Integral Equations Satisfied by Lamé-Wangerin Functions

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Summary.

Integral equations are obtained with nuclei $(1 - zt/a)^{2n}$ and $(z-t)^{2n}$ which are satisfied by characteristic solutions of the transformed Lamé-Wangerin equation of order n, and each of the two characteristic solutions is expressed in terms of the other by a contour integral.

1. *Introduction.*

Lamé-Wangerin functions are the solutions of Lamé's differential equation of order n when n is half of an odd integer. If $n = m + \frac{1}{2}$, where m is an integer, Lamé's equation in algebraic form is

$$4x(x-1)(x-a)\frac{d^2u}{dx^2} + 2\{3x^2 - 2(a+1)x + a\}\frac{du}{dx}$$
$$- \{(m+\tfrac{1}{2})(m+\tfrac{3}{2})x + h\}u = 0 \qquad (1.1)$$

and there is no loss of generality in taking the finite singularities as 0, 1 and a.

Writing $\quad x = (z^2 - a)^2 / 4z(z-1)(z-a)$

and $\quad u(x) = \{z(z-1)(z-a)\}^{(2m+1)/4}\, v(z),$

we have the transformed equation considered by Halphen and others: [1]

$$z(z-1)(z-a)\frac{d^2 v}{dz^2} - m\left\{3z^2 - 2(a+1)z + a\right\}\frac{dv}{dz}$$
$$+ \{(m+\tfrac{1}{2})(m+\tfrac{3}{2})z - h - (m+\tfrac{1}{2})^2(a+1)\}\,v = 0. \qquad (1.2)$$

If h has one of a set of $m+1$ characteristic values this equation has two solutions which are polynomials in z. I shall denote them by

$$v_1(z) = \sum_{\nu=0}^{m} c_\nu a^{m+\frac{1}{2}-\nu} z^\nu,$$

$$v_2(z) = \sum_{\nu=0}^{m} c_\nu z^{2m+1-\nu}, \qquad (1.3)$$

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[1] G. H. Halphen, *Traité des Fonctions Elliptiques* (Paris, 1888), t.2, p. 471. Whittaker and Watson, *Modern Analysis* (Cambridge, 1920, 3rd ed.), §§ 23·7.

the coefficients c_ν being the same in the two solutions and $c_0 = 1$. Substitution of either of these solutions in (1.2) leads to the recurrence relations

$$(\nu - m)(\nu + 1)\, c_{\nu+1} - \{h + (m + \tfrac{1}{2} - \nu)^2 (a + 1)\} c_\nu + a\,(m + 1 - \nu)(2m + 2 - \nu)\, c_{\nu-1} = 0 \tag{1.4}$$

for $\nu = 0, 1, \ldots, m$.

The first m relations determine the coefficients c_1 to c_m, the coefficient c_ν being a polynomial of degree ν in h; and putting $\nu = m$ we have

$$-\{h + (a + 1)/4\}\, c_m + (m + 2) a\, c_{m-1} = 0. \tag{1.5}$$

The expression in (1.5) is a polynomial of degree $m + 1$ in h determining $m + 1$ characteristic values of h which are real and distinct.

We have also

$$v_2(z) = \frac{z^{2m+1}}{a^{m+\frac{1}{2}}} v_1(a/z), \qquad v_1(z) = \frac{z^{2m+1}}{a^{m+\frac{1}{2}}} v_2(a/z). \tag{1.6}$$

2. *Types of integral equation.*

The differential equation (1.2) in Riemannian form is

$$v(z) = P\left\{\begin{matrix} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & -m & z \\ m+1 & m+1 & m+1 & -2m-1 & \end{matrix}\right\}. \tag{2.1}$$

A general theorem [1] on integral equations associated with differential equations of this type shows that characteristic and certain other solutions of (1.2) satisfy an integral equation whose nucleus is the hypergeometric function

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & -m & zt/a \\ m+1 & 2m+1 & -2m-1 & \end{matrix}\right\}.$$

We may take this function as being $(1 - zt/a)^{2m+1}$ and consider integrals of the type

$$u(z) = \frac{\lambda}{2\pi i} \int_C \left(1 - \frac{zt}{a}\right)^{2m+1} \frac{v(t)\, dt}{\{t(t-1)(t-a)\}^{m+1}} \tag{2.2}$$

where C is a closed curve encircling one or more of the singularities $t = 0$, $t = 1$, $t = a$.

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[1] C. G. Lambé and D. R. Ward, *Quart. J. of Math.* (Oxford), 5 (1934), 81 and A. Erdélyi, *Quart. J. Math.* (Oxford), 13 (1942), 107.

The general theorem shows that if $v(z)$ is a solution of (1.2) corresponding to a characteristic value h_ν of h, then $u(z)$ is also a solution of (1.2) for the same value of h.

From the relations (1.6) it follows that if z is replaced by a/z,

$$u(a/z) = a^{m+\frac{1}{2}} z^{-2m-1} w(z),$$

where $w(z)$ is a solution of (1.2). Hence integrals of the type

$$w(z) = \frac{\lambda}{2\pi i} \int_C (z-t)^{2m+1} \frac{v(t)\, dt}{\{t(t-1)(t-a)\}^{m+1}} \tag{2.3}$$

are also solutions of (1.2).

3. *Contours enclosing one singularity.*

If C is a contour enclosing only the singularity $t = 0$, the value of the integral (2.2) is the residue at $t = 0$. Hence $u(z)$ must be of degree m in z and therefore must be a multiple of $v_1(z)$.

If C encloses only the singularity $t = 1$, then, since powers of $(t-1)$ higher than the m^{th} in the expansion of

$$(1 - zt/a)^{2m+1} = \{(1 - z/a) - (t-1)z/a\}^{2m+1}$$

give zero residue at $t = 1$, it follows that $(z-a)^{m+1}$ is a factor of $u(z)$ and hence

$$u(z) = A\{v_1(a) v_2(z) - v_2(a) v_1(z)\}$$

where A is a constant.

If C encloses the singularity $t = a$, then, since

$$(1 - zt/a)^{2m+1} = \{(1-z) - (t-a)z/a\}^{2m+1},$$

it follows that $(z-1)^{m+1}$ is a factor of $u(z)$ and hence

$$u(z) = B\{v_1(1) v_2(z) - v_2(1) v_1(z)\}$$

where B is a constant.

Hence for these contours, if the equation (2.2) is to be an integral equation the function $v(t)$ in the integrand must be the same function of t as the corresponding $u(z)$ is of z. Using the relations (1.6) we have, therefore, solutions of the integral equation.

$$v(z) = \frac{\lambda}{2\pi i} \int_C (1 - zt/a)^{2m+1} \frac{v(t)\, dt}{\{t(t-1)(t-a)\}^{m+1}} \tag{3.1}$$

for contours encircling one singularity only as:

$$\begin{aligned} &\text{around } t = 0,\ V_0(z) = v_2(0) v_2(z) - v_1(0) v_1(z), \\ &\quad ,, \quad t = 1,\ V_1(z) = v_2(1) v_2(z) - v_1(1) v_1(z), \\ &\quad ,, \quad t = a,\ V_a(z) = v_2(a) v_2(z) - v_1(a) v_1(z). \end{aligned} \tag{3.2}$$

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4. *Value of* λ.

The constant λ involves the characteristic value of h for any solution of the integral equation (3.1) and may be evaluated in terms of the coefficient c_m of z^m in $v_1(z)$. Let λ_0, λ_1, λ_a denote the values of λ for the three contours considered.

Substituting $V_0(z)$ in (3.1) and comparing coefficients of z^m, we have

$$- v_1(0)\, a^{\frac{1}{2}} c_m = \frac{\lambda_0}{2\pi i} \int_{(0+)} \frac{(2m+1)!}{m!\,(m+1)!} \left(-\frac{1}{a}\right)^m \frac{V_0(t)\, dt}{t\,(t-1)^{m+1}(t-a)^{m+1}}$$

$$= \lambda_0 \frac{(2m+1)!}{m!\,(m+1)!} \left(-\frac{1}{a}\right)^m \frac{V_0(0)}{a^{m+1}},$$

$$V_0(0) = - v_1^2(0) = - a^{m+\frac{1}{2}}\, v_1(0),$$

and hence
$$\lambda_0 = (-)^m \frac{\Gamma(\frac{1}{2})\,(m+1)!}{2^{2m+1}\,\Gamma(m+\frac{3}{2})}\, a^{m+1} c_m. \qquad (4.1)$$

We have seen that $V_1(z)$ contains the factor $(z-a)^{m+1}$, and hence $V_1(t)/(t-a)^{m+1}$ is a polynomial in $(t-1)$ of which the first term is $v_2(1)\,(t-1)^m$. Substituting $V_1(t)$ in (3.1) and comparing coefficients of z^{m+1}, we have

$$v_2(1)\, c_m = \frac{\lambda_1}{2\pi i} \int_{(1+)} \frac{(2m+1)!}{m!\,(m+1)!} \left(-\frac{1}{a}\right)^{m+1} \frac{v_2(1)\,(t-1)^m + \dots}{(t-1)^{m+1}}\, dt.$$

$$= \lambda_1 \frac{(2m+1)!}{m!\,(m+1)!} \left(-\frac{1}{a}\right)^{m+1} v_2(1).$$

Hence $\lambda_1 = -\lambda_0$.

Similarly, substituting $V_a(z)$ in (3.1) and comparing coefficients of z^{m+1}, we find that $\lambda_a = -\lambda_0$.

Therefore

$$\lambda_a = \lambda_1 = -\lambda_0 = (-)^{m+1} \frac{\Gamma(\frac{1}{2})\,(m+1)!}{2^{2m+1}\,\Gamma(m+\frac{3}{2})}\, a^{m+1} c_m. \qquad (4.2)$$

5. *Contour enclosing two singularities.*

Since $V_1(t)$ has the factor $(t-a)^{m+1}$ and $V_a(t)$ the factor $(t-1)^{m+1}$, we have

$$A\ V_1(z) = \frac{\lambda_1}{2\pi i} \int_{(1+)} \left(1 - \frac{zt}{a}\right)^{2m+1} \frac{A\ V_1(t) + B\ V_2(t)}{\{t(t-1)(t-a)\}^{m+1}}\, dt$$

$$B\ V_a(z) = \frac{\lambda_1}{2\pi i} \int_{(a+)} \left(1 - \frac{zt}{a}\right)^{2m+1} \frac{A\,V_1(t) + B\,V_2(t)}{\{t\,(t-1)\,(t-a)\}^{m+1}}\, dt,$$

where A and B are any constants.

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Hence

$$A V_1(z) + B V_a(z) = \frac{\lambda_1}{2\pi i}\int_{(1+,a+)}\left(1-\frac{zt}{a}\right)^{2m+1}\frac{A V_1(t) + B V_2(t)}{\{t(t-1)(t-a)\}^{m+1}}\,dt.$$

It follows that any solution of the differential equation (1.2) when h has a characteristic value satisfies the integral equation

$$v(z) = \frac{\lambda_1}{2\pi i}\int_{(1+,a+)}\left(1-\frac{zt}{a}\right)^{2m+1}\frac{v(t)\,dt}{\{t(t-1)(t-a)\}^{m+1}}\,dt. \qquad (5.1)$$

6. *Nucleus* $(z-t)^{2m+1}$.

Substituting a/z for z in (3.1), we have

$$z^{2m+1}v(a/z) = \lambda\int_C (z-t)^{2m+1}\frac{v(t)\,dt}{\{t(t-1)(t-a)\}^{m+1}}.$$

Hence for contours around one singularity we have, using (1.6),

$$a^{m+\frac{1}{2}}v_2(z) = \frac{\lambda_0}{2\pi i}\int_{(0+)}(z-t)^{2m+1}\frac{v_1(t)\,dt}{\{t(t-1)(t-a)\}^{m+1}}$$

$$-V_a(z) = \frac{\lambda_1}{2\pi i}\int_{(1+)}(z-t)^{2m+1}\frac{V_1(t)\,dt}{\{t(t-1)(t-a)\}^{m+1}}$$

$$-a^{2m+1}V_1(z) = \frac{\lambda_a}{2\pi i}\int_{(a+)}(z-t)^{2m+1}\frac{V_a(t)\,dt}{\{t(t-1)(t-a)\}^{m+1}}.$$

Combining these results as before, we have

$$v_2(z) = a^{-m-\frac{1}{2}}\frac{\lambda_1}{2\pi i}\int_{(1+,a+)}(z-t)^{2m+1}\frac{v_1(t)\,dt}{\{t(t-1)(t-a)\}^{m+1}} \qquad (6.1)$$

$$v_1(z) = a^{-m-\frac{1}{2}}\frac{\lambda_1}{2\pi i}\int_{(1+,a+)}(z-t)^{2m+1}\frac{v_2(t)\,dt}{\{t(t-1)(t-a)\}^{m+1}}. \qquad (6.2)$$

Hence the solution of the integral equation

$$v(z) = \frac{\mu}{2\pi i}\int_{(1+,a+)}(z-t)^{2m+1}\frac{v(t)\,dt}{\{t(t-1)(t-a)\}^{m+1}} \qquad (6.3)$$

is $\qquad v(z) = v_2(z) \pm v_1(z)$

and $\qquad \mu = \pm(-)^{m+1}\dfrac{\Gamma(\frac{1}{2})(m+1)!}{2^{2m+1}\Gamma(m+\frac{3}{2})}a^{\frac{1}{2}}c_m.$

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