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THE STABILITY THEOREMS FOR SUBGROUPS OF $\mathcal{A}$ AND $\mathcal{K}$

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ABSTRACT. In singularity theory, J. Damon gave elegant versions of the unfolding and determinacy theorems for geometric subgroups of $\mathcal{A}$ and $\mathcal{K}$. In this work, we propose a unified treatment of the smooth stability of germs and the structural stability of versal unfoldings for a large class of such subgroups.

0. **Introduction.** Smooth (C^∞) singularity theory investigates properties of mappings under equivalence relations induced by various diffeomorphisms acting on the source and target space. These diffeomorphisms form subgroups of $\mathcal{A}$ (right-left equivalence) or $\mathcal{K}$ (contact equivalence) and since they are defined by *geometric* objects ([D 84] §8), they are referred to as *geometric subgroups* $\mathcal{G}$ of $\mathcal{A}$ and $\mathcal{K}$.

For the local theory, Damon ([D 84]) answered the two fundamental questions of finite determinacy and the existence of versal unfoldings for the geometric subgroups of $\mathcal{A}$ and $\mathcal{K}$. In the paper we take up two other fundamental issues, not addressed in [D 84], the first being the equivalence between the smooth *stability* of germs of mappings (in the sense of openness of the $\mathcal{G}$-orbit) and *infinitesimal stability* ($\mathcal{G}$-codimension zero), the second being the equivalence of versality of unfoldings and stability in the category of unfoldings. This is specially important from the point of view of applications and is referred to as the *structural stability* of versal unfoldings.

The actual infinitesimal computations are algebraic in nature and involve the structure of various tangent spaces. In this regard, the two key tools are the well known Mather's lemma and the Malgrange preparation theorem. One difficulty is that the tangent spaces one encounters are not necessarily modules over a ring; rather, they are sums of modules over systems of rings. Therefore the classical versions of the above results are not valid in general. To surmount such obstacles, [D 84] has gathered and developed the necessary algebraic concepts and tools. We use them throughout this work and recall in the Appendix only what is needed for our purpose, while referring to this book for more explanations and advising at least the reading of its Part I (Preliminaries) to clarify the notation, definitions and concepts.

To prove the stability theorem for germs, finite determinacy is used to factor the group action through an action on the space of jets. Then the key intermediate step in the proof is the passage from the algebraic criterion of infinitesimal stability to a geometric characterization involving the transversality of the r-jet extension of the germ to the orbit of

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its r-jet. This is accomplished by means of the generalized version of Mather's lemma for systems of DA-algebras: see Theorem 2 of the Appendix.

By an analogous proof we can establish the structural stability of versal unfoldings. The additional algebraic tool needed here is a generalized version of the preparation theorem for systems of DA-algebras: see Theorem 1 of the Appendix.

We make however several restrictive assumptions:

Let $\mathbb{R}^n$ denote the source and $\mathbb{R}^p$ the target space. We assume throughout this paper, that

$$T\mathcal{G}_e \subset \mathbb{R}^n \oplus T\mathcal{G} \oplus \mathbb{R}^p,$$

where $T\mathcal{G}$ and $T\mathcal{G}_e$ are respectively the tangent and the extended tangent space to $\mathcal{G}$. It turns out that this is not all that restrictive: all the geometric subgroups considered in [D 84] satisfy this condition. Damon states ([D 90]) that all the geometric subgroups of $\mathcal{A}$ or $\mathcal{K}$ that he knows also satisfy it. In a sense, one may view this as a *natural* condition.

We further assume that the induced action on the jet spaces is given by the action of a Lie group and that some version of Thom's transversality theorem holds for the given category within which we are working. We shall expand on these in Section 1, where it will be noted that almost all of the "useful" equivalence groups in singularity theory do satisfy these additional requirements.

In particular, our stability theorems encompass all the equivalence groups in the work of Arnold *et al.* [AGV 85], Wasserman [W 74], Tougeron and Gervais [G 84], the case of multigerms with Nakai [N 89], the equivariant case of Bierstone [B 77] and [B 80], Izumiya [I 80], Roberts [R 86], Wall [W 85], cases with distinguished parameters (Wasserman [W 75]) and in addition equivariance: Golubitsky *et al.* [GSS 88], Gervais [G 88], Lari-Lavassani and Lu [LL 92] and [LL 93].

Thus in this work we give a unified treatment of the smooth stability of germs and the structural stability of their versal unfoldings for a large class of subgroups of $\mathcal{A}$ and $\mathcal{K}$. In spite of the great generality of the situation under consideration, our proofs of the stability theorems turn out to be very simple.

Whereas [D 84] uses the notion of filtration, our work concentrates on r-jets. The passage to the former case can be easily performed.

1. **Preliminaries.**

1.1 *Notation.* The reader not familiar with the machinery related to differentiable algebras (DA-algebras), developed in [D 84], can find more details in the Appendix. We write $(\mathbb{R}^n, 0)$ for the germ of $\mathbb{R}^n$ at 0, and respectively x, y and u for an element in $(\mathbb{R}^n, 0)$, $(\mathbb{R}^p, 0)$ and $(\mathbb{R}^q, 0)$. The locale algebra of real valued germs on $(\mathbb{R}^n, 0)$ is denoted by C_x and its maximal ideal by m_x, while $C_{x,u}$ denotes the algebra of germs on $(\mathbb{R}^{n+q}, 0)$ and $C_{x,y}$ that of germs on $(\mathbb{R}^{n+p}, 0)$.

Let $C_{n,p}$ be the set of smooth germs $f: (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$, $\mathcal{F}$ be a linear subspace of $C_{n,p}$ and $\mathcal{G}$ be a geometric subgroup of $\mathcal{A}$ or $\mathcal{K}$ acting on $\mathcal{F}$. Let $\{R_\alpha\}$ be a system of DA-subalgebras of $C_{x,y}$ (see Appendix), such that by the tangent space structure (*cf.*

[D 84] p. 40), the tangent spaces $T\mathcal{F}$, $T\mathcal{F}_e$, $T\mathcal{G}$ and $T\mathcal{G}_e$ are finitely generated $\{R_\alpha\}$-modules (the subscript e denotes the extended tangent spaces): see also the Appendix. Furthermore for $f \in \mathcal{F}$

$$(1.1) \qquad da_f : T\mathcal{G}_e \longrightarrow T\mathcal{F}_e$$

is a homomorphism of $\{R_\alpha\}$-modules. Here d denotes the derivative and a_f is the orbit map

$$a_f : \mathcal{G} \longrightarrow \mathcal{F}, \quad a_f(\phi) = \phi \cdot f.$$

We assume throughout this paper that

$$\text{(H1)} \qquad T\mathcal{G}_e \subset \mathbb{R}^n \oplus T\mathcal{G} \oplus \mathbb{R}^p.$$

1.2 *Jet spaces.* The usual space of k-jets from $\mathbb{R}^n$ to $\mathbb{R}^p$ admits a natural fibration over $\mathbb{R}^n \times \mathbb{R}^p$. However, when dealing with specific subspaces $\mathcal{F}$, the k-jets of germs in $\mathcal{F}$ may not necessarily fiber over $\mathbb{R}^n \times \mathbb{R}^p$. We suppose

(H2) There exist $\mathcal{V}_n \subset \mathbb{R}^n$ and $\mathcal{W}_p \subset \mathbb{R}^p$ such that the space of k-jets of elements of $\mathcal{F}$ fibers over $\mathcal{V}_n \times \mathcal{W}_p$.

More precisely, following Arnold *et al.* [AGV 85], we set the following definitions: The *medium space* of k-jets of elements of $\mathcal{F}$ is:

$$\mathcal{F}_0^k \equiv \mathcal{W}_p \times \mathcal{F}/\{m_\alpha\}^k \cdot \mathcal{F}.$$

The *large space* of k-jets is:

$$\mathcal{F}^k \equiv \mathcal{V}_n \times \mathcal{F}_0^k.$$

For $f \in T\mathcal{F}_e$, the k-jet of f at 0 is an element of $\mathcal{F}_0^k$ and is denoted by $j_0^k f$. The *medium* and *large* k-jet extensions of a germ $f \in \mathcal{F}$ are defined to be respectively

$$j^k f : \mathcal{V}_n \longrightarrow \mathcal{F}_0^k \quad \text{and} \quad J^k f : \mathcal{V}_n \longrightarrow \mathcal{F}^k.$$

Note that $J_x^k f = (x, j_x^k f)$.

REMARK. In the equivariant case where a compact Lie group G acts linearly on $\mathbb{R}^n$ and $\mathbb{R}^p$, and $\mathcal{F}$ is the module of G-equivariant germs of mappings from $\mathbb{R}^n$ to $\mathbb{R}^p$, (H2) holds. For definitions of equivariant jets and the appropriate $\mathcal{V}_n$ and $\mathcal{W}_p$ see [B 77] 3.3. Furthermore (H2) is also valid for the case of multigerms (*cf.* [N 89], pp. 475–478).

We also assume

$$\text{(H3)} \qquad T\mathcal{F}_e \subset T\mathcal{V}_n \oplus T\mathcal{F} \oplus T\mathcal{W}_p.$$

(H4) Thom's transversality theorem is valid in $\mathcal{F}^k$.

REMARKS. i) If $\mathcal{F} = C_{n,p}$, the usual transversality theorem can be used. In the equivariant context, the usual transversality theorem is available in the space of equivariant jets ([W 85] 2.1). Finally (H4) is again satisfied ([N 89] 0.3.5) in the case of multigerms.

ii) Hypotheses (H1) and (H3) together with (1.1) have for natural consequence that

$$d\alpha_f(\mathbb{R}^n) \subset T\mathcal{V}_n \quad \text{and} \quad d\alpha_f(\mathbb{R}^p) \subset T\mathcal{W}_p.$$

Denote by $\mathcal{G}^k$ the group of k-jets of elements of $\mathcal{G}$ at 0. The action of $\mathcal{G}$ on $\mathcal{F}$ induces a quotient action of $\mathcal{G}^k$ on $\mathcal{F}/\{m_\alpha\}^k \cdot \mathcal{F}$: for $\phi \in \mathcal{G}$ and $f \in \mathcal{F}$,

$$(j_0^k\phi) \cdot (j_0^k f) \equiv j_0^k(\phi \cdot f). \tag{1.2}$$

We finally make the assumption

(H5) The induced action (1.2) is that of a Lie group and the orbits are submanifolds.

We denote the orbit of $j_0^k f$ under the action (1.2) by $\mathcal{G}_{0,0}^k f$ and define the *medium* orbit of $j_0^k f$ to be

$$\mathcal{G}_0^k f \equiv \mathcal{W}_p \times \mathcal{G}_{0,0}^k f$$

and the *large* orbit of $j_0^k f$ to be

$$\mathcal{G}^k f \equiv \mathcal{V}_n \times \mathcal{W}_p \times \mathcal{G}_{0,0}^k f.$$

Observe that $\mathcal{G}_0^k f$ and $\mathcal{G}^k f$ are respectively included in $\mathcal{F}_0^k$ and $\mathcal{F}^k$.

REMARKS. i) The second part of hypothesis (H5) enables us to speak of the transversality of the k-jet extensions of f to the orbits of the k-jet of f at 0. It is in this context that we will use Thom's transversality theorem from (H4).

ii) All the geometric subgroups discussed in [D 84] satisfy (H1) through (H5); this covers the majority of equivalence groups in singularity theory.

The (extended) tangent space $T\mathcal{G}_e f$ to the orbit $\mathcal{G}f$ is defined to be $d\alpha_f(T\mathcal{G}_e)$. To define the tangent space $T\mathcal{G}_0^k f$ to $\mathcal{G}_0^k f$ we proceed as above, except we keep the origin in $\mathbb{R}^n$ fixed but allow that of $\mathbb{R}^p$ to move; this leads to

$$T\mathcal{G}_0^k f \equiv j_0^k\big(d\alpha_f(T\mathcal{G} \oplus \mathbb{R}^p)\big). \tag{1.3}$$

For example, if $\mathcal{G} = \mathcal{A}$ and $\mathcal{F} = C_{n,p}$, then ($\mathcal{W}_p = \mathbb{R}^p$) and by [D 84] p. 6, the extended tangent space to the orbit $\mathcal{A}f$ is

$$T\mathcal{A}_e f = \left\{\eta \circ f - df(\delta) \mid \delta \in C_x\left\{\frac{\partial}{\partial x_i}\right\}, \eta \in C_y\left\{\frac{\partial}{\partial y_i}\right\}\right\}.$$

The non-extended tangent space $T\mathcal{A}f$ is defined as $T\mathcal{A}_e f$ except that η and δ vanish at the origin. Finally (1.3) becomes

$$T\mathcal{A}_0^k f = \left\{j_0^k\big(\eta \circ f - df(\delta)\big) \mid \delta \in m_x\left\{\frac{\partial}{\partial x_i}\right\}, \eta \in C_y\left\{\frac{\partial}{\partial y_i}\right\}\right\}.$$

Let us finally recall the following two results which will be needed later.

The derivative of the medium k-jet extension at the origin is (*cf.* [AGV 85] Proposition 2, p. 141)

$$(1.4)\qquad \begin{aligned} d(j^k f)_0 &: T_0 \mathcal{V}_n \longrightarrow T\mathcal{F}_0^k \cong \mathcal{F}_0^k \\ d(j^k f)_0(\xi) &= j_0^k\big(df_0(\xi)\big). \end{aligned}$$

Taking the identification $J_x^k f = (x, j_x^k f)$ into consideration, it follows at once that if P is a subspace of $\mathcal{F}_0^k$,

$$(1.5)\qquad j^k f \pitchfork P \text{ if and only if } J^k f \pitchfork (\mathcal{V}_n \times P).$$

2. **Stability.** Let $f \in \mathcal{F}$ and assume f is the germ at 0 of the C^∞ mapping $\tilde{f}: \mathbb{R}^n \longrightarrow \mathbb{R}^p$ of the same category, *e.g.*, G-equivariant in the equivariant context. Denote the space of such mappings by $C^\infty_{\mathcal{F}}(\mathbb{R}^n, \mathbb{R}^p)$ and equip it with the induced C^∞ topology. Note that one has to move to the global space $C^\infty_{\mathcal{F}}(\mathbb{R}^n, \mathbb{R}^p)$ since there is no natural topology on $C_{n,p}$ (and hence on $\mathcal{F}$).

The germ f is said to be *$\mathcal{G}$-stable* if its orbit $\mathcal{G}f$ is open. More precisely, for every neighborhood $\mathcal{U}$ of 0 in $\mathbb{R}^n$, there exists a neighborhood $\mathcal{N}$ of $\tilde{f}$ in $C^\infty_f(\mathbb{R}^n, \mathbb{R}^p)$ such that for every $\tilde{f}' \in \mathcal{N}$, there exist $x' \in \mathcal{U} \cap \mathcal{V}_n$, a ϕ defined exactly as an element of $\mathcal{G}$, except that ϕ maps $0 \in \mathcal{U}$ to $x' \in \mathcal{U}$ (in lieu of mapping 0 to 0), and satisfying

$$(2.1)\qquad f = \phi \cdot f',$$

where f' is the germ of $\tilde{f}'$ at x'.

A heuristic approach to the stability problem consists of the following. Recall that the orbit $\mathcal{G}f$ is the image of the orbit map a_f. If a_f is a submersion, in a finite dimensional setting, the implicit function theorem would imply that $\mathcal{G}f$ is open and hence stability would follow. The implicit function theorem does not hold in this context; however the result remains valid as it will be established below. This discussion leads to the definition: The germ f is said to be *$\mathcal{G}$-infinitesimally stable* if da_f (see 1.1) is surjective. That is to say, if $T\mathcal{G}_e f = T\mathcal{F}_e$.

In general, one defines the *$\mathcal{G}$-codimension* of f to be

$$\mathcal{G}\text{-}\operatorname{cod} f \equiv \dim_{\mathbf{R}} T\mathcal{F}_e / T\mathcal{G}_e f.$$

Then $\mathcal{G}$-infinitesimal stability coincides with $\mathcal{G}$-codimension zero.

Note that unlike the notion of $\mathcal{G}$-stability which is geometrical and not practical to verify, establishing $\mathcal{G}$-infinitesimal stability is (at least in theory) an algebraic computation.

It turns out that if $\mathcal{G}$-infinitesimal stability holds modulo certain higher order terms, then it holds in general. This paves the way to the reduction to the space of jets. More precisely using Mather's lemma for systems of DA-algebras (Theorem 2 of the Appendix) with $M_0 = M = T\mathcal{F}_e$, $N = T\mathcal{G}_e$ and $a = da_f$ we have:

LEMMA 2.2. *The following conditions are equivalent:*

a) $T\mathcal{G}_e f = T\mathcal{F}_e$.

b) There exists a number ℓ *such that*

$$T\mathcal{F}_e \subset T\mathcal{G}_e f + \{m_\alpha\}^\ell T\mathcal{F}_e.$$

We can now state the smooth stability theorem for germs:

THEOREM 2.3. *Let* $\mathcal{F}$ *be a linear subspace of* $C_{n,p}$ *and let* $\mathcal{G}$ *be a geometric subgroup of* $\mathcal{A}$ *or* $\mathcal{K}$ *acting on* $\mathcal{F}$*, with an adequately ordered system of* DA*-algebras* $\{R_\alpha\}$*. Assume (H1) through (H5) hold and* $f \in \mathcal{F}$*. Then the following conditions are equivalent:*

i) f *is* $\mathcal{G}$*-infinitesimally stable.*

ii) All unfoldings of f *are trivial (i.e.,* f *is its own universal unfolding).*

iii) With ℓ *defined in Lemma 2.2, for some* $k \geq \ell$, $J^k f \pitchfork \mathcal{G}^k f$.

iv) f *is* $\mathcal{G}$*-stable.*

3. **Proof of the stability theorem.** By virtue of the Unfolding Theorem 9.3 of [D 84] p. 44 (see also (4.1)), condition (ii) is equivalent to $\mathcal{G}$-cod$f = 0$. Whence (i) is equivalent to (ii). A key step in the proof is to go from the algebraic criterion (i) to the geometric characterization (iii). For this we need:

LEMMA 3.1. *Condition iii) is equivalent to: For any* $h \in T\mathcal{F}_e$*, there exist* $\beta \in (T\mathcal{G} \oplus \mathbb{R}^p)$ *and* $\xi \in \mathbb{R}^n$ *such that*

$$h(x) = d\alpha_f\big(\beta(x)\big) + df(\xi) \ \operatorname{mod}\{m_\alpha\}^{k+1} \cdot T\mathcal{F}_e. \qquad \text{(T. k)}$$

PROOF. We first move down to the space of medium jets: observe by virtue of (1.5), that (iii) is equivalent to $j^k f \pitchfork \mathcal{G}_0^k f$. Next using (1.4) and (1.3), this transversality condition can be expressed by: any $j_0^k h \in T\mathcal{F}_0^k \cong \mathcal{F}_0^k$ has a decomposition

$$j_0^k h = j_0^k\big(df(\xi) + d\alpha_f(\beta)\big)$$

for some $\xi \in \mathbb{R}^n$ and $\beta \in (T\mathcal{G} \oplus \mathbb{R}^p)$. Note that by the remark following (H4), $df(\xi) \in T\mathcal{V}_n$. ■

Now assume (i) is true, then any $h \in T\mathcal{F}_e$ admits a decomposition

$$h(x) = d\alpha_f\big(\tilde{\beta}(x)\big)$$

for some $\tilde{\beta} \in T\mathcal{G}_e$. By assumption (H1) we can write $\tilde{\beta} = \xi_1 + \beta$ for some $\xi_1 \in \mathbb{R}^n$ and $\beta \in T\mathcal{G} \oplus \mathbb{R}^p$.

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LEMMA 3.2. *With the above notation we have*

$$d\alpha_f(\tilde{\beta}) = d\alpha_f(\beta) - df(\xi_1).$$

PROOF. Let us only consider the proof for subgroups of $\mathcal{A}$ since the argument is analogous and somewhat simpler for $\mathcal{K}$. We will give the details for $\mathcal{A}$ itself, it suffices to adapt it for particular subgroups $\mathcal{G}$. Recall that (*cf.* [D 84] pp. 5, 6),

$$d\alpha_f \colon T\mathcal{A}_e f \cong C_x\Big\{\frac{\partial}{\partial x_i}\Big\} \oplus C_y\Big\{\frac{\partial}{\partial y_i}\Big\} \longrightarrow C_x\Big\{\frac{\partial}{\partial y_i}\Big\}$$
$$\tilde{\beta} = (\tilde{\eta}_1, \tilde{\eta}_2) \longrightarrow \tilde{\eta}_2 \circ f - df(\tilde{\eta}_1).$$

Separating off the constant term in $\tilde{\eta}_1$ yields

$$\tilde{\eta}_1 = \xi_1 + \eta_1,$$

with $\xi_1 \in \mathbb{R}^n$ and $\eta_1 \in m_x\{\frac{\partial}{\partial x_i}\}$; whence $\beta = (\eta_1, \tilde{\eta}_2)$. Now

$$d\alpha_f(\tilde{\beta}) = d\alpha_f(\beta) - df(\xi_1).$$ ∎

Let us return to the proof of the implication from (i) to (iii). Let $\xi = -\xi_1$, by Lemma 3.2,

$$h(x) = d\alpha_f\big(\beta(x)\big) + df(\xi),$$

hence (T.k) is true for all $k \geq 0$; then (iii) follows from Lemma 3.1.

Conversely assume (iii) is true. Again by Lemma 3.1, any $h \in T\mathcal{F}_e$ admits a decomposition

$$h(x) = d\alpha_f\big(\beta(x)\big) + df(\xi) \mod\{m_\alpha\}^{k+1}T\mathcal{F}_e.$$

As above, using Lemma 3.2, the constant ξ can be incorporated in $\beta(x)$, yielding

$$h(x) = d\alpha_f\big(\tilde{\beta}(x)\big) \mod\{m_\alpha\}^{k+1}T\mathcal{F}_e,$$

for some $\tilde{\beta} \in T\mathcal{G}_e$. Since $k \geq \ell$, Lemma 2.2 implies $T\mathcal{G}_e f = T\mathcal{F}_e$; *i.e.*, (i) holds.

It remains to verify that (iii) is equivalent to (iv). Assume (iii) holds, then f is $\mathcal{G}$-infinitesimally stable as proved earlier. Since f has finite $\mathcal{G}$-codimension, by the finite determinacy theorem ([D 84] Theorem 10.2, p. 49) f is r $\mathcal{G}$-determined for some integer r. Recall that this means each time

$$j_0^r f = j_0^r g,$$

then f and g are $\mathcal{G}$-equivalent.

Let $\mathcal{U}$ be an open neighborhood of 0 in $\mathbb{R}^n$ as in §2. Choose a neighborhood $\mathcal{N}$ of $\tilde{f}$ in $C^\infty_{\mathcal{F}}(\mathbb{R}^n, \mathbb{R}^p)$ (with f being the germ of $\tilde{f}$ at 0) small enough so that for any $\tilde{f}' \in \mathcal{N}$,

$$J^r f' \pitchfork G^r f,$$

at some jet Z close to $J_0^r f$. Let x' be the source of Z, by shrinking $\mathcal{N}$ if necessary, we may always assume $x' \in \mathcal{U} \cap \mathcal{V}_n$. Thus $j_{x'}^r f' \in \mathcal{G}_0^r f$. Whence there exists (passing to medium jets) a $\tilde{\phi}$ defined exactly as an element of $\mathcal{G}$, except it maps 0 to x', such that

$$j_0^r f = (j_0^r \tilde{\phi}) \cdot (j_{x'}^r f'). \tag{3.1}$$

Now define the germ g by

$$g \equiv \tilde{\phi} \cdot f'. \tag{3.2}$$

Taking the r-jet of (3.2) at 0 and comparing it to (3.1) yields $j_0^r f = j_0^r g$. Since f is r $\mathcal{G}$-determined, $f = \psi \cdot g$ for some $\psi \in \mathcal{G}$. Let

$$\phi \equiv \psi \cdot \tilde{\phi},$$

then (2.1) becomes true, *i.e.*, f is $\mathcal{G}$-stable.

Conversely, assume iv) holds. With the notation preceding (2.1), fix any $k \geq \ell$. Since $\mathcal{N}$ is open, by Thom's Transversality theorem (*cf.* (H4)), there exists $\tilde{f}' \in \mathcal{N}$ such that

$$J^k f' \pitchfork \mathcal{G}^k f.$$

Taking (2.1) into account it follows $J^k f \pitchfork \mathcal{G}^k f$.

This concludes the proof of Theorem 2.3.

4. **Structural stability of unfoldings.** For a discussion of the unfolding theory, we refer to [D 84], Part III. Denote the space of unfoldings by $\mathcal{F}_{\text{un}}$. Given $f_0 \in \mathcal{F}$, to specify that $f \in \mathcal{F}_{\text{un}}$ is a q-parameter unfolding of f_0, we write $(f, q) \in \mathcal{F}_{\text{un}}$ and let $u = (u_1, \ldots, u_q) \in \mathbb{R}^q$ be the unfolding parameter. By definition an unfolding $(f, q) \in \mathcal{F}_{\text{un}}$ of f_0 is a germ $f(x, u) = \big(\bar{f}(x, u), u\big)$, where $\bar{f}(x, 0) = f_0(x)$.

In the setting of §1, we also assume that the group of unfoldings $\mathcal{G}_{\text{un}}$ ([D 84] p. 39) acts on the space of unfolding germs $\mathcal{F}_{\text{un}}$. By the tangent space structure ([D 84], §8) $\mathcal{G}_{\text{un}}$ is a geometric subgroup of $\mathcal{A}$ or $\mathcal{K}$, but this time the adequately ordered system of DA-algebras to be used is $\{R_{\alpha,u}\}$: see the Appendix.

Let (f, q) and (g, t) be two unfoldings of f_0. *Cf.* [D 84] p. 6; a *$\mathcal{G}$-mapping of unfoldings* $\Phi: (g, t) \longrightarrow (f, q)$ consists of a change of parameters $\lambda: (\mathbb{R}^t, 0) \longrightarrow (\mathbb{R}^q, 0)$ and:

- If $\mathcal{G}$ is a subgroup of $\mathcal{K}$, of an additional t-parameter unfolding $H \in \mathcal{G}_{\text{un}}$, which unfolds the identity and such that if one writes $H(x, y, v) = \big(\bar{h}(x, v), H_1(x, y, v), v\big)$ then
$$H_1\big(x, \bar{g}(x, v), v\big) = \bar{f}\big(\bar{h}(x, v), \lambda(v)\big).$$
- If $\mathcal{G}$ is a subgroup of $\mathcal{A}$, replace H above by $(\alpha, \beta) \in \mathcal{G}_{\text{un}}$, unfolding the identity and impose that $\beta \circ g \circ \alpha^{-1}(x, v) = \Big(\bar{f}\big(x, \lambda(v)\big), v\Big)$.

The unfolding (f, q) of f_0 is called *$\mathcal{G}$-versal* if for any unfolding (g, t) of f_0 there is a $\mathcal{G}$-mapping of unfoldings $\Phi: (g, t) \longrightarrow (f, q)$.

As it is known, the classification of unfoldings requires the use of an equivalence group slightly larger than $\mathcal{G}_{\text{un}}$. *Cf.* [D 84] Definition 8.3 p. 41; two unfoldings (f, q) and

(g,q) of f_0 are said to be $\mathcal{G}$*-equivalent* if there exist $\mathcal{G}$-mappings of unfoldings $\Phi\colon (f,q) \longrightarrow (g,q)$ and $\Psi\colon (g,q) \longrightarrow (f,q)$ such that $\Phi\cdot\Psi$ and $\Psi\cdot\Phi$ are identity. The group inducing this equivalence is denoted by $\mathcal{G}_{\mathrm{eq}}$. It follows from the definition of $\mathcal{G}_{\mathrm{eq}}$ (see the above reference) that

$$T\mathcal{G}_{\mathrm{eq},e} \cong T\mathcal{G}_{\mathrm{un},e} \oplus C_u\Big\{\frac{\partial}{\partial u_1},\dots,\frac{\partial}{\partial u_q}\Big\}.$$

The unfolding theorem ([D 84] p. 44) states that (f,q) is a $\mathcal{G}$-versal unfolding of f_0 if and only if it is $\mathcal{G}$*-infinitesimally versal, i.e.*,

$$T\mathcal{G}_e f_0 + \mathbb{R}\{\partial_1,\dots,\partial_q\} = T\mathcal{F}_e, \tag{4.1}$$

where $\partial_i = \frac{\partial \bar f}{\partial u_i}\big|_{u=0}$, $i=1,\dots,q$.

For $f\in\mathcal{F}_{\mathrm{un}}$, one can define as in the case of $\mathcal{F}$, the orbit map a_f and its derivative da_f: see [D 84] p. 40.

To obtain the notions of $\mathcal{G}$*-stability* and $\mathcal{G}$*-infinitesimal stability* for an unfolding $f\in\mathcal{F}_{\mathrm{un}}$ it suffices to replace in the definitions given for the case of a germ $f_0\in\mathcal{F}$ in §2: $\mathcal{F}$ by $\mathcal{F}_{\mathrm{un}}$, $\mathcal{G}$ by $\mathcal{G}_{\mathrm{eq}}$, n by $n+q$ and p by $p+q$.

The second main result of this paper is on the (structural) stability of versal unfoldings.

THEOREM 4.2. *Let $\mathcal{G}$ be a geometric subgroup of $\mathcal{A}$ or $\mathcal{K}$, with an adequately ordered system of* DA*-algebras $\{R_\alpha\}$ and let $\mathcal{G}_{\mathrm{un}}$ be the corresponding group of unfoldings. Assume hypotheses equivalent to (H1) through (H5) hold, with obvious modifications for the unfolding category. Then for an unfolding $(f,q)\in\mathcal{F}_{\mathrm{un}}$ of a germ $f_0\in\mathcal{F}$, the following conditions are equivalent:*

i) f is $\mathcal{G}$-versal.

ii) f is $\mathcal{G}$-stable.

PROOF. We first review a result similar to the so-called algebraic lemma of unfolding theory.

LEMMA 4.3. *With the hypothesis of Theorem 4.2, the following conditions are equivalent*

1) $T\mathcal{G}_{\mathrm{eq},e}f = T\mathcal{F}_{\mathrm{un},e}$.

2) f_0 is $\mathcal{G}$-infinitesimally versal (i.e., (4.1) holds).

PROOF. Consider

$$\Omega\colon T\mathcal{G}_{\mathrm{eq},e} \cong T\mathcal{G}_{\mathrm{un},e} \oplus C_u\Big\{\frac{\partial}{\partial u_1},\dots,\frac{\partial}{\partial u_q}\Big\} \longrightarrow T\mathcal{F}_{\mathrm{un},e} \tag{4.2}$$

where Ω equals da_f on $T\mathcal{G}_{\mathrm{un},e}$ and sends $\frac{\partial}{\partial u_i}$ to $\frac{\partial \bar f}{\partial u_i}$ on the second set of components. Then Ω is a homomorphism over the system of DA-algebras $\{C_u\}\cup\{R_{\alpha,u}\}$ (similar to [D 84] p. 45). Note that condition 1) signifies that Ω is surjective. By the assumptions on $\{R_\alpha\}$ and the generalized version of the preparation theorem for systems of DA-algebras

(Theorem 2 of the Appendix), $\{C_u, m_u\} \cup \{R_{\alpha,u}, m_u \cdot R_{\alpha,u}\}$ is adequate. Therefore the surjectivity of Ω is equivalent to

$$\Omega\Big(T\mathcal{G}_{\mathrm{un},e} \oplus C_u\Big\{\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_q}\Big\}\Big) + m_u T\mathcal{F}_{\mathrm{un},e} = T\mathcal{F}_{\mathrm{un},e}.$$

This in turn is equivalent to

$$\Omega\Big[(T\mathcal{G}_{\mathrm{un},e}/m_u T\mathcal{G}_{\mathrm{un},e}) \oplus \mathbb{R}\Big\{\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_q}\Big\}\Big] = T\mathcal{F}_{\mathrm{un},e}/m_u T\mathcal{F}_{\mathrm{un},e}. \tag{4.3}$$

Now by the tangent space structure ([D 84] p. 40) and the definition of Ω, condition (4.3) is nothing but (4.1). ∎

Let us return to the proof of Theorem 4.2. The group $\mathcal{G}_{\mathrm{un}}$ is itself a geometric subgroup of $\mathcal{A}$ or $\mathcal{K}$; therefore the stability Theorem 2.3 applies to it. We contend that it also applies to $\mathcal{G}_{\mathrm{eq}}$. Indeed, in the proof of Theorem 2.3, Lemma 2.2 is crucial. Its proof is based on the version of Mather's lemma for systems of DA-algebras (Theorem 2 of the Appendix) to the homomorphism of DA-algebras $d\alpha_f: T\mathcal{G}_e \longrightarrow T\mathcal{F}_e$. As was noted above, Ω in (4.2) is also a homomorphism of DA-algebras and therefore by the same argument, Lemma 2.2 holds (with obvious modifications) for $T\mathcal{G}_{\mathrm{eq},e}f$. The remainder of the proof of the stability theorem holds identically for $\mathcal{G}_{\mathrm{eq}}$ if one replaces in it $\mathcal{G}$ by $\mathcal{G}_{\mathrm{eq}}$, n by $n+q$, *etc.* ... as mentioned before.

Therefore we can establish that the unfolding f is $\mathcal{G}$-stable if and only if $T\mathcal{G}_{\mathrm{eq},e}f = T\mathcal{F}_{\mathrm{un},e}$. By Lemma 4.3 this is equivalent to f being $\mathcal{G}$-infinitesimally versal. Now the unfolding theorem recalled above (see (4.1)) concludes the proof. ∎

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Appendix. In the notation of §1.1, for a germ $f: (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$, $y = f(x)$, we denote by f^* the pull-back morphism

$$f^*: C_y \longrightarrow C_x, \quad f^*(g) = g \circ f$$

and define (*cf.* [D 84], §4) a differentiable algebra (DA-algebra) A as consisting of an $\mathbb{R}$-algebra A and a surjective algebra morphism $\theta: C_x \longrightarrow A$. In addition A is a local ring with maximal ideal m_A. Let $\rho: C_y \longrightarrow B$ be another DA-algebra B. Then a homomorphism of DA-algebras $\alpha: B \longrightarrow A$ is an algebra morphism which lifts to $f^*: C_y \longrightarrow C_x$, such that $\theta \circ f^* = \alpha \circ \rho$, for some germ f. In particular B is a DA-subalgebra of C_x if the inclusion $i: B \hookrightarrow C_x$ is a homomorphism of DA-algebras, with $A = C_x$ and $\theta = \mathrm{id}$.

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Now let u denote the element in the germ of a parameter space $(\mathbb{R}^q, 0)$. The algebra B_u of u-parametrized germs in B is defined as follows. Let f^* cover i as above and define $h = f \times \mathrm{id}: \mathbb{R}^n \times \mathbb{R}^q \longrightarrow \mathbb{R}^p \times \mathbb{R}^q$. We define $B_u = h^* C_{y,u} \subseteq C_{x,u}$: see [D 84] §4 for more.

By [D 94] §6, a system of DA-algebras associated to a finite partially ordered set $(\mathcal{D}, <)$, consists of a set of DA-algebras $\{R_\alpha, \alpha \in \mathcal{D}\}$ together with ring homomorphisms $\Phi_{\alpha\beta}: R_\alpha \longrightarrow R_\beta$ defined for $\alpha \leq \beta$ so that $\Phi_{\beta\gamma} \circ \Phi_{\alpha\beta} = \Phi_{\alpha\gamma}$ for $\alpha \leq \beta \leq \gamma$ and $\Phi_{\alpha\alpha} = \mathrm{id}$. A system of ideals $\{I_\alpha\}$ of $\{R_\alpha\}$ consists of ideals $I_\alpha \subseteq R_\alpha$ so that $\Phi_{\alpha\beta}(I_\alpha) \subseteq I_\beta$ for $\alpha \leq \beta$; one then writes $\{R_\alpha, I_\alpha\}$. This system is called *Jacobson* if $1 + a$ is invertible in R_α for all a in I_α. An $\{R_\alpha\}$-module M consists of a direct sum $\oplus M_\alpha$ (summed for α over $\mathcal{D}$) where M_α is an R_α-module. M is said to be *finitely* (respectively *almost finitely*) *generated* if M_α is finitely generated over R_α for all α (respectively α nonmaximal in $\mathcal{D}$). For each R_α one can construct the ring $R_{\alpha,u}$ (as B_u above), this yields the system of DA-algebras $\{R_{\alpha,u}\}$. Each $R_{\alpha,u}$ is a C_u-algebra. One says $\{R_{\alpha,u}\}$ is over C_u if each $\Phi_{\alpha\beta}: R_{\alpha,u} \longrightarrow R_{\beta,u}$ is a C_u-algebra homomorphism.

The importance of the following versions of the preparation theorem and Mather's lemma lies in the fact that they are expressed in terms of homomorphisms of DA-algebras and only knowledge of the connecting homomorphisms is sufficient to establish them.

By definition ([D 84] p. 30) a system of DA-algebras and Jacobson ideals $\{R_\alpha, I_\alpha\}$ is called *adequate* if the following conditions hold:

i) each element of $\mathcal{D}$ has at most one immediate predecessor.
ii) if $\Psi: N \longrightarrow M$ is a homomorphism of modules over $\{R_\alpha\}$ with M finitely generated and N almost finitely generated such that

$$\Psi(N) + \{I_\alpha\} \cdot M = M.$$

Then,

$$\Psi(N) = M \quad \text{and} \quad \Psi(\{I_\alpha\} \cdot N) = \{I_\alpha\} \cdot M.$$

The system of DA-algebras $\{R_\alpha\}$ is called *adequately ordered* if condition (i) holds.

We can now present a corollary to the preparation theorem for systems of DA-algebras ([D 84] Corollary 6.16, p. 33).

THEOREM 1. *Consider an adequately ordered system of* DA-*algebras* $\{R_{\alpha,u}\}$ *over* C_u. *Then* $\{(C_u, m_u)\} \cup \{(R_{\alpha,u}, m_u \cdot R_{\alpha,u})\}$ *is adequate.*

We finally turn to the version of Mather's lemma for systems of DA-algebras ([D 84] p. 35).

THEOREM 2. *Let* $\{R_\alpha\}$ *be an adequately ordered system of* DA-*algebras,* $a: N \longrightarrow M$ *be a homomorphism of finitely generated* $\{R_\alpha\}$*-modules and* $M_0 \subseteq M$ *be an* $\{R_\alpha\}$*-submodule of finite codimension* c *over* $\mathbb{R}$. *There is a number* ℓ *which depends on* c *and the dimensions of* $N_\alpha / m_\alpha N_\alpha$, $M_\alpha / m_\alpha M_\alpha$ *and* m_α / m_α^2 *so that*

$$M_0 \subseteq \alpha(N) + \{m_\alpha^\ell\} M_0$$

implies

$$M_0 \subseteq \alpha(N).$$

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