m}} \in C$ such that $(\overline{a}-\overline{\lambda_{\mathbf{m}}})^{[\frac{n+1}{2}]} = 0$; that is, $(a-\lambda_{\mathbf{m}})^{[(n+1)/2]} \in \mathbf{m}Q$. In particular, $(a-\lambda_{\mathbf{m}})^{[(n+1)/2]+1} \in \mathbf{m}Q$.

In either case, if $\mathbf{m} \in \mathrm{Spec}(\mathbf{B})$, there exists $\lambda_{\mathbf{m}} \in C$ such that $(a-\lambda_{\mathbf{m}})^{[(n+1)/2]+1} \in \mathbf{m}Q$. That is, $\sum_{i=0}^{q} g_i(\lambda_{\mathbf{m}})a^i \in \mathbf{m}Q$, where $q := [\frac{n+1}{2}]+1$ and $g_i(t) := (-1)^{q-i}\binom{q}{i}t^{q-i} \in C[t]$ for $0 \le i \le q$. In view of Theorem 2.2, $\sum_{i=0}^{q} g_i(\lambda)a^i = 0$ for some $\lambda \in C$, *i.e.*, $(a-\lambda)^{[\frac{n+1}{2}]+1} = 0$ for some $\lambda \in C$. Since $a^* = -a$, we have $(a+\lambda^*)^{[(n+1)/2]+1} = 0$. Thus, $\lambda^* + \lambda$ is nilpotent as $\lambda^* + \lambda = (a+\lambda^*) - (a-\lambda)$. Hence, $\lambda^* = -\lambda$ by the semiprimeness of Q. ∎

Lemma 2.4 *Let R be a semiprime ring with involution $*$ and $\lambda \in C$. Then $C\,\mathrm{E}[\lambda] = C\lambda$ and $\mathrm{E}[\lambda^*] = \mathrm{E}[\lambda]^*$. Moreover, if $C\lambda = C\lambda^*$, then $\mathrm{E}[\lambda]^* = \mathrm{E}[\lambda]$.*

Proof Since C is a regular ring, $\lambda\lambda_1\lambda = \lambda$ for some $\lambda_1 \in C$. Then $e := \lambda\lambda_1$ is a central idempotent. We claim that $e = \mathrm{E}[\lambda]$. Indeed, $\mathrm{E}[\lambda]e = \mathrm{E}[\lambda]\lambda\lambda_1 = \lambda\lambda_1 = e$, implying $e \le \mathrm{E}[\lambda]$. On the other hand, $e\lambda = \lambda\lambda_1\lambda = \lambda$, implying $\mathrm{E}[\lambda] \le e$. Thus, $e = \mathrm{E}[\lambda]$, as asserted. Clearly, $C\,\mathrm{E}[\lambda] = C\lambda\lambda_1 \subseteq C\lambda$. On the other hand, $C\lambda = C\lambda\lambda'\lambda \subseteq C\lambda\lambda' = C\,\mathrm{E}[\lambda]$. Thus, $C\,\mathrm{E}[\lambda] = C\lambda$.

We have $C\,\mathrm{E}[\lambda]^* = C\lambda^*$. However, $C\lambda^* = C\,\mathrm{E}[\lambda^*]$ and so $C\,\mathrm{E}[\lambda]^* = C\,\mathrm{E}[\lambda^*]$, implying $\mathrm{E}[\lambda]^* = \mathrm{E}[\lambda^*]$, as asserted. Finally, suppose that $C\lambda = C\lambda^*$. Then $C\,\mathrm{E}[\lambda] = C\,\mathrm{E}[\lambda^*]$ and so $\mathrm{E}[\lambda] = \mathrm{E}[\lambda^*] = \mathrm{E}[\lambda]^*$. ∎

Let R be a semiprime ring with involution $*$. An ideal I of R is called a $*$-ideal if $I = I^*$.

Lemma 2.5 *Let R be a semiprime ring with involution $*$. Suppose that $(\mathrm{ad}_a)^n = 0$ on K, where $a \in K$ and n is a positive integer. If R is 2-torsion free, then $(\mathrm{ad}_{\lambda a})^n = 0$ and $(\mathrm{ad}_{\mathrm{E}[\lambda]a})^n = 0$ on Q for $\lambda^* = -\lambda \in C$.*

Proof Suppose that R is 2-torsion free. Choose an essential $*$-ideal I of R such that $\lambda I \subseteq R$. Let $x \in I$. Then $2x = s + k$, where $s = x + x^* \in I$ and $k = x - x^* \in I$. Then $\lambda s \in K$ and so

$$2(\mathrm{ad}_{\lambda a})^n(x) = (\mathrm{ad}_{\lambda a})^n(s+k) = \lambda^{n-1}(\mathrm{ad}_a)^n(\lambda s) + \lambda^n(\mathrm{ad}_a)^n(k) = 0.$$

Thus, $(\mathrm{ad}_{\lambda a})^n(x) = 0$. This proves that $(\mathrm{ad}_{\lambda a})^n = 0$ on I. In view of [2, Theorem 1.4.1], I and Q satisfy the same $*$-GPIs with coefficients in Q. Thus, $(\mathrm{ad}_{\lambda a})^n = 0$ on Q. By Lemma 2.4, $\mathrm{E}[\lambda] = \lambda\lambda_1$ for some $\lambda_1 \in C$. Then $(\mathrm{ad}_{\mathrm{E}[\lambda]a})^n = \lambda_1^n(\mathrm{ad}_{\lambda a})^n = 0$ on Q. ∎

Proof of Theorem 1.5 In view of [22, Theorem 2.2], there exists an idempotent $e \in C$ such that $(1-e)Q$ is an S_4-ring and eQ is a faithful S_4-free ring. Moreover, $R \cap (1-e)Q$ is the largest ideal of R satisfying S_4 (see [22, Theorem 2.2(3)]). Since $(1-e)Q$ satisfies S_4, so does $(1-e^*)Q$. Thus, $R \cap (1-e^*)Q \subseteq R \cap (1-e)Q$, implying that $e(1-e^*) = 0$ and so $e = e^*$.

Thus, $ea \in K(eQ)$. Since $(\mathrm{ad}_a)^n = 0$ on $K(Q)$ (see the proof of Lemma 2.3), we have $(\mathrm{ad}_{ea})^n = 0$ on $K(eQ)$. But eQ is an $n!$-torsion free, faithful S_4-free semiprime ring. By Lemma 2.3, there exists $\lambda \in eC \subseteq C$ such that $(ea - \lambda)^{[(n+1)/2]+1} = 0$. Since $(ea)^* = -ea$, we have $(ea + \lambda^*)^{[(n+1)/2]+1} = 0$, which implies that $\lambda + \lambda^*$ is a nilpotent element in C. By the semiprimeness of Q, we get $\lambda^* = -\lambda$.

By Lemma 2.4, we have $C\lambda = C\,\mathrm{E}[\lambda]$ and $\mathrm{E}[\lambda]^* = \mathrm{E}[\lambda]$. In view of Lemma 2.5, $(\mathrm{ad}_{\mathrm{E}[\lambda]a})^n = 0$ on Q. By Theorem 1.3, there exists $\mu \in C$ such that

$$\big(\mathrm{E}[\lambda]ea - \mu\big)^{[\frac{n+1}{2}]} = 0 \quad \text{and} \quad \big((1-\mathrm{E}[\lambda])ea\big)^{[\frac{n+1}{2}]+1} = \big(1-\mathrm{E}[\lambda]\big)(ea-\lambda)^{[\frac{n+1}{2}]+1} = 0,$$

as $(1-\mathrm{E}[\lambda])\lambda = 0$. Since $(ea-\lambda)^{[(n+1)/2]+1} = 0$, it follows that $(\mathrm{E}[\lambda]ea - \lambda)^{[\frac{n+1}{2}]+1} = 0$ as $\mathrm{E}[\lambda]\lambda = \lambda$. This implies that $\lambda = \mu$. That is, $(\mathrm{E}[\lambda]ea - \lambda)^{[(n+1)/2]} = 0$. ∎

We now turn to the proof of Theorem 1.6. Given an ideal I of R, for $q \in R$ we have $qI = 0$ if and only if $Iq = 0$. Thus, $\mathrm{Ann}_R(I) := \{q \in R \mid qI = 0\}$ is an ideal of R. An ideal J of R is called *essential* if $\mathrm{Ann}_R(J) = 0$. An ideal J of R is called an *annihilator ideal* of R if $J = \mathrm{Ann}_R(I)$ for some ideal I of R. The following is well known in the literature (see, for instance, [17, Lemma 2.10]).

Lemma 2.6 *Let R be a semiprime ring. Then every annihilator ideal of Q is generated by one central idempotent.*

Given additive subgroups A, B of R, let AB (resp. $[A,B]$) denote the additive subgroup of R generated by all ab (resp. $[a,b]$) for $a \in A$ and $b \in B$. If A is generated by one element, say a, we write aB (resp. $[a,B]$) to stand for AB (resp. $[A,B]$).

Theorem 2.7 *Let R be a semiprime ring, $a_i \in Q$ and $g_i(t) \in C[t]$ for $1 \le i \le n$. Suppose that, given any $\mathbf{m} \in \mathrm{Spec}(\mathbf{B})$, there exists $\lambda_{\mathbf{m}} \in C$ such that $[\sum_{i=1}^n g_i(\lambda_{\mathbf{m}})a_i, Q] \subseteq \mathbf{m}Q$. Then $\sum_{i=1}^n g_i(\lambda)a_i \in C$ for some $\lambda \in C$.*

Proof The proof is analogous to that of Theorem 2.2. We only sketch it. Let

$$\Sigma := \Big\{ e \in \mathbf{B} \mid e\Big(\sum_{i=1}^{n} g_i(\beta) a_i \Big) \in C \text{ for some } \beta \in C \Big\}.$$

Applying an analogous argument as given in the proof of Theorem 2.2, we get that Σ is an ideal of the complete Boolean algebra $\mathbf{B}$. If $1 \in \Sigma$, then we are done. Suppose on the contrary that $1 \notin \Sigma$. Then there exists a maximal ideal $\mathbf{m}$ of $\mathbf{B}$ such that $\Sigma \subseteq \mathbf{m}$. By hypothesis, there exists $\lambda_{\mathbf{m}} \in C$ such that $[\sum_{i=1}^{n} g_i(\lambda_{\mathbf{m}})a_i, Q] \subseteq \mathbf{m}Q$. Note that $[\sum_{i=1}^{n} g_i(\lambda_{\mathbf{m}})a_i, Q]$ is an orthogonally complete subset of Q. In view of [1, Proposition 3.1.11], there exists $e \in \mathbf{B} \setminus \mathbf{m}$ such that $e[\sum_{i=1}^{n} g_i(\lambda_{\mathbf{m}})a_i, Q] = 0$. This implies that $e\sum_{i=1}^{n} g_i(\lambda_{\mathbf{m}})a_i \in C$ and so $e \in \Sigma$, contradicting to the fact that $\Sigma \subseteq \mathbf{m}$. ■

Let R be a semiprime S_{2n}-ring. Recall that R and Q satisfy the same GPIs with coefficients in Q. Thus, Q is also a semiprime S_{2n}-ring. It is known that every nilpotent element in a semiprime S_{2n}-ring has nilpotence index $\le n$. Thus, $a^n = 0$ for any nilpotent element $a \in Q$. We will use this fact in the proof below.

Proof of Theorem 1.6 By Lemma 2.6, $\mathrm{Ann}_Q(Q[a,K]Q) = e_1Q$ for some $e_1 \in \mathbf{B}$. Since a is a skew element, $Q[a,K]Q$ is a $*$-ideal of Q and so $e_1^* = e_1$. This implies that $[e_1a, e_1K] = 0$; that is, $e_1a \in Z(e_1K)$. Let $e_2 := 1 - e_1$. For simplicity of notation, let $R_2 := e_2Q \cap R$, $a_2 := e_2a$ and $Q_2 := e_2Q$. Then Q_2 is equal to the Martindale symmetric ring of quotients of R_2 (see [1, Proposition 2.3.14]). By assumption, we have $(\mathrm{ad}_{e_2a})^n(K(R_2)) = 0$, implying $(\mathrm{ad}_{e_2a})^n(K(Q_2)) = 0$ (see [2, Theorem 1.4.1]). By a direct computation, we get

$$(\mathrm{ad}_{e_2a})^{2n-1}(K(Q_2)^2) = 0. \tag{2.1}$$

Let $\mathbf{B}_2 := e_2\mathbf{B}$. Let $\mathbf{m} \in \mathrm{Spec}(\mathbf{B}_2)$. Note that Q_2 is a $(2n-1)!$-torsion free semiprime S_4-ring with involution $*$. By Lemma 2.1, $\mathrm{char}(Q_2/\mathbf{m}Q_2) = 0$ or a prime $p > 2n-1$.

Case 1: $\mathbf{m} = \mathbf{m}^*$. Then $*$ canonically induces an involution, denoted by $*$ also, on the prime ring $Q_2/\mathbf{m}Q_2$. That is, $\overline{x}^* := \overline{x^*}$ for $x \in Q$. We claim that $K(Q_2/\mathbf{m}Q_2) = (K(Q_2)+\mathbf{m}Q_2)/\mathbf{m}Q_2$. Clearly, $(K(Q_2)+\mathbf{m}Q_2)/\mathbf{m}Q_2 \subseteq K(Q_2/\mathbf{m}Q_2)$. For the reverse inclusion, let $\overline{y} \in K(Q_2/\mathbf{m}Q_2)$, where $y \in Q_2$. Since $\frac{1}{2} \in (Ce_2 + \mathbf{m}Q_2)/\mathbf{m}Q_2$, there exists $\overline{z} \in K(Q_2/\mathbf{m}Q_2)$, where $z \in Q_2$, such that

$$\overline{y} = 2\overline{z} = \overline{z} - \overline{z}^* = \overline{z - z^*} \in \big(K(Q_2) + \mathbf{m}Q_2\big)/\mathbf{m}Q_2.$$

Thus, $K(Q_2/\mathbf{m}Q_2) \subseteq (K(Q_2) + \mathbf{m}Q_2)/\mathbf{m}Q_2$, as asserted. By (2.1), we get

$$(\mathrm{ad}_{\overline{a_2}})^{2n-1}(\overline{K(Q_2)}^2) = 0.$$

Note that $\overline{K(Q_2)}^2$ is a Lie ideal of $\overline{Q_2}$ (see [14, Lemma 2.1]). Suppose first that $\overline{K(Q_2)}^2$ is noncentral. In view of [6, Theorem], $(\mathrm{ad}_{\overline{a_2}})^{2n-1}(\overline{Q_2}) = 0$. By Theorem 1.1, there exists $\lambda \in e_2C$ such that $(\overline{a_2} - \overline{\lambda})^n = 0$. But $Q_2/\mathbf{m}Q_2$ is a prime S_4-ring. This implies that $(\overline{a_2} - \overline{\lambda})^2 = 0$. That is, $(a_2 - \lambda)^2 \in \mathbf{m}Q_2$. Suppose next that $\overline{K(Q_2)}^2$ is a central Lie ideal. In particular, $\overline{a_2}^2 \in \overline{Ce_2}$.

Case 2: $\mathbf{m} \ne \mathbf{m}^*$. Then $\mathbf{m}^*Q_2 + \mathbf{m}Q_2/\mathbf{m}Q_2$, which is contained in $K(Q_2) + \mathbf{m}Q_2/\mathbf{m}Q_2$, is a nonzero ideal of the prime ring $Q_2/\mathbf{m}Q_2$. Thus, by (2.1),

$$(\mathrm{ad}_{\overline{a_2}})^{2n-1}(\overline{\mathbf{m}^*Q_2}^2) = 0,$$

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where $\overline{\mathbf{m}^* Q_2}$ is a nonzero ideal of $Q_2/\mathbf{m}Q_2$. Note that $\overline{\mathbf{m}^* Q_2}^2$ and $Q_2/\mathbf{m}Q_2$ satisfy the same GPIs (see [1, Theorem 6.4.1] or [7, Theorem 2]). Therefore, $(\mathrm{ad}_{\overline{a_2}})^{2n-1}(\overline{Q_2}) = 0$ (see also [9, Theorem]). Since $\mathrm{char}(Q_2/\mathbf{m}Q_2) = 0$ or a prime $p > 2n-1$. By Theorem 1.1, there exists $\lambda_{\mathbf{m}} \in Ce_2$ such that $(\overline{a_2} - \overline{\lambda_{\mathbf{m}}})^n = \overline{0}$. But $Q_2/\mathbf{m}Q_2$ is a prime S_4-ring. We have $(\overline{a_2} - \overline{\lambda_{\mathbf{m}}})^2 = \overline{0}$. That is, $(a_2 - \lambda_{\mathbf{m}})^2 \in \mathbf{m}Q_2$.

In either case, we have proved that given an $\mathbf{m} \in \mathrm{Spec}(\mathbf{B}_2)$, there exists $\lambda_{\mathbf{m}} \in Ce_2$ such that $[(a_2 - \lambda_{\mathbf{m}})^2, Q_2] \subseteq \mathbf{m}Q_2$. In view of Theorem 2.7, there exists $\lambda \in Ce_2$ such that $(a_2 - \lambda)^2 \in Ce_2$.

We claim that $(a_2 - \lambda)^2 = 0$. Suppose not. Let $b := a_2 - \lambda$ and $\beta := b^2$. Then $0 \neq \beta \in Ce_2$. Note that $(\mathrm{ad}_b)^n = (\mathrm{ad}_{a_2})^n = 0$ on K_2. Given any $k \in K_2$, we expand $(\mathrm{ad}_b)^n(k) = 0$ to get $2^{n-1}\beta^q k = 2^{n-1}\beta^{q-1}bkb$ if $n = 2q$ and $2^{n-1}\beta^q bk = 2^{n-1}\beta^q kb$ if $n = 2q+1$ for some positive integer q, where we have used the fact that

$$1 + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}.$$

Since Q_2 is 2-torsion free, we see that either $\beta^q k = \beta^{q-1}bkb$ or $\beta^q bk = \beta^q kb$. Since $\beta = b^2 \in C$, we get $\beta^q(bk - kb) = 0$ for all $k \in K_2$. By [2, Theorem 1.4.1], $\beta^q(bk - kb) = 0$ for all $k \in K(Q_2)$.

Let $\mathbf{m} \in \mathrm{Spec}(\mathbf{B}_2)$. Then $\overline{\beta}^q[\overline{b}, K(\overline{Q_2})] = 0$, where $\overline{Q_2} := Q_2/\mathbf{m}Q_2$. This implies that either $\beta \in \mathbf{m}Q_2$ or $[b, K_2] \subseteq \mathbf{m}Q_2$. Thus, $\beta Q_2[b, K_2]Q_2 \subseteq \mathbf{m}Q_2$. Note that $\bigcap_{\mathbf{m}\in\mathrm{Spec}(\mathbf{B}_2)} \mathbf{m}Q_2 = 0$. Therefore, $\beta Q_2[b, K_2]Q_2 = 0$. That is, $(e_2 a - \lambda)^2 Q[a, K]Q = 0$, implying that $(e_2 a - \lambda)^2 \in e_1 Q$ and so $(e_2 a - \lambda)^2 = 0$, as asserted. ∎

Lemma 2.8 *Suppose that R is a faithful f-free semiprime ring. Then eR is also a faithful f-free ring for any nonzero $e \in \mathbf{B}$.*

Proof Let N be a nonzero ideal of eR. Choose an essential ideal J of R such that $eJ \subseteq R$. Then eJR is a nonzero ideal of R contained in eR. Then $JN = eJN$, which is a nonzero ideal of R. Since R is faithful f-free, JN does not satisfy f. Note that $JN = eJN \subset N$. In particular, N does not satisfy f. This proves that eR is a faithful f-free ring. ∎

Proof of Theorem 1.7 By [22, Theorem 2.2], there exists orthogonal idempotents $g_1, g_2 \in C$, $g_1 + g_2 = 1$, such that $g_1 Q$ is faithful S_4-free and $g_2 Q$ is an S_4-ring. Since the ideal of Q generated by $S_4(x_1, \ldots, x_4)$ for all $x_i \in Q$ is a $*$-ideal, it follows that g_1 and g_2 are symmetric. In view of Theorems 1.5 and 1.6, there exist orthogonal symmetric idempotents $f_1, \ldots, f_4 \in C$, $f_1, f_2 \in g_1 C$, $f_3, f_4 \in g_2 C$, $f_1 + f_2 = g_1$, $f_3 + f_4 = g_2$, and $\mu_1, \mu_2 \in C$ such that

(i) $(f_1 a - \mu_1)^{[\frac{n+1}{2}]} = 0$;
(ii) $(f_2 a)^{[\frac{n+1}{2}]+1} = 0$ and $(f_1 + f_2)R$ is an faithful S_4-free ring;
(iii) $[f_3 a, K] = 0$;
(iv) $(f_4 a - \mu_2)^2 = 0$ and $(f_3 + f_4)R$ is an S_4-ring.

It follows from Lemma 2.6 that $\mathrm{Ann}_Q(QaQ) = (1 - e)Q$ for some symmetric idempotent $e \in C$. Thus, $RaR \subseteq eR$ and $\mathrm{Ann}_{eR}(RaR) = 0$. That is, RaR is an essential ideal of eR. Set $e_i = f_i e$ for $1 \le i \le 4$, $e_5 = 1 - e$ and $\lambda_i = e_i \mu_i$ for $i = 1, 2$.

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Then $(e_1 a - \lambda_1)^{[\frac{n+1}{2}]} = 0$, $(e_2 a)^{[\frac{n+1}{2}]+1} = 0$, $[e_3 a, K] = 0$, $(e_4 a - \lambda_2)^2 = 0$, and $e_5 a = 0$. Since $(e_1 + e_2)R = (e_1 + e_2)(f_1 + f_2)R$, it follows from Lemma 2.8 that $(e_1 + e_2)R$ is a faithful S_4-free ring. Finally, it is obvious that $(e_3 + e_4)R$ is an S_4-ring since $(e_3 + e_4)R \subseteq (f_3 + f_4)R$ and $(f_3 + f_4)R$ is an S_4-ring. This proves (i)–(v). ■

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