A NOTE ON UNIVERSALLY ZERO-DIVISOR RINGS

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In this note we consider commutative rings with identity over which every unitary module is a zero-divisor module. We call such rings Universally Zero-divisor (UZD) rings. We show (1) a Noetherian ring $R$ is a UZD if and only if $R$ is semilocal and the Krull dimension of $R$ is at most one, (2) a Prüfer domain $R$ is a UZD if and only if $R$ has only a finite number of maximal ideals, and (3) if a ring $R$ has Noetherian spectrum and descending chain condition on prime ideals then $R$ is a UZD if and only if Spec $(R)$ is a finite set. The question of ascent and descent of the property of a ring being a UZD with respect to integral extension of rings has also been answered.

INTRODUCTION

Let $R$ be a commutative ring with identity. Let $M$ be a unitary $R$-module. Recall that $M$ is said to be a Zero-divisor $R$-module if for every submodule $N$ of $M$, $N \neq M$, the set of zero divisors of $M/N$ (that is, $\{x \in R : zm \in N$ for some $m \in M \setminus N\}$) denoted by $Z_R(M/N)$ is the union of a finite number of prime ideals of $R$. $R$ is said to be a Zero-divisor ring (Z.D. ring) if $R$ is a Z.D. $R$-module [4]. In this note we study the properties of those commutative rings $R$ with identity for which every $R$-module is a Z.D. $R$-module.

All rings considered here are assumed to be commutative and with identity. If $A \subseteq B$ are rings we assume that $A$ and $B$ have the same identity element. By dimension of a ring we mean the Krull dimension. Modules are assumed to be unitary. Whenever a set $A$ is a subset of a set $B$ and $A \neq B$ we denote this symbolically as $A \subset B$.

We begin with the following definition.

We say a ring $R$ is a Universally Zero-divisor (UZD) ring if every $R$-module is a Z.D. $R$-module.

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233
**PROPOSITION 1.** Let $R$ be a ring. Then $R$ is a UZD if and only if the union of any family of prime ideals of $R$ is the union of a finite number of prime ideals of $R$ (not necessarily belonging to the same family).

**PROOF:** Assume that $R$ is a UZD. Let $\{P_\alpha\}_{\alpha \in A}$ be any family of prime ideals of $R$. Let $M = \bigoplus_{\alpha \in A} R/P_\alpha$ (that is, direct sum of the $R$-modules $R/P_\alpha$). It is easy to see that $Z_R(M) = \bigcup_{\alpha \in A} P_\alpha$. Since $R$ is a UZD, $M$ is a Z.D. $R$-module and so $Z_R(M) = \bigcup_{i=1}^t Q_i$ for some finite number of prime ideals $Q_1, \ldots, Q_t$ of $R$. Thus $Z_R(M) = \bigcup_{\alpha \in A} P_\alpha = \bigcup_{i=1}^t Q_i$.

Conversely assume that the union of any family of prime ideals of $R$ is the union of a finite number of prime ideals of $R$. Let $M$ be any $R$-module. Let $N$ be a submodule of $M$, $N \neq M$. Notice that $R \setminus Z_R(M/N)$ is a saturated multiplicatively closed subset of $R$. Hence by [1, Exercise 7 (i), p.44] $Z_R(M/N)$ is a union of prime ideals of $R$. By assumption it follows that $Z_R(M/N)$ is the union of a finite number of prime ideals of $R$. Thus $M$ is a Z.D. $R$-module. Hence we obtain that $R$ is a UZD. \(\square\)

**REMARK 2.** Using the above Proposition we see that $R$ is a UZD implies that any homomorphic image of $R$ is a UZD and $S^{-1}R$ is a UZD for every multiplicatively closed subset $S$ of $R$, $S \subseteq R \setminus \{0\}$.

**PROPOSITION 3.** Let $R$ be an integral domain with quotient field $K$. Then $R$ is a UZD if and only if $K$ is a Z.D. $R$-module.

**PROOF:** The “only if” part is clear. The “if” part follows from [13, Remark 2.1] and Proposition 1. \(\square\)

**PROPOSITION 4.**

(i) Let $R$ be a Noetherian ring. Then $R$ is a UZD if and only if $R$ is semilocal and the dimension of $R$ is at most 1.

(ii) A Prüfer domain $R$ is a UZD if and only if $R$ has only a finite number of maximal ideals.

The proof of Proposition 4 makes use of the following results.

**LEMMA 5.** If a ring $R$ is a UZD then $R$ has only a finite number of maximal ideals.

**PROOF:** Let $\{M_\alpha\}_{\alpha \in A}$ be the family of all maximal ideals of $R$. By Proposition 1, $\bigcup_{\alpha \in A} M_\alpha = \bigcup_{i=1}^s Q_i$ for some finite number of prime ideals $Q_1, \ldots, Q_s$ of $R$. Let $M_i$ $(i = 1, \ldots, s)$ be maximal ideals of $R$ such that $Q_i \subseteq M_i$ (for $i = 1, \ldots, s$). Then it is clear that $\bigcup_{\alpha \in A} M_\alpha = \bigcup_{i=1}^s Q_i = \bigcup_{i=1}^s M_i$. It is now evident that distinct elements among $M_1, \ldots, M_s$ are all the maximal ideals of $R$. \(\square\)
RESULT 6. In a Noetherian ring every prime ideal has finite height [1, Corollary 11.12].

RESULT 7. In a Noetherian ring any prime ideal of height 2 contains an infinite number of height 1 prime ideals [11, Theorem 144].

RESULT 8. Let \( I \) be any ideal of a Noetherian ring \( R \), \( I \neq R \). Then the set of prime ideals of \( R \) which are minimal over \( I \) is finite.

Result 8 follows by applying [1, Exercise 9, p.79] to the Noetherian ring \( R/I \).

PROOF OF PROPOSITION 4: (i) Assume that \( R \) is a Noetherian ring and \( R \) is a UZD. By Lemma 5, \( R \) is semilocal. We prove that the dimension of \( R \) is at most 1. Suppose that the dimension of \( R \) is at least 2. Then by Result 6 it follows that there exists a prime ideal \( p \) of \( R \) such that height \( p = 2 \).

Let \( \{Q_\alpha\}_{\alpha \in A} \) be the set of all height one prime ideals of \( R_p \). Note that \( \{Q_\alpha\}_{\alpha \in A} = \{P_\alpha R_p\}_{\alpha \in A} \) where \( \{P_\alpha\}_{\alpha \in A} \) are prime ideals of \( R \) such that height \( P_\alpha = 1 \) and \( P_\alpha \subset p \) for each \( \alpha \in A \). By Result 7 it follows that \( A \) is an infinite set. Result 8 and [2, Exercise 2, p.121] imply that there exists an element \( y \in pR_p \) which is not in any of the minimal prime ideals of \( R_p \). Let them be \( \{P_{\alpha_i}R_p\}_{i=1}^t \). Let \( \Lambda = A \setminus \{\alpha_1, \ldots, \alpha_t\} \). Then it is easy to see that \( \bigcup_{\alpha \in \Lambda} P_\alpha R_p \) cannot be equal to the union of any finite number of prime ideals of \( R_p \). This is in contradiction to the fact that \( R_p \) is a UZD. Thus \( R \) is semilocal and the dimension of \( R \) is at most 1.

Conversely if \( R \) is semilocal and the dimension of \( R \) is at most 1 then any prime ideal of \( R \) is either a maximal ideal of \( R \) or a minimal prime ideal of \( R \). Since the set of minimal prime ideals of a Noetherian ring is finite we obtain that \( R \) has only a finite number of prime ideals. It is then clear that \( R \) is a UZD.

(ii) In view of Lemma 5, we need only prove the “if part” of (ii). Assume that \( R \) is a Prüfer domain with only a finite number of maximal ideals \( M_1, \ldots, M_t \). Let \( \{P_\alpha\}_{\alpha \in A} \) be any family of prime ideals of \( R \). Let \( C_i \) be the union of those \( P_\alpha \)'s which are contained in \( M_i \) (for \( i = 1, \ldots, t \)). Now \( R_{M_i} \) is a valuation ring and so in the case \( C_i \neq 0 \), \( C_i \) is the union of some pairwaise comparable prime ideals of \( R \) and hence \( C_i \in \text{Spec}(R) \). This is true for \( i = 1, \ldots, t \). Further it is clear that \( \bigcup_{\alpha \in A} P_\alpha = \bigcup_{i=1}^t C_i \). Hence by Proposition 1, \( R \) is a UZD.

REMARK 9. We have noted in Proposition 3 that an integral domain \( R \) is a UZD if and only if the quotient field of \( R \) is a Z.D. \( R \)-module. We now mention an example which shows (for an arbitrary ring \( R \)) that “the total quotient ring of \( R \) is a Z.D. \( R \)-module” need not imply that \( R \) is a UZD. Consider \( T = \mathbb{Q}(\sqrt{2}) [[X, Y, Z]] \), the power series ring in three indeterminates \( X, Y, Z \) over \( \mathbb{Q}(\sqrt{2}) \) where \( \mathbb{Q} \) denotes the field of rationals. Let \( M \) denote the unique maximal ideal of \( T \). Let \( S = \mathbb{Q} + M \). Notice that
the dimension of $T$ is 3 and $T$ is a finite integral extension of $S$. Hence by [6, 11.8, p.106] and [3, Theorem 2] it follows that the dimension of $S$ is 3 and $S$ is Noetherian. Consider the chain of prime ideals $(0) \subset P_1 \subset P_2 \subset M$ of $S$ where $P_1 = XT$ and $P_2 = XT + YT$. Let $R = S/(XS)$. We now show that the unique maximal ideal $M/(XS)$ of $R$ is full of zero divisors. For an element $m \in M$, let $m + XS \in M/(XS)$. Note that $(m + XS)(\sqrt{2}X + XS) = \sqrt{2}(mX) + XS = X(\sqrt{2}m) + XS = XS$ since $\sqrt{2}m \in M \subset S$. But $\sqrt{2}X \notin XS$. For if $\sqrt{2}X \in XS$ then we obtain $\sqrt{2} \in S$ which in turn implies that $\sqrt{2} = q + y$ for some $q \in Q$, $y \in M$. This implies that $\sqrt{2} - q = y \in Q(\sqrt{2}) \cap M = (0)$ and so $\sqrt{2} = q \in Q$ which is not true. Thus $\sqrt{2}X \notin XS$. This proves that $M/(XS)$ is full of zero divisors. Hence $R$ equals the total quotient ring of $R$. Since $R$ is a Noetherian ring, $R$ is a Z.D. $R$-module. Since the dimension of $R$ is 2, it follows from Proposition 4 (i) that $R$ is not a UZD.

**Proposition 10.** Let $R$ be a ring with Noetherian spectrum and descending chain condition on prime ideals. Then $R$ is a UZD if and only if $\text{Spec}(R)$ is a finite set.

**Proof:** Assume that $R$ has Noetherian spectrum and has descending chain condition on prime ideals and $R$ is a UZD. The argument that we shall give below to show that $\text{Spec}(R)$ is a finite set closely follows an argument of Heinzer and Lantz [10, Proposition 3.7]. By Lemma 5, $R$ has only a finite number of maximal ideals say $M_1, \ldots, M_t$. Let, if possible, $\text{Spec}(R)$ be an infinite set. Then $\text{Spec}(R_{M_i})$ is an infinite set for some $i \in \{1, \ldots, t\}$. Now $R_{M_i}$ has Noetherian spectrum and so $M_i R_{M_i} = (y_1, \ldots, y_h) R_{M_i}$ for some $y_j \in M_i R_{M_i}$ ($j = 1, \ldots, h$) [12, Corollary 2.4]. It is then clear that $\text{Spec}(R_{M_i}[1/y_j])$ is an infinite set for some $j \in \{1, \ldots, h\}$. Since $R_{M_i}[1/(y_j)]$ is a UZD, it has only a finite number of maximal ideals say $N_1, \ldots, N_s$. Note that each $N_g$ ($g = 1, \ldots, s$) is of the form $Q_g R_{M_i}[1/y_j]$ for some prime ideal $Q_g R_{M_i}$ of $R_{M_i}$ such that $Q_g R_{M_i} \subset M_i R_{M_i}$. Notice that $\text{Spec}(R_{M_i}[1/y_j])$ is an infinite set for some $g \in \{1, \ldots, s\}$. Further observe that $M_i \supset Q_g$. Now $R_{M_i}[1/y_j]$ $N_g \cong R_{Q_g}$ by [2, Proposition 11 (iii), p.70] and thus $\text{Spec}(R_{Q_g})$ is an infinite set and $R_{Q_g}$ has Noetherian spectrum and is a UZD. Hence applying the above argument to the ring $R_{Q_g}$ yields $H \in \text{Spec}(R)$ such that $Q_g \supset H$ and $\text{Spec}(R_H)$ is infinite. So by repeating the above procedure we obtain a strictly descending sequence of prime ideals of $R$. This is in contradiction to the assumption that $R$ has descending chain condition on prime ideals. Therefore $\text{Spec}(R)$ is a finite set.

The converse is obvious.

**Remark 11.** (i) We mention an example to show that the hypothesis in Proposition 10 that $R$ has Noetherian spectrum cannot be dropped. There exists a valuation ring $V$ such that the set of prime ideals of $V$ forms an infinite ascending chain $(0) \subset P_1 \subset$
$P_2 \subset \cdots \subset M = \bigcup_{i=1}^{\infty} P_i$ [5, Example 5, p.578]. Thus $\text{Spec}(V)$ is an infinite set but by Proposition 4 (ii), $V$ is a UZD. Further, note that $V$ has descending chain condition on prime ideals.

(ii) We now mention an example to show that the hypothesis in Proposition 10 that $R$ has descending chain condition on prime ideals cannot be dropped.

Let $F$ be a field and $\{X_i\}_{i=1}^{\infty}$ be a set of elements algebraically independent over $F$. Let $K = F(\{X_i\}_{i=1}^{\infty})$. Let $G$ be the direct sum of countably many copies of $\mathbb{Z}$, the additive group of integers. We order $G$ with reverse lexicographic ordering. Then there exists a valuation ring $W$ on $K$ with value group $G$ by [7, Example 2.6]. It is easy to verify that the set of all prime ideals of $W$ forms an infinite descending chain $M \supset P_1 \supset P_2 \supset \ldots$. But $W$ is a UZD and $W$ has Noetherian spectrum.

Next we consider the ascent and descent of UZD with respect to integral extension of rings.

**PROPOSITION 12.** (i) Let $R \subset T$ be rings. Let $T$ be integral over $R$. If $T$ is a UZD then $R$ is a UZD.

(ii) Let $B$ be a finite integral extension ring of a ring $A$. If $B$ has finitely many minimal prime ideals and if $A$ is a UZD then $B$ is a UZD.

**Proof:** (i) Let $\{P_\alpha\}_{\alpha \in A}$ be any family of prime ideals of $R$. Now for each $P_\alpha$, there exists $Q_\alpha \in \text{Spec}(T)$ such that $Q_\alpha \cap R = P_\alpha$ by [1, Theorem 5.10]. Since $T$ is UZD, by Proposition 1, $\bigcup_{\alpha \in A} Q_\alpha = \bigcup_{i=1}^{s} H_i$ for some $H_i \in \text{Spec}(T)$ ($i = 1, \ldots, s$). Now it follows that $\bigcup_{\alpha \in A} P_\alpha = \bigcup_{\alpha \in A} (Q_\alpha \cap R) = \bigcup_{i=1}^{s} (H_i \cap R)$. Hence $R$ is a UZD.

(ii) By hypothesis $B$ has only a finite number of minimal prime ideals, say $Q_1, \ldots, Q_t$. Notice that each $B/Q_i$ ($i = 1, \ldots, t$) is a finite integral extension of $A/(Q_i \cap A)$ and $A/(Q_i \cap A)$ is a UZD (for $i = 1, \ldots, t$). We prove that $B/Q_i$ is a UZD for each $i \in \{1, \ldots, t\}$. Then it will follow that $B$ is a UZD. Hence it suffices to prove (ii) in the case in which $B$ is an integral domain. Let $K$ denote the quotient field of $B$. Let $X$ be an indeterminate over $K$. Consider $V = K[[X]] = K + M$ where $M = XK[[X]]$. Let $B_1 = B + M$; $A_1 = A + M$. Since $B$ is a finite integral extension of $A$, it follows that $B_1$ is a finite integral extension of $A_1$. As $A$ is a UZD, $A_1$ is a Z.D. ring by [13, Remark 2.1]. Hence $B_1$ is a Z.D. ring by [9, Theorem 2.9]. Again by [13, Remark 2.1], $B$ is a UZD. This completes the proof of (ii).

**REMARK 13.** We mention an example to show that Proposition 12 (ii) does not extend to infinite integral extensions. Gilmer and Huckaba in [8, Example p.211] have constructed for a fixed prime $p$ an infinite algebraic extension $L$ of the field of rationals $\mathbb{Q}$ such that the integral closure $\overline{Z}_p$ of $Z_p$ in $L$ has an infinite number of maximal ideals.
Since $Z_p$ is a 1-dimensional quasilocal domain it is clear that $Z_p$ is a UZD. As $\overline{Z_p}$ has an infinite number of maximal ideals, $\overline{Z_p}$ is not a UZD.

We conclude this note with the following Proposition which determines when every overring of an integral domain is a UZD.

**Proposition 14.** Let $R$ be an integral domain with quotient field $K$. Then each overring of $R$ is a UZD if and only if the integral closure of $R$ in $K$ is a Prüfer domain with only finitely many maximal ideals.

**Proof:** ($\Rightarrow$) Let $\overline{R}$ denote the integral closure of $R$ in $K$. Let $Q \in \text{Spec}(\overline{R})$. Let $\alpha \in K$, $\alpha \neq 0$. Let $X$ be an indeterminate over $\overline{R}_Q$. Let $g$ denote the $\overline{R}_Q$ homomorphism from $\overline{R}_Q[X]$ to $\overline{R}_Q[\alpha]$ determined by $g(X) = \alpha$. Now $\overline{R}_Q[\alpha]$ is a UZD and hence it has only a finite number of maximal ideals. We assert that $\ker g \subseteq Q\overline{R}_Q[X]$. For if $\ker g \subsetneq Q\overline{R}_Q[X]$ then $(\overline{R}_Q[X])/(Q\overline{R}_Q[X]) \cong (\overline{R}_Q)/(Q\overline{R}_Q)[X]$ becomes a homomorphic image of $\overline{R}_Q[\alpha]$ which would force $(\overline{R}_Q)/(Q\overline{R}_Q)[X]$ to have only a finite number of maximal ideals, a contradiction. Hence $\ker g \subseteq Q\overline{R}_Q[X]$. So by [6, Lemma 19.14] either $\alpha$ or $\alpha^{-1}$ is in $\overline{R}_Q$. Thus $\overline{R}_Q$ is a valuation ring for each $Q \in \text{Spec}(\overline{R})$. Hence $\overline{R}$ is a Prüfer domain. Since $\overline{R}$ is a UZD, $\overline{R}$ has only a finite number of maximal ideals.

($\Leftarrow$) Let $A$ be any overring of $R$. Let $\overline{A}$ denote the integral closure of $A$ in $K$. Then $\overline{A}$ is a Prüfer domain with only a finite number of maximal ideals by [6, Theorem 26.1 (a) and Exercise 14, p. 331]. So $\overline{A}$ is a UZD by Proposition 4(ii). Now Proposition 12 (i) implies that $A$ is a UZD.

**References**


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