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CONVEX SETS, CANTOR SETS AND A MIDPOINT PROPERTY

BY
HAROLD REITER[1]

1. **Introduction.** It is well known that every point of the closed unit interval I can be expressed as the midpoint of two points of the Cantor ternary set D. See [2, p. 549] and [3, p. 105]. Regarding I as a one dimensional compact convex set, it seems natural to try to generalize the above result to higher dimensional convex sets. We prove in section 3 that every convex polytope K in Euclidean space R^d contains a topological copy C of D such that each point of K is expressible as a midpoint of two points of C. Also, we give necessary and sufficient conditions on a planar compact convex set for it to contain a copy of D with the midpoint property above. In the final section we prove a result on minimal midpoint sets.

2. **Notation and definitions.** In the sequel, the term Cantor set will mean any non-empty compact totally disconnected metric space with no isolated points; that is, any homeomorph of the Cantor ternary set. If A and B are subsets of R^d (or any Banach space), and α is a real number, then, as usual, $A+B=\{a+b \mid a\in A \text{ and } b\in B\}$ and $\alpha A=\{\alpha a \mid a\in A\}$. If K is a compact subset of metric space, the collection 2^K of closed subsets of K can be metrized by the Hausdorff metric. For the most part, the definitions and notation correspond to that of Grunbaum [1].

DEFINITION. Let Y be a subset of a linear space L. We call a subset X of Y *a pole set for* Y provided each point of Y is a midpoint of a pair of (not necessarily distinct) points of X. If X is a pole set for Y and X is a Cantor set, then Y is said to contain a Cantor pole set (for itself). Thus X is pole set for Y if $\frac{1}{2}X+\frac{1}{2}X\supset Y$. Of course, if Y is convex $\frac{1}{2}X+\frac{1}{2}X\subset Y$. We use conv S and int S to denote the convex hull of S and the interior of S respectively.

3. Our strategy for proving that every convex polytope has a Cantor pole set begins by showing that for every d, the d-simplex has this property. We then express each convex polytope as a finite union of simplices, and from this fact the desired conclusion easily follows. First, let us record a few remarks, the proofs of which are left to the reader. The cube $C^d=\{(x_1,\ldots,x_d) \mid 0\le x_i\le 1$ for $i=1,2,\ldots,d\}$ contains a Cantor pole set. The property of having a Cantor

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pole set is an affine invariant; that is, if Y has a pole set homeomorphic with D and ϕ is a non-singular affine mapping, then $\phi(D)$ is Cantor pole set for $\phi(Y)$.

LEMMA 1. *Let d be a positive integer. For each point P of the d-simplex T^d, there is a non-singular affine mapping ϕ of the d-cube C^d into T^d such that $\phi(C^d)$ is a neighborhood of P.*

Proof. We may take T^d to be the convex hull, $\text{conv}\{0, e_1, e_2, \ldots, e_d\}$, where 0 is the zero vector and $\{e_i\}_{i=1}^d$ is the standard basis for R^d. Because of the homogeneity of the j-faces of T^d under non-singular affine maps, we may assume that (1) each face containing P has 0 as a vertex and that (2) if $P=(x_1, x_2, \ldots, x_d)$, then $x_j=0$ implies $x_{j+1}=0$. Now, if $P=0$, the (unique) linear map satisfying $\phi(0)=0$ and $\phi(e_1)=e_i/2$ fulfills the requirements. We proceed by induction on the number of non-zero coordinates of P. If P has one non-zero coordinate, then by (2), it must be x_1e_1 and by (1), we have $0<x_1<1$. Then the mapping defined by $\psi(Q)=(\frac{4}{3}-\frac{4}{3}x_1)\phi(Q)+(\frac{4}{3}x_1-\frac{1}{3})e_1$ is a non-singular affine map of C^d into T^d of the desired type since ϕ is such a map. Now assume that for every point having n non-zero coordinates ($n<d$), there is a non-singular affine mapping ϕ of C^d into T^d such that $\phi(C^d)$ is a neighborhood of the point in T^d. Let $P=(x_1, \ldots, x_n, x_{n+1}, \ldots, x_d)$ have $n+1$ non-zero coordinates. Apply the induction hypothesis to the point $P'=(x_1, \ldots, x_n, 0, 0, \ldots)$ to obtain a mapping ϕ. Note that the directed segment L from P' through P intersects the boundary of T^d in a point R different from P, since $R=P$ would violate assumption (1). Select a point S in $\phi(C^d)\cap L-\{P'\}$ so that $S\in \text{relint}[\phi(C^d)\cap F_n]$ where F_n is the face $\{(x_1, x_2, \ldots, x_d) \mid x_i=0 \text{ when } i>n+1\}$. Now suppose $P=\alpha S+(1-\alpha)R$. Then the mapping ψ defined by $\psi(Q)=\alpha\phi(Q)+(1-\alpha)R$ has the desired properties. This completes the proof of the lemma.

THEOREM 1. *Let T^d be a d-simplex. Then T^d contains a Cantor set D of measure zero which is a pole set for T^d.*

Proof. The following proof was suggested to the author by V. Klee. For each P in T^d, let $C^d(P)$ be the non-singular affine image of C^d containing P in its interior (relative to T^d) guaranteed by Lemma 1. Since T^d is compact, a finite subcollection $C^d(P_1)$, $C^d(P_2), \ldots, C^d(P_n)$ can be selected which covers T^d. By the remarks at the beginning of this section, each of the $C^d(P_i)$ contains a Cantor pole set, D_i. Then $D=\bigcup_{i=1}^n D_i$ is a Cantor set and D is clearly a pole set for T^d. Also, since each D_i can be chosen to have measure zero, D can be so chosen.

THEOREM 2. *Let K be a convex d-polytope in R^d, $d\geq 1$. Then K contains a Cantor pole set C. Moreover, C can be chosen to have measure zero.*

Proof. We prove, by induction on d, that each convex d-polytope K can be expressed as a finite union of d-simplices. For $d=1$, the result follows from Theorem 1 because a 1-polytope is a 1-simplex. Now assume that each d-1-polytope admits a decomposition into d-1-simplices and let K be a d-polytope. Then each facet (maximal proper face) F_i of K, being a d-1-polytope, can be expressed as the union of a finite number t_i of d-1-simplices $S_{i,1}, S_{i,2}, \ldots, S_{i,t_i}$. Then

$$F_i=\{S_{i,n} \mid n=1,2,\ldots,t_i\}, \quad \text{for} \quad i=1,2,\ldots,N.$$

If P is a point of the interior of K, then

$$K_{i,n}=\operatorname{conv}(\{P\}\cup S_{i,n})$$

is a d-simplex and we have

$$K=\bigcup_{i=1}^{N}\bigcup_{n=1}^{t_i} K_{i,n}.$$

By Theorem 1, each $K_{i,n}$ contains a Cantor pole set of measure zero. The desired pole set is formed by taking the union of the pole sets for the $K_{i,n}$.

Let K be a compact convex subset of R^d, and let S be a subset of K whose set of midpoints is K. Then every extreme point of K must belong to S. Thus, if a compact convex set is to have a Cantor pole set, its set of extreme points must be zero dimensional. For planar compact convex sets, that condition is also sufficient.

THEOREM 3. *Let K be a compact convex subset of R^2. Then K contains a Cantor pole set if and only if the set* $\operatorname{ext} K$ is zero dimensional.

Proof. The proof of the necessity is implicit in the remarks above. To prove the sufficiency, let $\mathcal{F}=\{F_\alpha:\alpha\in A\}$ be the family of proper faces of K. Each F_α is a segment or single point. If P_0 is a point of $\operatorname{int} K$, then each set $K_\alpha=\operatorname{conv}(\{P_0\}\cup F_\alpha)$ is a segment or a triangular region. We construct in each K_α a Cantor pole set for K_α as follows. Let D be any Cantor pole set for the triangular region T^2.

Let $\phi_1,\phi_2,\ldots,\phi_6$ be the 6 affine mappings of R^2 onto R^2 leaving T^2 invariant. Let $D'=\bigcup_{i=1}^{6}\phi_i(D)$. If K_α is a triangular region and ϕ is an affine map of T^2 onto K_α, let $C_\alpha=\phi(D')$. If K_α is a segment let $C_\alpha=\phi(D'\cap[0,e_1])$, where $[0,e_1]=\{te_1:0\le t\le 1\}$. Now let $C=\bigcup\{C_\alpha:\alpha\in A\}$. We must show that C is a Cantor set and that C is a pole set for K. The latter statement is clear since $K=\bigcup\{K_\alpha:\alpha\in A\}$.

To prove the former, note first that the subspace $\{C_\alpha:\alpha\in A\}$ of 2^K is homeomorphic with the subspace $\mathcal{F}$ of 2^K. But $\mathcal{F}$ is compact. In fact, if $\{F_i\}$ is a convergent sequence of faces, then the sequence $\{\delta(F_i)\}$ has limit zero, and the sequence $\{\operatorname{ext}(F_i)\}$ of extreme points of the F_i converges to a singleton which

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must also be a face. This is due to the fact that the set of extreme points of K is closed and $\text{ext } K = \bigcup\{\text{ext } F_\alpha : \alpha \in A\}$. Thus, the set $C = \bigcup\{C_\alpha : \alpha \in A\}$ is a union of a compact family of compact sets. Such a union is known to be compact.

Since none of the C_α has isolated isolated points, C also does not. It remains to show that C is zero-dimensional. We do this by showing that C is zero dimensional at each of its points. There is no problem if the point P of C belongs to the relative interior of a triangular K_α. If P does not belong to the relative interior of some triangular region K_α and $P \neq P_0$, then there is some singleton face $F_\beta = \{Q_0\}$ such that $P = \lambda P_0 + (1-\lambda)Q_0$ for $\lambda \in [0, 1)$.

Let U be an open ball in K centered at P. We must find an open subset V of K contained in U, containing P and having empty boundary. Since C_β is zero-dimensional, there exists points R and S of $K_\beta \setminus C_\beta \cap U$ such that P is between R and S. We will handle only the case where $P \neq Q_0$, since the other case follows similarly. Let U_R and U_S be open balls about R and S respectively whose intersection with C is empty. It can be shown that there exist four points R_1, R_2, S_1, S_2 satisfying the following properties:

1. $\{R_1, S_1\} \subset \text{rel int } K_{\alpha_1} \cap U$ and $\{R_2, S_2\} \subset \text{rel int } K_{\alpha_2} \cap U$, for triangular regions K_{α_1} and K_{α_2}.
2. $\{R_1, R_2\} \subset U_R$ and $\{S_1, S_2\} \subset U_S$.
3. P belongs to the convex hull of $\{R_1, S_1, R_2, S_2\}$.

Now let A_1 be an arc in $U \cap K_{\alpha_1} \setminus C$ with endpoints R_1 and S_1, let A_2 be an arc in $U \cap K_{\alpha_2} \setminus C$ with endpoints R_2 and S_2, let A_3 be an arc in $U_R \cap U$ with endpoints R_1 and R_2 and finally let A_4 be an arc in $U_S \cap U$ with endpoints S_1 and S_2. Then $A_1 \cup A_2 \cup A_3 \cup A_4$ contains a simple closed curve γ with P in its interior. The interior of γ is the desired neighborhood of P. This proves that $C \setminus \{P_0\}$ is zero-dimensional. The addition of P_0 does not change the dimension, hence C_α is a Cantor set. Since each C is a pole set for K_α, $\cup C_\alpha$ is a pole set for $K = \bigcup K_\alpha$. This completes the proof of the theorem.

In this section we turn out attention to the notion of minimal pole sets. A closed pole set S for K shall be called minimal (for K) provided no proper closed subset of S is a pole set for K. We prove the existence of minimal pole sets for any compact convex subset of a Banach space. We end the paper with two examples of minimal pole sets and a question.

THEOREM 4. *Let X be a pole set for a compact convex subset K of a Banach space. Then X contains a minimal pole set for K.*

Proof. The proof is by Zorn's lemma. Let X_t be a chain of pole sets for K each contained in X. The mapping $\phi: 2^K \to 2^K$ defined by $\phi(Y) = \frac{1}{2}Y + \frac{1}{2}Y$ is continuous in the topology of 2^K. Also $\phi(X_t) = K$ for all t. Regarding X_t as a net in 2^K we have $\lim_t X_t = \bigcap X_t$. Hence $\phi(\lim_t X_t) = \lim_t K = K$.

It is tempting to conjecture that if K is a compact convex set with a Cantor pole set, then each midpoint set for K contains a Cantor pole set. This is seen to be false by the following two examples.

The set $[0, 1/n] \cup \{2/n, 3/n, 4/n, \ldots, n-2/n\} \cup [n-1/n, 1]$ is seen to be a minimal pole set for $[0, 1]$. Also, the boundary C of a circular disc D is a pole set for D. The proof of this uses the intermediate value theorem. The details are left to the reader. In fact, C is a minimal pole set for the D. Now for the question. Let K be a planar compact convex set with non-empty interior and boundary B. Then B is a pole set for K. Is B necessarily a minimal pole set? It can be seen that if K is a polygon, then B minimal.

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Dept. of Math.
University of North Carolina
UNCC Station, Charlotte, N.C.
28223 U.S.A.

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