PRIME NUMBERS OF THE FORM \([n^c]\)

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Abstract. In 1953, Pjateckiǐ-Šapiro has proved that there are infinitely many primes of the form \(\lfloor n^c \rfloor\) for \(1 < c < \frac{12}{11}\) (with an asymptotic result). This range, which measures our progress in the technique of exponential sums, has been improved by many authors. In this paper we obtain \(1 < c < \frac{243}{205}\).


1. Introduction. One of the most odd problems in prime number theory is to represent infinitely many prime numbers by polynomials. Though this problem has been completely solved for linear polynomials by the Dirichlet theorem on primes in arithmetic progressions, until now it remains open for polynomials of degree greater than 1 and it seems to be extremely difficult. Iwaniec [6] has proved that there exists an infinity of integers \(n\) such that \(n^2 + 1\) has at most two prime factors. As another direction of approaching this problem, Pjateckiǐ-Šapiro [9] has proposed to investigate the prime numbers of the form \([n^c]\), where \(c > 1\) and \([t]\) denotes the integer part of \(t\). Clearly \([n^c]\) can be regarded as “polynomials of degree \(c\)”. Define

\[
\pi_c(x) := |\{n \leq x : [n^c] \text{ is prime number}\}|
\]

Pjateckiǐ-Šapiro [9] has shown for \(1 < c < \frac{12}{11} = 1.09\) that

\[
\pi_c(x) \sim \frac{x}{c \log x} \quad (x \to \infty).
\]  

(1.1)

This range has been improved by many authors. A detailed history can be found in [10]. The best record known is due to Rivat and Sargos [11]: \(1 < c < \frac{2817}{2426} \approx 1.16117\).

Jia and Rivat have proved, independently, that

\[
\pi_c(x) \gg \frac{x}{c \log x} \quad (x \geq x_0)
\]  

(1.2)

for \(1 < c < \frac{7}{6} = 1.16\) (Rivat [10]) and \(1 < c \leq \frac{20}{17} \approx 1.17647\) (Jia [6, I], and independently Baker, Harman and Rivat [I]). This result has been improved to \(1 < c < \frac{13}{11} = 1.18\) by Jia [6, II] and to \(1 < c \leq \frac{45}{38} \approx 1.18421\) by Kumchev [8]. The admissible range for \(c\) provides a measure of our progress towards the quadratic case, and it is interesting to extend it further.

We shall establish the following result.

Theorem 1. The inequality (1.2) is true for \(1 < c \leq \frac{243}{205} \approx 1.18536\) and \(x \geq x_0(c)\).
We shall prove Theorem 1 by sieving the following two sequences

\[ A := \{ m : m = [n^r], \ X \leq n^r < 2X \}, \ B := \{ n : X \leq n < 2X \}. \]

For this it is necessary to present some standard notations in the sieve. For any finite set \( E \subset \mathbb{N}, \ d \in \mathbb{N} \) and \( z \geq 2 \), we define \( E_d := \{ n \in E : n \equiv 0 \pmod{d} \} \) and \( S(E, z) := |\{ n \in E : p | n \Rightarrow p \geq z \}|. \) In order to prove Theorem 1, it suffices to show that

\[ S(A, (2X)^{1/2}) \gg \frac{\delta X}{\log X} \quad (X \geq X_0), \quad (1.3) \]

where and in the sequel we put \( \gamma := 1/c \) and \( \delta := X^{\gamma - 1}(2^\gamma - 1). \)

Following Baker–Harman–Rivat [1], we shall repeatedly use the Buchstab identity

\[ S(E, z_1) = S(E, z_1) - \sum_{z_2 \leq p < z_1} S(E_p, p) \quad (z_2 \leq z_1) \quad (1.4) \]

together with asymptotic formulas in Lemmas 3.4 and 3.5 below. For some of them we can obtain asymptotic formulas by these Lemmas and the rest may be discarded; (see (4.7) below). In order to get such asymptotic formulas, we must treat the following exponential sums:

\[ S_I := \sum_{1 \leq h \leq H} \left| \sum_{m \sim M} \sum_{n \sim N} b_n e(h(mm)^{\gamma}) \right|, \]
\[ S_H := \sum_{1 \leq h \leq H} \left| \sum_{m \sim M} \sum_{n \sim N} a_m b_n e(h(mm)^{\gamma}) \right|, \]

where \( e(t) := e^{2\pi it}, \ |a_m| \leq 1, \ |b_n| \leq 1, \ H = X^{1-\gamma + \epsilon}, \ MN = X, \ \epsilon \) is an arbitrarily small positive number and \( m \sim M \) means \( M < m \leq 2M. \) In all previous works ([1], [7], [8], [10]), these two sums were estimated by the method of Fouvry–Iwaniec [2] and by the method of Heath-Brown [5].

Here we shall combine the method of Rivat–Sargos [11] with a “Weyl shift” (Lemma 2.5 of [3]) to improve that of Heath-Brown [5]; (see Theorem 2 below). This allows us to extend \( X^{1-\gamma + \epsilon} \leq N \leq X^{5\gamma - 4 - \epsilon} \) of ([1], [7], [8], [10]) to \( X^{1-\gamma + \epsilon} \leq N \leq X^{(61\gamma - 49)/11 - \epsilon}, \) which is the key to proving Theorem 1. In addition the work of Sargos and Wu [12, Theorem 7] concerning the method of Fouvry–Iwaniec enables us to extend the range \( \frac{10}{19} < \gamma < 1 \) in Lemmas 3 and 4 of [8] to \( \frac{37}{44} < \gamma < 1. \) Finally we shall obtain (see (4.16) below)

\[ S(A, (2X)^{1/2}) \geq \{ F(\gamma) + O(\epsilon) \} \frac{\delta X}{\log X}, \]

where \( F(\gamma) \) is a very complicated function (for the definition of \( F(\gamma) \), see (4.17) below). If we use \( \gamma_0 \) to denote the zero of \( F(\gamma) \), then \( c_0 := 1/\gamma_0 \) will be our desired value. It seems quite tedious to find the exact value of \( \gamma_0 \), but using Maple we have been able to obtain a good approximation guaranteed by a sufficient precision. We shall provide the details of these numerical computations in a separate section.
Throughout the paper (except for § 2), we suppose that $\frac{45}{58} < c \leq \frac{243}{305}$ and write $\gamma := 1/c$. We use $\varepsilon$ to denote an arbitrarily small positive number and put $\eta := \varepsilon^2$. Finally we define $L := \log X$. The notations of § 2 are independent of other sections.

2. Estimates on exponential sums. This section is devoted to investigating multiple exponential sums. The notations of this section are independent. Firstly we combine the method of Rivat–Sargos [11] with a “Weyl shift” [3, Lemma 2.5] to treat the following triple exponential sum of type II:

$$S := \sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} a_h b_{m,n} e \left( X \frac{h^\alpha m^\beta n^\gamma}{H^\alpha M^\beta N^\gamma} \right).$$ (2.1)

**Theorem 2.** Let $k \geq 2$ and $K := 2^{k+1} - 2$. Let $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha(\alpha - 1) \neq 0$ and $(\gamma - 1)/(1 - \alpha) \neq 1, \cdots, k$. Let $X > 0$, $H \geq 1$, $M \geq 1$, $N \geq 1$, $L := \log (2 + XHMN)$. $|a_h| \leq 1$ and $|b_{m,n}| \leq 1$. Let $S$ be defined as in (2.1). If $X \leq H^2$, then

$$S \ll \left( X^K H^{4K-2k} M^{5K-k} N^{5K-3k} \right)^{1/(6K-2k)} + (XH^2 M^2 N^2)^{1/4} + H(MN)^{1/2} + H^{3/4} MN + X^{-1/2} H M N L.$$

**Proof.** In view of Theorem 2 in [2], we can suppose that $X \geq MN$. Let $c_0, c_1, \cdots$ be some suitable constants depending on $\alpha, \beta, \gamma$ at most and $\Xi := c_0 M^\beta N^\gamma$. Let $Q \geq 10$ be a parameter to be chosen later. By the Cauchy inequality and by Lemma 5 in [11] (or [13, Lemma 2.5]) with $x_i = m^\beta n^\gamma / \Xi$ and $z_i = b_{m,n} e(Xh^\alpha m^\beta n^\gamma / H^\alpha M^\beta N^\gamma)$, we have

$$|S|^2 \ll H \sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} b_{m,n} e \left( X \frac{h^\alpha m^\beta n^\gamma}{H^\alpha M^\beta N^\gamma} \right)^2 \ll HQ \sum_{m \sim M} \sum_{m \sim M} \sum_{n \sim N} \sum_{n \sim N} \left| \sum_{h \sim H} e \left( \frac{Xh^\alpha u}{H^\alpha M^\beta N^\gamma} \right) \right|,$$

where $u = u_m(n) := m^\beta n^\gamma - \tilde{m}^\beta \tilde{n}^\gamma$, $m = (m, \tilde{m})$ and $n = (n, \tilde{n})$.

According to Lemma 1 in [2], the contribution of $|u| \leq \Xi L / MN$ is $O(H^2 M N Q L)$. Thus after a splitting-up argument, we find, for some $\Delta \in (L / MN, 1 / Q)$, that

$$|S|^2 \ll HQ L \sum_{m \sim M} \sum_{m \sim M} \sum_{n \sim N} \sum_{n \sim N} \left| \sum_{h \sim H} e \left( \frac{Xh^\alpha u}{H^\alpha M^\beta N^\gamma} \right) \right| + H^2 MNQL.$$ (2.2)

If $H^\prime := X\Delta / H \leq \varepsilon$, by Theorem 2.1 in [3] and Lemma 1 in [2], the first term on the right-hand side of (2.2) is $\ll X^{-1} (MN)^2 Q L \ll H^2 MNQL$. Next we suppose that $H^\prime \geq \varepsilon$. By applying Lemma 2.2 in [12], the innermost sum of (2.2) is equal to

$$(X\Delta)^{-1/2} H^\delta \sum_{h \sim H} \hat{a}_h e(f(h^\prime)) + O(L + \Psi_1(u) + \Psi_2(u),)$$
where $\Psi_t(u) := \min \{ (X\Delta)^{-1/2} H, 1/\|c_i X u / H \Xi\| \}$, $\|t\| := \min_{n \in \mathbb{Z}} |t - n|$, $|\tilde{a}_h| \leq 1$, $|\tilde{b}_u| \leq 1$, $f(h') := c_3 X \Delta (u / \Xi)^{1/(1-\alpha)} (h' / H)^{\alpha/(\alpha-1)}$ and $I'$ is a subinterval of $[H', 2H]$. Inserting into (2.2), we deduce, for some $\Delta \in (\mathcal{L} / MN, 1 / Q)$ and some $h' \sim H'$, the following inequality:

$$|S|^2 \ll (X\Delta)^{-1/2} H^2 Q S(\Delta) + HQ(T_1(\Delta) + T_2(\Delta)) + (H^2 MNQ + H(MN)^2)\mathcal{L},$$  

(2.3)

where

$$S(\Delta) := \sum_{m = M}^{m - M} \sum_{n = N}^{n - N} \sum_{\Delta < |u| / \Xi \leq 2\Delta} |\sum_{k \in I'} \tilde{a}_h e(f(h'))|, \quad T_i(\Delta) := \sum_{m = M}^{m - M} \sum_{n = N}^{n - N} \sum_{\Delta < |u| / \Xi \leq 2\Delta} \Psi_t(u).$$

According to (2.13) in [13], we have

$$T_i(\Delta) \ll \left\{ X^{1/2} \Delta M^2 N + X^{1/2} MN + \Delta (MN)^2 \right\} \mathcal{L}^2.$$

(2.4)

Next we shall apply the method of [11] to treat $S(\Delta)$. For this we must smooth the coefficient of $u$. By the Cauchy inequality and a “Weyl shift” [3, Lemma 2.5], we have, for any $R \in (0, \varepsilon H']$, that

$$|S(\Delta)|^2 \ll \langle \Delta H' \rangle^2 (MN)^4 R^{-1} + \Delta H'(MN)^2 R^{-1} \sum_{1 \leq i \leq R, h' \sim H} \sum_{\Delta < |u| / \Xi \leq 2\Delta} |S_{h', i}(\Delta)|,$$

(2.5)

where

$$S_{h', i}(\Delta) := \sum_{m = M}^{m - M} \sum_{n = N}^{n - N} \sum_{\Delta < |u| / \Xi \leq 2\Delta} e(g(u)),$$

$$g(u) := Yu^{1/(1-\alpha)} \text{ and } Y := c_3 X \Delta [(h' + r)^{\alpha/(\alpha-1)} - h^{\alpha/(\alpha-1)}] / H^{\alpha/(\alpha-1)} (\Delta \Xi)^{1/(1-\alpha)}.$$  

Clearly $c_4 \leq (m / \bar{m})^{b / \gamma} \leq c_5$ for $m \sim M$ and $\bar{m} \sim M$. We split the interval $I := [c_4, c_5]$ into $J := [|I| / \varepsilon \Delta] + 1$ subintervals $I_j$ of length $\delta := |I| / J \sim \varepsilon \Delta$. For each $d \in \mathbb{N}$, we put

$$\mathcal{M}_j(d) := \{ m : m \sim M, \bar{m} \sim M, (m, \bar{m}) = d, (m / \bar{m})^{b / \gamma} \in I_j \},$$

and for each $m \in \mathcal{M}_j(d)$ we define $\mathcal{N}(m) := \{ n : n \sim N, \bar{n} \sim N, \Delta < |u| / \Xi \leq 2\Delta \}$. Thus

$$S_{h', j}(\Delta) = \sum_{d \leq 2M} \sum_{j \leq J} \sum_{m \in \mathcal{M}_j(d)} |S_{h', j}(\Delta, m)|, \quad S_{h', j}(\Delta, m) := \sum_{n \in \mathcal{N}(m)} e(g(u)).$$  

(2.6)

We pick $a_j / b_j \in I_j \cup \mathbb{Q}$ with $b_j$ least and subject to $b_j \geq \delta^{-1/2}$ for $d_j \in \mathbb{Z}$ we define $\Gamma_{d_j} := \{(x, y) \in \mathbb{R}^2 : y = (a_j x - d_j) / b_j \}$. Clearly when $d_j$ runs over $\mathbb{Z}$, $\mathbb{Z}^2 \cap \Gamma_{d_j}$ constitutes a partition of $\mathbb{Z}^2$. Hence we can write

$$S_{h', j}(\Delta, m) \ll \sum_{d_j \in \mathbb{Z}} \sum_{n \in \mathcal{N}(m) \cap \Gamma_{d_j}} e(g(u)).$$  

(2.7)
Let \((n_0, \tilde{n}_0)\) be the starting point of \(N(m) \cap \Gamma_d \neq \emptyset\). Then \(b_j \tilde{n}_0 = a_j n_0 - d_j\). Now we write the parameter equation of this straight line as follows:

\[
n = b_j l + n_0, \quad \tilde{n} = a_j l + \tilde{n}_0 \quad (0 \leq l \ll N/b_j \sim N/a_j).
\]

Define

\[
u(t) := m^\beta (b_j t + n_0)^\gamma - \tilde{m}^\beta (a_j t + \tilde{n}_0)^\gamma, \quad F(t) := g(u(t)).
\]

From (2.7) and Lemma 2.7 in [13], we deduce that

\[
S_{k, r}(\Delta, m) \ll \begin{cases} \Delta N b_j \left| \sum_{l \ll N/b_j} e(F(l)) \right| & \text{if } b_j \leq N, \\ \Delta N b_j & \text{if } b_j > N. \end{cases} \tag{2.8}
\]

Similarly to Lemma 8 in [11] (or [13, Lemma 2.8]), we easily prove the following result.

**Lemma 2.1.** If \((\gamma - 1)/(1 - \alpha) \neq 0, 1, \ldots, k - 1\), then

\[
u(t) = A(b_j t + n_0)^{\gamma - 1}(1 + O(\varepsilon + \Delta)), \quad F(i)(t) = B(b_j t + n_0)^{(\gamma - 1)/(1 - \alpha) - k} b_j^k (1 + O_k(\varepsilon + \Delta)) \quad (0 \leq i \leq k), \tag{2.9, 2.10}
\]

where

\[
A := \gamma \tilde{m}^\beta (a_j/b_j)^{\gamma - 1} (d_j/b_j) \ll \Delta M^\beta N, \\
B := \prod_{i=0}^{k-1} ((\gamma - 1)/(1 - \alpha) - i) YA^{1/(1 - \alpha)} \ll HRN^{-(\gamma - 1)/(1 - \alpha)}.
\]

Applying Theorem 2.9* in [3] to the sum over \(l\) in (2.8) with \((F, N) = (HR, N/b_j)\) yields

\[
S_{k, r}(\Delta, m) \ll \Delta N^2 \Theta_j,
\]

where

\[
\Theta_j := \begin{cases} \min \{1, \Theta(b_j)\} & \text{if } b_j \leq N, \\ b_j/N & \text{if } b_j > N, \end{cases}
\]

and \(\Theta(b_j) := (HRN^{-k} b_j^k)^2/(k - 2)\). Inserting into (2.6) leads to

\[
S_{k, r}(\Delta) \ll \Delta N^2 \mathcal{L} \sum_{d \leq 2M} \sum_{j \leq L} \sum_{b_j \leq N} \sum_{m \in M(d)} \Theta_j + \Delta N^2 \mathcal{L} \sum_{d \leq 2M} \sum_{j > L} \sum_{b_j > N} \sum_{m \in M(d)} \Theta_j. \tag{2.11}
\]

By using Lemma 2.6 in [13], we have

*Clearly the condition \(I \subset [N, 2N]\) in this theorem can be relaxed to \(I \subset (1, 2N]\).
\[ \sum_{j \leq J, b_j > N} \Theta_j \ll \sum_{0 \leq b \in \mathcal{L}} \sum_{j \leq J, 1 \leq b_j / 2N} b_j / N \ll \sum_{0 \leq b \in \mathcal{L}} 2^{-i(N\delta)^{-2}} \ll (N\delta)^{-2}. \]

In addition since the points in \( \mathcal{M}_j(d) \) are spaced by \( (d/M)^2 \), we have \( |\mathcal{M}_j(d)| \ll \delta(M/d)^2 + 1 \). Thus the second member on the right-hand side of (2.11) is

\[ \ll \Delta N^2(\delta M^2 + M(N\delta)^{-2}) \mathcal{L} \ll (M^2 + \Delta^{-1}M) \mathcal{L}. \] (2.12)

Similarly Lemma 2.6 in [13] allows us to deduce that

\[ \sum_{j \leq J, b_j \leq N} \Theta_j \ll \sum_{0 \leq b \in \mathcal{L}} \sum_{j \leq J, 1 \leq b_j / 2\delta^{-1/2}} \Theta(b_j) \ll \sum_{i \geq 0} 4^{-i} \delta^{-1} \Theta(2^i \delta^{-1/2}) \ll \delta^{-1} \Theta(\delta^{-1/2}). \]

Thus the first member on the right-hand side of (2.11) is

\[ \ll \Delta N^2 \mathcal{L} \sum_{d_1 \leq \sqrt{\delta}} \sum_{j \leq J, b_j \leq N} \sum_{m \in \mathcal{M}_j(d)} \Theta(b_j) + \Delta N^2 \mathcal{L} \sum_{d_2 > \sqrt{\delta}} \sum_{j \leq J, b_j \leq N} \sum_{m \in \mathcal{M}_j(d)} 1 \]

\[ \ll \Delta N^2 \mathcal{L} \sum_{d_1 \leq \sqrt{\delta}} \delta(M/d)^2 \sum_{j \leq J, b_j \leq N} \Theta(b_j) + \Delta N^2 \mathcal{L} \sum_{d_2 > \sqrt{\delta}} \sum_{m \in \mathcal{M}_j(d)} (M/d)^2 \]

\[ \ll \Delta(MN)^2 \Theta(\delta^{-1/2}) \mathcal{L} + \Delta \delta^{-1/2} MN^2 \mathcal{L} \]

\[ \ll \Delta(MN)^2 \Theta(\Delta^{-1/2}) \mathcal{L} + \Delta^{1/2} MN^2 \mathcal{L}. \] (2.13)

Inserting (2.12) and (2.13) into (2.11), we obtain

\[ S_{H', r}(\Delta) \ll \left\{ \left( \Delta^{K-k-2} H^2 M^{2K-4} N^{2k-4} r^2 \right)^1/(K-2) + \Delta^{1/2} MN^2 + M^2 + \Delta^{-1} M \right\} \mathcal{L}. \]

Inserting into (2.5) leads to the inequality

\[ |S(\Delta)|^2 \ll \left\{ \left( \Delta^{2K-k} H^2 M^{2K} N^{K-2k} r^2 \right)^1/(K-2) \right. \]

\[ + \left( \Delta H' \right)^2 (MN)^4 R^{-1} + \left( \Delta^3 H^4 M^6 N^8 \right)^{1/2} + \Delta H'^2 M^4 N^2 + H'^2 M^3 N^2 \} \mathcal{L}. \]

Optimizing \( R \) over \( (0, \epsilon H'] \) we get

\[ |S(\Delta)|^2 \ll \left\{ \left( \Delta^{2K-k} H^2 M^{2K} N^{K-2k} \right)^{1/K} \right. \]

\[ + \left( \Delta^2 H' (MN)^4 + \left( \Delta^3 H^4 M^6 N^8 \right)^{1/2} + \Delta H'^2 M^4 N^2 + H'^2 M^3 N^2 \right\} \mathcal{L}. \]

Replacing \( H' \) by \( X\Delta/H \) yields

\[ |S(\Delta)|^2 \ll \left\{ \left( \Delta^{K-k} H^{2K+2} M^{4K} N^{K-2k} \right)^{1/K} + \Delta^3 H^{-1} (MN)^4 \right. \]

\[ + \left( \Delta^4 H^{4} M^{6} N^{8} \right)^{1/2} + X^2 \Delta^{3} H^{-2} M^{4} N^{2} + X^2 \Delta^{2} H^{-2} M^{3} N^{2} \} \mathcal{L}. \] (2.14)

Inserting (2.14) and (2.4) into (2.3) and using \( \Delta \leq 1/Q \), we have

\[ |S|^2 \ll \left\{ \left( \Delta^{K} H^{2K+2} M^{4K} N^{K-2k} Q^{-(K-k)} \right)^{1/2K} \right. \]

\[ + \left( H^2 M^{4} N^{4} \right)^{1/2} + \left( X^2 H^4 M^{6} N^{8} \right)^{-1/4} + \left( X^2 H^2 M^{4} N^{2} \right)^{1/2} \]

\[ + \left( X H^2 M^{3} N^{2} Q \right)^{1/2} + \left( X H^2 M^{2} N^{2} Q^{2} \right)^{1/2} + H^2 MNQ + H(MN)^2 \} \mathcal{L}^2. \]
Since \( X \leq H^2 \), we have \((XH^2M^2N^2Q^2)^{1/2} \leq H^2MNQ\). If \( Q \geq Q_0 := \max\{10, XH^{-2}M\} \), then \((XH^2M^3N^2Q)^{1/2} \leq H^2MNQ\). Thus the previous estimation can be simplified as follows:

\[
|S|^2 \ll \left( (XH^2^{K+2}M^6N^4K^{-2}Q^{-K})^{1/2} + (X^2H^4M^6N^8Q^{-1})^{1/4} \right. \\
\left. + H^2MNQ + H^{3/2}M^2N^2 + X^{1/2}HM^2N + H(MN)^2 \right) \mathcal{L}^2.
\]

Now optimizing \( Q \) over \([Q_0, \infty)\) and noticing that \((XM^2N)^{1/2} \leq (XH^2M^4N^2)^{1/4}\), we obtain the required result. This completes the proof.

We also need another result on triple exponential sum of type II; see [12, Theorem 7].

**Lemma 2.2.** Let \( \alpha, \beta, \gamma \in \mathbb{R} \) with \( \alpha \beta \gamma(\alpha - 1)(\alpha - 2) \neq 0 \), \( X > 0 \), \( H \geq 1 \), \( M \geq 1 \), \( N \geq 1 \), \( \mathcal{L} := \log(2 + XHMN) \), \( |a_h| \leq 1 \) and \( |b_{m,n}| \leq 1 \). Let \( S \) be defined as in (2.1). Then

\[
S \ll \left\{ (X^4H^{15}M^{22}N^{22})^{1/26} + (XH^2M^3N^3)^{1/4} \right. \\
\left. + H^{11/18}MN + H(MN)^{3/4} + X^{-1/2}HMN \right\} \mathcal{L}^2.
\]

From Theorem 2 and Lemma 2.2, we immediately deduce the following result.

**Corollary 1.** Let \( \frac{37}{44} < \gamma < \frac{51}{71} \), \( H = X^{1-\gamma+4\eta} \), \( MN = X \), \( |a_m| \leq 1 \) and \( |b_n| \leq 1 \). If one of the following three conditions (i) \( X^{3-3\gamma+\varepsilon} \leq N \leq X^{3\gamma-2-\varepsilon} \), (ii) \( X^{1-\gamma+\varepsilon} \leq N \leq X(61\gamma-49)/11-\varepsilon \), (iii) \( X^{60-61\gamma)/11+\varepsilon} \leq N \leq X^{\gamma-\varepsilon} \) holds, then

\[
\sum_{1 \leq h \leq H} \left| \sum_{m \sim M} \sum_{n \sim N} a_mb_ne(h(mn)^\gamma) \right| \ll X^{1-5\eta}.
\]

**Proof.** By using Lemma 2.2 with \((H, M, N) = (X/N, H_0, N) \) \((H_0 \leq H)\), we see that (2.15) holds provided \( \frac{36}{43} \leq \gamma < 1 \) and \( X^{3-3\gamma+\varepsilon} \leq N \leq X^{(22\gamma-15)/7-\varepsilon} \). Interchanging \( X/N \) and \( N \), we find \( X^{22(\gamma-1)/7+\varepsilon} \leq X^{3\gamma-2-\varepsilon} \). Since \( 22(\gamma-1)/7 \leq 1/2 \leq (22\gamma - 15)/7 \) for \( \gamma > \frac{37}{44} \), this proves (2.15) provided (i) holds and \( \frac{37}{44} < \gamma < 1 \).

If (ii) holds and \( \frac{5}{6} < \gamma < \frac{61}{71} \), we can get (2.15) by using Theorem 2 with \( k = 5 \) and \((H, M, N) = (X/N, H_0, N) \) \((H_0 \leq H)\). Similarly if (iii) holds and \( \frac{5}{6} < \gamma < \frac{61}{71} \), we can get (2.15) by using Theorem 2 with \( k = 5 \) and \((H, M, N) = (N, H_0, X/N) \) \((H_0 \leq H)\). □

We also need a result on triple exponential sums of type I [12, Theorem 8].

**Lemma 2.3.** Let \( \alpha, \beta, \gamma \in \mathbb{R} \) with \( \alpha \beta \gamma(\gamma - 2)(\alpha + 1 - \gamma)(\alpha + 2\gamma - 2) \neq 0 \), \( X > 0 \), \( H \geq 1 \), \( M \geq 1 \), \( N \geq 1 \), \( \mathcal{L} := \log(2 + XHMN) \), \( |a_h| \leq 1 \) and \( |b_{m,n}| \leq 1 \). Define
\[ S := \sum_{h \sim H} \sum_{m \sim M} \sum_{n \sim N} a_h b_m e \left( X \frac{h^\theta m^\theta n^\nu}{H^\alpha M^\beta N^\gamma} \right). \]

Then
\[ S \ll \left\{ (X^{13} H^{15} M^{22} A^4)^{1/26} + (X^2 H^3 M^3 N)^{1/4} + (X H^4 M^3 N)^{1/4} + (X H^1 M^{18})^{1/18} + HM + X^{-1} HMN \right\} C^2. \]

**Corollary 2.** Let \( \frac{37}{44} < \gamma < \frac{28}{33}, H = X^{1-\gamma+4\eta}, MN = X, |b_n| \leq 1 \). If \( N \leq X^{3\gamma-2-\epsilon} \), then
\[ \sum_{1 \leq h \leq H} \left| \sum_{m \sim M} \sum_{n \sim N} b_n e(h(mn)\gamma) \right| \ll X^{1-5\eta}. \]

**Proof.** By Lemma 2.3 with \((H, M, N) = (N, H_0, X/N) (H_0 \leq H)\), we see that (2.16) holds provided \( N \leq X^{(22\gamma-13)/11-\epsilon} \) and \( \frac{37}{44} < \gamma < \frac{28}{33} \). Combining this and Corollary 1(i) and noticing that \( (22\gamma-13)/11 > 3 - 3\gamma \) for \( \gamma > \frac{46}{33} \), we deduce the desired result.

\[ \square \]

### 3. An alternative sieve

The aim of this section is to describe simply the alternative sieve of Baker–Harman–Rivat [1]. We insert new results on exponential sums in §2 into their alternative sieve. This allows us to improve all results there. Since the proof is very similar, we shall state our results. For more details see [1], [8] and [4].

Let \( \frac{37}{44} < \gamma < \frac{28}{33}, \theta_0 := 6\gamma - 5, \theta_1 := 1 - \gamma, \theta_2 := (61\gamma - 49)/11, \theta_3 := 3 - 3\gamma, \theta_4 := 3\gamma - 2, \theta_5 := (60 - 61\gamma)/11, \theta_6 := \gamma, \phi_i := X^{\theta_i} \) for \( i = 1, \ldots, 6 \) and \( z := (2X)^{1/2} \). Define \( I_i(\gamma) := [\phi_{2i-1}X^\gamma, \phi_{2i}X^{-\gamma}] \) for \( i = 1, 2, 3 \).

**Lemma 3.1.** Let \( |b_n| \ll X^\eta \) and \( \frac{37}{44} < \gamma < \frac{28}{33} \). If \( N \leq X^{\theta_4-\epsilon} \), then
\[ \sum_{n \leq N} b_n |A_n| = \delta X \sum_{n \leq N} b_n / n + O(\delta X^{1-3\eta}). \]

**Proof.** We can proceed as in Lemma 6 in [1], by replacing Lemma 5 in [1] by Corollary 2.

\[ \square \]

**Lemma 3.2.** Let \( \frac{37}{44} < \gamma < \frac{61}{44}, |a_m| \ll X^{\nu/2} \) and \( |b_n| \ll X^{\nu/2} \). If \( N \) lies in one of \( I_i(\gamma) \) (\( i = 1, 2, 3 \)), then
\[ \sum_{m \sim M} \sum_{n \sim N} a_m b_n = \sum_{m \sim M} \sum_{n \sim N} a_m b_n \gamma(mn)^{\gamma-1} + O(\delta X^{1-3\eta}). \]

**Proof.** We can proceed as in Lemma 7 in [1], by replacing Lemma 4 in [1] by Corollary 1.

\[ \square \]

**Lemma 3.3.** Suppose that \( \frac{37}{44} < \gamma < \frac{28}{33} \) and \( N \leq X^{\theta_4-\epsilon} \). Then we have
\[ \sum_{n \sim N} b_n S(A_n, \phi_0) = \delta \sum_{n \sim N} b_n S(B_n, \phi_0) \{1 + O(1/L)\} + O(\delta X^{1-2\nu}), \]

where \(0 \leq b_n \ll X^\nu\) and \(b_n = 0\) unless \(p|n \Rightarrow p \geq \phi_0\).

Proof. This can be proved in the same way as in Lemma 12 in [1], by using Lemmas 3.2 and 3.3 in place of Lemmas 7 and 9 in [1]. \(\Box\)

**Lemma 3.4.** Let \(k\) be a given positive integer and let \(D \subset \{1, \cdots, k\}\). Assuming that \(N\) lies in one of \(I_i(\gamma)\) \((i = 1, 2, 3)\), then

\[ \sum_{p_1} \cdots \sum_{p_k} S(A_{p_1, \cdots, p_k}, p_1) = \delta \sum_{p_1} \cdots \sum_{p_k} S(B_{p_1, p_k}, p_1) \{1 + O(1/L)\} + O(\delta X^{1-2\nu}), \]

where \(*\) means that \((p_1, \cdots, p_k)\) satisfy \(X^\nu \leq p_1 < p_2 < \cdots < p_k, \prod_{j \in D} p_j \asymp N\) together with no more than \(\nu^{-1}\) further conditions which take the form \(R \leq \prod_{j \in J} p_j \leq S\).

Proof. We can proceed as in Lemma 11 in [1], by replacing Lemma 7 in [1] by Lemma 3.2. \(\Box\)

**4. Decomposition of \(S(A, z)\).** As described in § 1, we repeatedly use the Buchstab identity (1.4) together with asymptotic formulas in Lemmas 3.4 and 3.5 to decompose \(S(A, z)\) such that the loss (i.e. the sums discarded) is as small as possible. Henceforth the letters \(p, q, r, s\) always denote prime numbers.

By (1.4), we first write

\[ S(A, z) = S(A, \phi_0) - \sum_{\phi_0 \leq p < \phi_1} S(A_p, p) - \sum_{\phi_1 \leq p < \phi_2} S(A_p, p) \]

\[ - \sum_{\phi_2 \leq p < \phi_3} S(A_p, p) - \sum_{\phi_3 \leq p < z} S(A_p, p) \]

\[ =: S_1 - S_2 - S_3 - S_4 - S_5, \tag{4.1} \]

where \(S_1\) can be evaluated asymptotically by Lemma 3.3 and \(S_3, S_5\) by Lemma 3.4. Applying three times (1.4) to \(S_2\) yields

\[ S_2 = \sum_{\phi_0 \leq p < \phi_1} S(A_p, \phi_0) - \sum_{\phi_0 \leq q < \phi_1} \sum_{p} S(A_{pq}, \phi_0) + \sum_{\phi_0 \leq r < \phi_1} \sum_{p} \sum_{q} S(A_{pqr}, \phi_0) \]

\[ - \sum_{\phi_0 \leq s < r < \phi_1} \sum_{p} \sum_{q} \sum_{s} S(A_{pqrs}, \phi_0) \]

\[ =: S_6 - S_7 + S_8 - S_9 - S_{10}, \tag{4.2} \]

where

\((\nabla_4)\) some subproducts of \(pqrs\) lie in one of the intervals \(I_i(\gamma)\) \((i = 1, 2, 3)\).

\((\nabla_4')\) no subproduct of \(pqrs\) lies in one of the intervals \(I_i(\gamma)\) \((i = 1, 2, 3)\).

We have asymptotic formulas for \(S_6, S_7, S_8\) by Lemma 3.3 and for \(S_9\) by Lemma 3.4.
In order to treat $S_4$ in (4.1), we first write, by (1.4),

$$S_4 = \sum_{\phi_2 \leq p < \phi_3} S(A_p, \phi_0) - \sum_{\phi_2 \leq p < \phi_3} \sum_{\phi_1 \leq q < \phi_2} S(A_{pq}, q)$$

$$- \sum_{\phi_1 \leq q < \phi_2} S(A_{pq}, q) - \sum_{\phi_2 \leq q < \phi_3} \sum_{\phi_1 \leq q < \phi_2} S(A_{pq}, q)$$

(4.3)

$$=: S_{11} - S_{12} - S_{13} - S_{14}.$$ 

Lemma 3.3 provides asymptotic formulae for $S_{11}$ and Lemma 3.4 for $S_{13}$.

For $S_{12}$, we split the range of summation into three parts:

$$S_{12} = \sum_{\phi_2 \leq p < \phi_3} \sum_{\phi_1 \leq q < \phi_2} S(A_{pq}, q) + \sum_{\phi_2 \leq p < \phi_3} \sum_{\phi_1 \leq q < \phi_2} S(A_{pq}, q) + \sum_{\phi_2 \leq p < \phi_3} \sum_{\phi_1 \leq q < \phi_2} S(A_{pq}, q)$$

$$=: S_{15} + S_{16} + S_{17}.$$ 

According to Lemma 3.4, we have asymptotic formulas for $S_{15}$.

By applying (1.4), we have

$$S_{16} = \sum_{\phi_2 \leq p < \phi_3} \sum_{\phi_1 \leq q < \phi_2} S(A_{pq}, (2X/pq)^{1/2}) + \sum_{\phi_2 \leq p < \phi_3} \sum_{\phi_1 \leq q < \phi_2} S(A_{pq}, r)$$

$$+ \sum_{\phi_2 \leq p < \phi_3} \sum_{\phi_1 \leq q < \phi_2} S(A_{pq}, q)$$

$$=: S_{18} + S_{19} + S_{20},$$

where

$(\nabla_3)$ some subproducts of $pqr$ lie in one of the intervals $I_i(\gamma)$ ($i = 1, 2, 3$),

$(\nabla_3')$ no subproduct of $pqr$ lies in one of the intervals $I_i(\gamma)$ ($i = 1, 2, 3$).

We can evaluate asymptotically $S_{19}$ by Lemma 3.3.

By applying (1.4) twice, we have

$$S_{17} = \sum_{\phi_2 \leq p < \phi_3} \sum_{\phi_1 \leq q < \phi_2} S(A_{pq}, \phi_0) - \sum_{\phi_2 \leq p < \phi_3} \sum_{\phi_1 \leq q < \phi_2} S(A_{pq}, \phi_0)$$

$$+ \sum_{\phi_2 \leq p < \phi_3} \sum_{\phi_1 \leq q < \phi_2} S(A_{pq}, s) + \sum_{\phi_2 \leq p < \phi_3} \sum_{\phi_1 \leq q < \phi_2} S(A_{pq}, s)$$

$$=: S_{21} - S_{22} + S_{23} + S_{24}.$$ 

Lemma 3.3 provides asymptotic formulae for $S_{21}, S_{22}$ and Lemma 3.4 for $S_{23}$.

Combining these we obtain

$$S_{12} = S_{15} + S_{18} + S_{19} + S_{20} + S_{21} - S_{22} + S_{23} + S_{24},$$

(4.4)

where $S_{15}, S_{19}, S_{21}, S_{22}, S_{23}$ can be evaluated asymptotically.
It remains to treat \( S_{14} \) in (4.3). We have

\[
S_{14} = \sum_{\phi_2 \leq q < p < \phi_3} S(A_{pq}, \phi) + \sum_{\phi_2 \leq q < p < \phi_3} S(A_{pq}, q) + \sum_{\phi_2 \leq q < p < \phi_3} S(A_{pq}, \phi_4)
\]

where \( S \) can be evaluated asymptotically, except for \( j = 18, 20, 24, 25, 27, 29 \).

Next we shall give upper bounds for \( S_j \) for \( j = 10, 18, 20, 24, 25, 27, 29 \). Define \( \alpha \) by the relations \( p := X^{\alpha_1}, q := X^{\alpha_2}, r := X^{\alpha_3} \) and \( s := X^{\alpha_4} \). Let \( \omega(t) \) be the Buchstab function.

By the prime number theorem, we have

\[
S(B, z) \sim \frac{X}{\log X}.
\]
A. Estimation of $S'_{10}$.

According to Lemma 5 in [8], we deduce that

$$S'_{10} = \{1 + O(\varepsilon)\} \sum_{\phi_0 < r < q < \phi_1} \sum_{(\mathbb{N}_1)} X \log s \left( \log (X/pqrs) / \log s \right).$$

By using the standard procedure for replacing sums over primes by integrals, we can prove that

$$S'_{10} = \frac{X}{\log X} \int_{D_{10}} \omega \left( \frac{1 - \alpha_1 - \cdots - \alpha_4}{\alpha_4} \right) \frac{d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4}{\alpha_1 \alpha_2 \alpha_3 \alpha_4^2} \{1 + O(\varepsilon)\},$$

where

$$D_{10} := \{ (\alpha_1, \alpha_2, \alpha_3, \alpha_4) : \theta_0 \leq \alpha_4 \leq \alpha_3 \leq \alpha_2 \leq \alpha_1 \leq \theta_1, (C)_4 \}$$

and

$$(C)_4$$

there is no subsum of $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ in $[\theta_1, \theta_2, [\theta_3, \theta_4]$.\)

Since $\omega(t) < 0.5672$ for $t \geq 2$ and $(1 - \alpha_1 - \cdots - \alpha_4)/\alpha_4 \geq 2$ for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in D_{10}$, we find that

$$S'_{10} \leq f_{10}(\gamma) \frac{X}{\log X},$$

where

$$f_{10}(\gamma) := 0.5672 \int_{D_{10}} \frac{d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4}{\alpha_1 \alpha_2 \alpha_3 \alpha_4^2}.$$

B. Estimation of $S'_{18}$.

According to the definition of $S'_{18}$, we have

$$S'_{18} = \sum_{\phi_2 < p < \phi_3} \sum_{\phi_0 < q < \phi_1} \sum_{X/pq < r < 2X/pq} \sum_{pq^2 > \phi_4, pq \notin \{\phi, \phi_4\}} \{1 + O(\varepsilon)\} X \log \left( \log (X/pq) \right),$$

from which we deduce that

$$S'_{18} = \{ f_{18}(\gamma) + O(\varepsilon) \} \frac{X}{\log X},$$

where

$$f_{18}(\gamma) := \int_{D_{18}} \frac{d\alpha_1 d\alpha_2}{\alpha_1 \alpha_2 (1 - \alpha_1 - \alpha_2)}.$$
and
\[ \mathbb{D}_{18} := \{ (\alpha_1, \alpha_2) : \theta_2 \leq \alpha_1 \leq \theta_3, \theta_0 \leq \alpha_2 \leq \theta_1, \alpha_1 + 2\alpha_2 > \theta_4, \alpha_1 + \alpha_2 \notin [\theta_3, \theta_4] \}. \]

C. Estimation of \( S_{20}' \).
Similarly to A, we have
\[ S_{20}' \leq f_{20}(\gamma) \frac{X}{\log X}, \tag{4.11} \]
where
\[ f_{20}(\gamma) := 0.5672 \int_{\mathbb{D}_{20}} \frac{\alpha_1 \alpha_2 \alpha_3}{\alpha_1 \alpha_2 \alpha_3^2}, \]
\[ \mathbb{D}_{20} := \{ (\alpha_1, \alpha_2, \alpha_3) : \theta_2 \leq \alpha_1 \leq \theta_3, \theta_0 \leq \alpha_2 \leq \theta_1, \alpha_2 \leq \alpha_3 \leq \frac{1}{2}(1 - \alpha_1 - \alpha_2), \]
\[ \alpha_1 + 2\alpha_2 \geq \theta_4, (C_3) \}

and
\( (C)_3 \) there is no subsum of \( \alpha_1 + \alpha_2 + \alpha_3 \) in \([\theta_1, \theta_2], [\theta_3, \theta_4], [\theta_5, \theta_6] \).

D. Estimation of \( S_{24}' \).
Similarly to A, we have
\[ S_{24}' \leq f_{24}(\gamma) \frac{X}{\log X}, \tag{4.12} \]
where
\[ f_{24}(\gamma) := 0.5672 \int_{\mathbb{D}_{24}} \frac{\alpha_1 \alpha_2 \alpha_3 \alpha_4}{\alpha_1 \alpha_2 \alpha_3 \alpha_4^2}, \]
\[ \mathbb{D}_{24} := \{ (\alpha_1, \alpha_2, \alpha_3, \alpha_4) : \theta_2 \leq \alpha_1 \leq \theta_3, \theta_0 \leq \alpha_4 \leq \alpha_3 \leq \alpha_2 \leq \theta_1, \alpha_1 + 2\alpha_2 \leq \theta_4, (C'_4) \}

and
\( (C)'_4 \) there is no subsum of \( \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \) in \([\theta_1, \theta_2], [\theta_3, \theta_4], [\theta_5, \theta_6] \).

E. Estimation of \( S_{25}' \).
Similarly to \{B\}, we have
\[ S_{25}' \leq \{ f_{25}(\gamma) + O(\epsilon) \} \frac{X}{\log X}, \tag{4.13} \]
where
\[ f_{25}(\gamma) := \int_{D_{25}} \frac{d\alpha_1 d\alpha_2}{\alpha_1 \alpha_2 (1 - \alpha_1 - \alpha_2)} \]

and

\[ D_{25} := \{(\alpha_1, \alpha_2) : \theta_2 \leq \alpha_2 \leq \alpha_1 \leq \theta_3, \alpha_1 + \alpha_2 \leq \theta_1 \}. \]

**F. Estimation of \( S'_{27} \).**

Similarly to B, we have

\[ S'_{27} \leq \{ f_{27}(\gamma) + O(\varepsilon) \} \frac{X}{\log X}, \tag{4.14} \]

where

\[ f_{27}(\gamma) := \int_{D_{27}} \frac{d\alpha_1 d\alpha_2 d\alpha_3}{\alpha_1 \alpha_2 \alpha_3 (1 - \alpha_1 - \alpha_2 - \alpha_3)}, \]

\[ D_{27} := \{(\alpha_1, \alpha_2, \alpha_3) : \theta_2 \leq \alpha_2 \leq \alpha_1 \leq \theta_3, \alpha_2 \leq \alpha_3 \leq \frac{1}{2}(1 - \alpha_1 - \alpha_2), \alpha_1 + \alpha_2 \leq \theta_3, (C)'_3 \} \]

and

\( (C)'_3 \) there is no subsum of \( \alpha_1 + \alpha_2 + \alpha_3 \) in \( [\theta_3, \theta_4] \).

**G. Estimation of \( S'_{29} \).**

Similarly to A, we have

\[ S'_{29} \leq \{ f_{29}(\gamma) + O(\varepsilon) \} \frac{X}{\log X}, \tag{4.15} \]

where

\[ f_{29}(\gamma) := \int_{D_{29}} \omega \left( \frac{1 - \alpha_1 - \alpha_2}{\alpha_2} \right) \frac{d\alpha_1 d\alpha_2}{\alpha_1^2} \]

and

\[ D_{29} := \{(\alpha_1, \alpha_2) : \theta_2 \leq \alpha_2 \leq \alpha_1 \leq \theta_3, \alpha_1 + \alpha_2 \geq \theta_4, \alpha_1 + 2\alpha_2 \leq 1 \}. \]

Inserting (4.8)–(4.15) into (4.7), we obtain

\[ S(\mathcal{A}, z) \geq \{ F(\gamma) + O(\varepsilon) \} \frac{\delta X}{\log X}, \tag{4.16} \]

where

\[ F(\gamma) := 1 - f_{10}(\gamma) - f_{18}(\gamma) - f_{20}(\gamma) - f_{24}(\gamma) - f_{25}(\gamma) - f_{27}(\gamma) - f_{29}(\gamma). \tag{4.17} \]
5. End of the proof of Theorem 1: numerical calculations. The problem of computing $\gamma_0$ with a sufficient precision remains to compute multiple integrals over a simplex domain with sufficient precision. In general, this is a difficult question of numerical analysis. The Monte Carlo method provides an approximation but the precision is not guaranteed. Hence we choose a safer method: symbolic integration and numerical integral in 1 variable (which is safe using Simpson or another method), using Maple. Since the use of min and max is not safe with Maple, we shall manually split the domains of integration in order to simplify them. (We have slightly enlarged them, by the way.)

We have

$$\mathbb{D}_{10} \subset \mathbb{D}^A_{10} \cup \mathbb{D}^B_{10} \cup \mathbb{D}^C_{10} \cup \mathbb{D}^D_{10} \cup \mathbb{D}^E_{10},$$

where

$$\mathbb{D}^A_{10} := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{D}_{10} : \alpha_3 + \alpha_4 > \theta_2, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 > \theta_4\},$$

$$\mathbb{D}^B_{10} := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{D}_{10} : \alpha_3 + \alpha_4 > \theta_2, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < \theta_3\},$$

$$\mathbb{D}^C_{10} := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{D}_{10} : \alpha_3 + \alpha_4 < \theta_1, \alpha_1 + \alpha_2 > \theta_2, \alpha_2 + \alpha_3 > \theta_2\},$$

$$\mathbb{D}^D_{10} := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{D}_{10} : \alpha_3 + \alpha_4 < \theta_1, \alpha_1 + \alpha_2 > \theta_2, \alpha_2 + \alpha_3 < \theta_1\},$$

$$\mathbb{D}^E_{10} := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{D}_{10} : \alpha_3 + \alpha_4 < \theta_1, \alpha_1 + \alpha_2 < \theta_1\}.$$

Hence

$$\mathbb{D}^A_{10} \subset \left\{\frac{1}{4} \theta_4 < \alpha_1 < \theta_1, \frac{1}{4} \theta_4 - \frac{1}{2} \theta_2 < \alpha_2 < \alpha_1, \frac{1}{2} \theta_2 < \alpha_3 < \alpha_2, \theta_2 - \alpha_3 < \alpha_4 < \alpha_3\right\},$$

$$\mathbb{D}^B_{10} \subset \left\{\frac{1}{2} \theta_2 < \alpha_1 < \theta_1 - \frac{1}{4} \theta_2 - \frac{1}{2} \theta_2 < \alpha_2 < \theta_2 - \alpha_3 - \alpha_1, \frac{1}{2} \theta_2 < \alpha_3 < \alpha_2, \theta_2 - \alpha_3 < \alpha_4 < \alpha_3\right\},$$

$$\mathbb{D}^C_{10} \subset \left\{\theta_2 - \frac{1}{2} \theta_1 < \alpha_1 < \theta_1, \theta_2 - \theta_1 + \theta_0 < \alpha_2 < \alpha_1, \theta_0 < \alpha_4 < \frac{1}{2} \theta_1, \alpha_4 < \alpha_3 < \theta_1 - \alpha_4\right\},$$

$$\mathbb{D}^D_{10} \subset \left\{\theta_2 - \frac{1}{2} \theta_1 < \alpha_1 < \theta_1, \frac{1}{2} \theta_2 < \alpha_2 < \theta_1 - \theta_0, \theta_2 - \theta_1 < \alpha_4 < \frac{1}{2} \theta_1, \alpha_4 < \alpha_3 < \frac{1}{2} \theta_1\right\},$$

$$\mathbb{D}^E_{10} \subset \left\{\frac{1}{2} \theta_2 < \alpha_1 < \theta_1 - \frac{1}{4} \theta_2, \frac{1}{2} \theta_2 < \alpha_2 < \alpha_1, \theta_2 - \theta_1 < \alpha_3 < \frac{1}{2} \theta_1, \theta_2 - \theta_1 < \alpha_4 < \alpha_3\right\}.$$

We have

$$\mathbb{D}_{18} = \mathbb{D}^A_{18} \cup \mathbb{D}^B_{18} \cup \mathbb{D}^C_{18},$$

where

$$\mathbb{D}^A_{18} := \{(\alpha_1, \alpha_2) \in \mathbb{D}_{18} : \alpha_1 + \alpha_2 < \theta_3, \alpha_2 < \frac{1}{2} (\theta_4 - \theta_2)\},$$

$$\mathbb{D}^B_{18} := \{(\alpha_1, \alpha_2) \in \mathbb{D}_{18} : \alpha_1 + \alpha_2 < \theta_3, \alpha_2 > \frac{1}{2} (\theta_4 - \theta_2)\},$$

$$\mathbb{D}^C_{18} := \{(\alpha_1, \alpha_2) \in \mathbb{D}_{18} : \alpha_1 + \alpha_2 > \theta_4\}.$$

Hence we have

$$\mathbb{D}^A_{18} = \{\theta_0 < \alpha_2 < \frac{1}{2} (\theta_4 - \theta_2), \theta_4 - 2 \alpha_2 < \alpha_1 < \theta_3 - \alpha_2\}.$$

$$\mathbb{D}^B_{18} = \left\{\frac{1}{2} (\theta_4 - \theta_2) < \alpha_2 < \theta_1, \theta_2 < \alpha_1 < \theta_3 - \alpha_2\right\},$$

$$\mathbb{D}^C_{18} = \{\theta_0 < \alpha_2 < \theta_1, \theta_4 - \alpha_2 < \alpha_1 < \theta_3\}.$$
We have
\[
\mathbb{D}_{20} = \mathbb{D}^A_{20} \cup \mathbb{D}^B_{20} \cup \mathbb{D}^C_{20} \cup \mathbb{D}^D_{20} \cup \mathbb{D}^E_{20} \cup \mathbb{D}^F_{20} \cup \mathbb{D}^G_{20} \cup \mathbb{D}^H_{20} \cup \mathbb{D}^I_{20} \cup \mathbb{D}^J_{20} \cup \mathbb{D}^K_{20} \cup \mathbb{D}^L_{20} \\
\cup \mathbb{D}^M_{20} \cup \mathbb{D}^N_{20},
\]

where
\[
\begin{align*}
\mathbb{D}^A_{20} & := \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_2 + \alpha_3 < \theta_1, \alpha_1 + \alpha_2 < \theta_3, \alpha_2 < \frac{1}{2}(\theta_1 - \theta_3 + \theta_4)\}, \\
\mathbb{D}^B_{20} & := \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_2 + \alpha_3 < \theta_1, \alpha_1 + \alpha_2 < \theta_3, \alpha_2 > \frac{1}{2}(\theta_1 - \theta_3 + \theta_4)\}, \\
\mathbb{D}^C_{20} & := \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_2 + \alpha_3 < \theta_1, \alpha_1 + \alpha_2 < \theta_4\}, \\
\mathbb{D}^D_{20} & := \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_2 + \alpha_3 > \theta_2, \alpha_3 < \theta_1, \alpha_1 + \alpha_3 < \theta_3, \alpha_3 < \frac{1}{2}(2\theta_2 + \theta_3 - \theta_4)\}, \\
\mathbb{D}^E_{20} & := \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_2 + \alpha_3 > \theta_2, \alpha_3 < \theta_1, \alpha_1 + \alpha_3 < \theta_3, \alpha_3 > \frac{1}{2}(2\theta_2 + \theta_3 - \theta_4)\}, \\
\mathbb{D}^F_{20} & := \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_2 + \alpha_3 > \theta_2, \alpha_3 < \theta_1, \alpha_1 + \alpha_3 > \theta_4, \alpha_1 + \alpha_2 < \theta_3, \alpha_2 < \frac{1}{2}\theta_1\}, \\
\mathbb{D}^G_{20} & := \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_2 + \alpha_3 > \theta_2, \alpha_3 < \theta_1, \alpha_1 + \alpha_3 > \theta_4, \alpha_1 + \alpha_2 < \theta_3, \alpha_2 > \frac{1}{2}\theta_1\}, \\
\mathbb{D}^H_{20} & := \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_2 + \alpha_3 > \theta_2, \alpha_3 < \theta_1, \alpha_1 + \alpha_3 > \theta_4, \alpha_1 + \alpha_2 > \theta_4\}, \\
\mathbb{D}^I_{20} & := \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_3 > \theta_2, \alpha_1 + \alpha_2 < \theta_3, \alpha_1 + \alpha_3 < \theta_3\}, \\
\mathbb{D}^J_{20} & := \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_3 > \theta_2, \alpha_1 + \alpha_2 < \theta_3, \alpha_1 + \alpha_3 > \theta_4, \alpha_2 < \frac{1}{2}\theta_2\}, \\
\mathbb{D}^K_{20} & := \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_3 > \theta_2, \alpha_1 + \alpha_2 > \theta_4, \alpha_1 + \alpha_3 > \theta_4, \alpha_2 < 1 - 2\theta_2 - \theta_3\}, \\
\mathbb{D}^L_{20} & := \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{D}_{20} : \alpha_3 > \theta_2, \alpha_1 + \alpha_2 > \theta_4, \alpha_1 + \alpha_3 > \theta_4, \alpha_2 > 1 - 2\theta_2 - \theta_3\}.
\end{align*}
\]

Hence
\[
\begin{align*}
\mathbb{D}^A_{20} & \subset \{\theta_0 < \alpha_2 < \frac{1}{2}(\theta_1 - \theta_3 + \theta_4), \alpha_2 < \alpha_3 < 2\alpha_2 + \theta_3 - \theta_4, \theta_4 - 2\alpha_2 < \alpha_1 < \theta_3 - \alpha_3\}, \\
\mathbb{D}^B_{20} & \subset \{\frac{1}{2}(\theta_1 - \theta_3 + \theta_4) < \alpha_2 < \frac{1}{2}\theta_1, \alpha_2 < \alpha_3 < \theta_1 - \alpha_2, \theta_4 - 2\alpha_2 < \alpha_1 < \theta_3 - \alpha_3\}, \\
\mathbb{D}^C_{20} & \subset \{\theta_0 < \alpha_2 < \frac{1}{2}\theta_1, \alpha_2 < \alpha_3 < \theta_1 - \alpha_2, \theta_4 - 2\alpha_2 < \alpha_1 < \theta_3 - \alpha_3\}, \\
\mathbb{D}^D_{20} & \subset \{\frac{1}{2}\theta_2 < \alpha_3 < \frac{1}{2}(2\theta_2 + \theta_3 - \theta_4), \theta_2 - \alpha_3 < \alpha_2 < \alpha_3, \theta_4 - 2\alpha_2 < \alpha_1 < \theta_3 - \alpha_3\}, \\
\mathbb{D}^E_{20} & \subset \{\frac{1}{2}(2\theta_2 + \theta_3 - \theta_4) < \alpha_3 < \frac{1}{2}(\alpha_3 + \theta_4 - \theta_3), \alpha_2 < \alpha_3, \theta_4 - 2\alpha_2 < \alpha_1 < \theta_3 - \alpha_3\}, \\
\mathbb{D}^F_{20} & \subset \{\theta_0 < \alpha_2 < \frac{1}{2}\theta_1, \theta_4 - 2\alpha_2 < \alpha_1 < \frac{1}{2}(\theta_4 - \theta_3 + \theta_2) < \alpha_3 < \theta_1\}, \\
\mathbb{D}^G_{20} & \subset \{\frac{1}{2}\theta_1 < \alpha_2 < \theta_1 + \theta_3 - \theta_4, \theta_4 - \theta_1 < \alpha_1 < \theta_3 - \alpha_2, \frac{1}{2}(\theta_4 - \theta_3 + \theta_2) < \alpha_3 < \theta_1\}, \\
\mathbb{D}^H_{20} & \subset \{\frac{1}{2}\theta_2 < \alpha_3 < \theta_1, \theta_2 - \alpha_3 < \alpha_2 < \alpha_3, \theta_4 - 2\alpha_2 < \alpha_1 < \theta_3\}, \\
\mathbb{D}^I_{20} & \subset \{\frac{1}{2}\theta_2 < \alpha_3 < \theta_1, \theta_2 - \alpha_3 < \alpha_2 < \alpha_3, \theta_4 - 2\alpha_2 < \alpha_1 < \theta_3\}, \\
\mathbb{D}^J_{20} & \subset \{\theta_0 < \alpha_2 < \frac{1}{2}\theta_2, \theta_4 - 2\alpha_2 < \alpha_1 < \theta_3 - \alpha_2, \frac{1}{2}(1 - \alpha_1 - \alpha_2)\}, \\
\mathbb{D}^K_{20} & \subset \{\frac{1}{2}\theta_2 < \alpha_2 < \theta_1, \theta_4 - 2\alpha_2 < \alpha_1 < \theta_3 - \alpha_2, \frac{1}{2}(1 - \alpha_1 - \alpha_2)\}, \\
\mathbb{D}^L_{20} & \subset \{\frac{1}{2}\theta_2 < \alpha_2 < \theta_1, \theta_4 - 2\alpha_2 < \alpha_1 < \theta_3 - \alpha_2, \frac{1}{2}(1 - \alpha_1 - \alpha_2)\}, \\
\mathbb{D}^M_{20} & \subset \{\theta_0 < \alpha_2 < 1 - 2\theta_2 - \theta_3, \theta_4 - 2\alpha_2 < \alpha_1 < \theta_3, \theta_2 < \alpha_3 < \frac{1}{2}(1 - \alpha_1 - \alpha_2)\}, \\
\mathbb{D}^N_{20} & \subset \{1 - 2\theta_2 - \theta_3 < \alpha_2 < \theta_1, \theta_4 - 2\alpha_2 < \alpha_1 < 1 - 2\theta_2 - \alpha_2, \theta_2 < \alpha_3 < \frac{1}{2}(1 - \alpha_1 - \alpha_2)\}.
\end{align*}
\]
We have

$$D_{24}^A \subset D_{24}^A \cup D_{24}^B \cup D_{24}^C \cup D_{24}^D,$$

where

$$D_{24}^A := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in D_{24} : \alpha_2 + \alpha_3 > \theta_2, \alpha_3 + \alpha_4 > \theta_2\},$$

$$D_{24}^B := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in D_{24} : \alpha_2 + \alpha_3 > \theta_2, \alpha_3 + \alpha_4 < \theta_1\},$$

$$D_{24}^C := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in D_{24} : \alpha_2 + \alpha_3 < \theta_1, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 > \theta_4\},$$

$$D_{24}^D := \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in D_{24} : \alpha_2 + \alpha_3 < \theta_1, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < \theta_3\}.$$

Hence

$$D_{24}^A \subset \{\frac{1}{2} \theta_2 < \alpha_3 < \frac{1}{2} (\theta_3 - \theta_2), \alpha_3 < \alpha_2 < \theta_3 - \frac{1}{2} \theta_2, \theta_2 < \alpha_1 < \theta_3 - \theta_2, \frac{1}{2} (3 \theta_2 - \theta_3) < \alpha_4 < \alpha_3\},$$

$$D_{24}^B \subset \{\theta_2 < \alpha_1 < \theta_3 - \theta_2, \theta_2 - \theta_1 < \alpha_4 < \frac{1}{2} \theta_1, \theta_2 - \alpha_4 < \theta_2 < \theta_1, \alpha_4 < \alpha_3 < \theta_1 - \alpha_4\},$$

$$D_{24}^C \subset \{\theta_0 < \alpha_4 < \alpha_3 < \frac{1}{2} \theta_2, \frac{1}{2} \theta_2 < \alpha_2 < \theta_1 - \alpha_3, \theta_4 - \frac{1}{2} (\alpha_2 + \alpha_3) < \alpha_1 < \theta_3 - \alpha_2 - \alpha_3\},$$

$$D_{24}^D \subset \{\theta_2 - \theta_1 < \alpha_3 < \frac{1}{2} \theta_1, \frac{1}{2} \theta_2 < \alpha_2 < \theta_1 - \alpha_3, \theta_2 < \alpha_1 < \theta_3 - \theta_2, \theta_2 - \theta_1 < \alpha_4 < \alpha_3\}.$$

We have

$$D_{25} = \{(\alpha_1, \alpha_2) : \theta_2 < \alpha_2 < \frac{1}{2} \theta_3, \alpha_2 < \alpha_1 < \theta_3 - \alpha_2\}.$$

Also, we have

$$D_{27} = \{(\alpha_1, \alpha_2, \alpha_3) : \theta_2 < \alpha_2 < \frac{1}{2} \theta_3, \alpha_2 < \alpha_1 < \theta_3 - \alpha_2, \alpha_2 < \alpha_3 < \theta_3 - \alpha_1\}.$$

Finally

$$D_{29} \subset D_{29}^A \cup D_{29}^B \cup D_{29}^C \cup D_{29}^D \cup D_{29}^E \cup D_{29}^F,$$

where

$$D_{29}^A := \{(\alpha_1, \alpha_2) : \frac{1}{2} \theta_4 < \alpha_1 < \frac{1}{2} (3 \theta_4 - 1), \theta_4 - \alpha_1 < \alpha_2 < \alpha_1\},$$

$$D_{29}^B := \{(\alpha_1, \alpha_2) : \frac{1}{2} (3 \theta_4 - 1) < \alpha_1 < 1 - \theta_2, \frac{1}{2} (1 - \alpha_1) < \alpha_2 < \alpha_1\},$$

$$D_{29}^C := \{(\alpha_1, \alpha_2) : 1 - \theta_2 < \alpha_1 < \frac{1}{2}, \theta_2 < \alpha_2 < \alpha_1\},$$

$$D_{29}^D := \{(\alpha_1, \alpha_2) : \frac{1}{2} < \alpha_1 < \theta_3, \theta_2 < \alpha_2 < \frac{1}{2} (1 - \alpha_1)\},$$

$$D_{29}^E := \{(\alpha_1, \alpha_2) : \frac{1}{2} (3 \theta_4 - 1) < \alpha_1 < \theta_4 - \alpha_2, \theta_4 - \alpha_1 < \alpha_2 < \frac{1}{2} (1 - \alpha_1)\},$$

$$D_{29}^F := \{(\alpha_1, \alpha_2) : \theta_4 - \theta_2 < \alpha_1 < 1 - \theta_2, \theta_2 < \alpha_2 < \frac{1}{2} (1 - \alpha_1)\}.$$

(For the integrals on $D_{29}^E$ and $D_{29}^F$ we make use of $(t \omega(t))' = \omega(t - 1)$ and $\omega(t) \leq 0.5644$ for $t \geq 3$.)

Using these domains we obtain $F(\frac{205}{243}) > 0.0003$. Hence it suffices to choose $\gamma \geq \frac{205}{243}$; i.e. $c \leq \frac{243}{205}$. 

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