BULL. AUSTRAL. MATH. SOC. MOS 7620, 7634, 7641, 7644

VOL. I (1969), 29-46.

Surface and internal waves in a liquid of variable depth

D. G. Hurley and J. Imberger

Consider a stably stratified liquid, whose density varies exponentially with the vertical co-ordinate, that is bounded above by a free surface and below by a bed whose height depends on only one of the horizontal co-ordinates. Suppose that a gravity wave, that may be either a surface or an internal one, is travelling in a direction normal to the lines of constant depth. It is shown that if the frequency is below a certain value an infinite number of waves, all of the same frequency but having differing wave lengths, are generated and expressions for their amplitude are given in terms of the changes in depth which are assumed to be small.

1. Introduction

It is well known that it is necessary to take account of the stratification when discussing the wave motions that occur in the atmosphere and the oceans. This is so even if the changes in fluid properties associated with the stratification are small, as is the case for the oceans, because internal waves, which are known to be of importance, do not occur in unstratified fluid.

The model considered herein consists of a stably stratified liquid, whose density varies exponentially with the vertical co-ordinate and which is bounded above by a free surface and below by a bed whose height depends on only one of the horizontal co-ordinates. The assumption of an exponential

Received 21 March 1969. Received by J. Austral. Math. Soc. 21 September 1967. Communicated by G.B. Preston.

https://doi.org/10.1017/S0004972700041277 Published online by Cambridge University Press

variation of density, which is made primarily for mathematical convenience, is more nearly satisfied by the atmosphere than the ocean in which case most of the density changes occur near the thermocline.

It is supposed that a small amplitude gravity wave, which may be either a surface or an internal one, is travelling in a direction normal to the lines of constant depth and the effect of bed irregularities on the motion is considered. In the case of the atmosphere these irregularities could represent mountain ranges and in the oceanic case obstacles on the floor of the ocean. It is supposed that the motion is strictly two-dimensional so that the effects of the Coriolis force are necessarily excluded. This should not introduce serious errors unless the period of the waves is as long as about half a day.

It is found that if the frequency of the primary wave is below a certain value an infinite number of waves, all of the same frequency, but having differing wave lengths, are generated and expressions are given for their amplitudes in terms of the variations in the depth of the liquid. The results agree with those of Cox and Sandstrom [1] in the limit of small stratification which is the only case considered therein.

2. Waves in a stratified liquid of uniform depth

Consider a layer of stably stratified liquid of uniform depth h, that lies above an infinite horizontal plane surface. Let Ox^*y^* be a set of rectangular axes with O in the undisturbed surface of the liquid and Oy^* vertically up. Then if the undisturbed density is

$$\rho = \rho_0 \, e^{-\beta y^*}$$

and small two-dimensional motions are occurring in the liquid, a stream function ψ such that

$$u = -\frac{\partial \psi^*}{\partial y^*}, \quad v = \frac{\partial \psi^*}{\partial x^*}$$

may be introduced. If the time dependence of the motion is given by

$$\psi^*(x^*,y^*,t) = \psi(x^*,y^*)e^{i\omega t} \tag{1}$$

then, see for example Lamb [2], ψ must satisfy

$$\frac{\partial^2 \psi}{\partial x^2}\left(1 - \frac{B}{\Omega^2}\right) + \frac{\partial^2 \psi}{\partial y^2} - B\frac{\partial \psi}{\partial y} = 0 \tag{2}$$

and the boundary conditions,

(3) $$\frac{\partial^2\psi}{\partial x^2} + \Omega^2 \frac{\partial\psi}{\partial y} = 0 , \quad y = 0 ,$$

and

(4) $$\psi = 0 , \quad y = -1 ,$$

where

$$B = \beta h , \quad \Omega^2 = \omega^2 h/g , \quad x = x^*/h , \quad y = y^*/h .$$

The solutions of (2) that satisfy (3) and (4) and represent waves whose phase velocity has a positive component in the direction of Ox^* are

(5) $$\psi(x , y) = e^{-inx + \frac{B}{2}y} \sinh \zeta(y + 1)$$

where

(6) $$\zeta = \left\{\frac{B^2}{4} - n^2 \left[\frac{B}{\Omega^2} - 1\right]\right\}^{\frac{1}{2}}$$

and n is any positive real number such that ζ as given by (6) is a root of the equation

(7) $$\tanh \zeta = \frac{\zeta(\Omega^2 - B)}{\zeta^2 - \frac{B}{4}(2\Omega^2 - B)} .$$

It may readily be shown that there is only one value of n , n_0 say, if $\Omega > B^{\frac{1}{2}}$ but infinitely many n_s $(s = 0,1,2,\ldots)$ if $\Omega < B^{\frac{1}{2}}$. Figure 1 shows how $\lambda_s = \frac{2\pi}{n_s}$ depends on Ω . If ζ_s as given by (6) with $n = n_s$ is real then the phase velocity of the wave is in the direction of Ox^* , its wave length is λ_s , and the wave is a surface one. The figure shows that there is one such wave if

$$\Omega > \left\{\frac{B(1 + B/4)}{1 + B/2}\right\}^{\frac{1}{2}}$$

and no such wave if Ω is less than this value.

In all the other cases ζ_s is pure imaginary, equal to $i\eta_s$ say, so that (5) can be written

https://doi.org/10.1017/S0004972700041277 Published online by Cambridge University Press

$$\psi(x\ ,\ y) = \frac{e}{2}^{-in_s x + \frac{By}{2}} \left\{ e^{i\eta_s(y+1)} - e^{-i\eta_s(y+1)} \right\} .$$

Thus ψ is the sum of two waves whose phase velocities are in the directions $(n_s\ ,\ \pm\ \eta_s)$. The waves are called internal ones and the value of s , the mode number, gives the number of times ψ (and v) vanishes in $-1 < y < 0$.

The actual values given in Figure 1 refer to the case when B is small. Then (7) gives approximately

$$(8) \qquad \frac{h}{\lambda_0} = \left(\frac{\Omega}{B^{\frac{1}{2}}}\right) \frac{B^{\frac{1}{2}}}{2\pi} ,$$

$$(9) \qquad \frac{h}{\lambda_s} = \frac{s}{2\left(\frac{B}{\Omega^2} - 1\right)^{\frac{1}{2}}} \qquad s = 1,2,3,\ldots \quad .$$

The curve marked $s = 0$ in the Figure is given by (8) with $B = 0.1$ and the curves $s = 1,2,3,\ldots$ are given by (9).

3. Waves in stratified liquid of non-uniform depth

Now consider the case when there are small non-uniformities in the depth that do not extend to $x = \pm\ \infty$. It is assumed that the equation of the bottom is

$$(10) \qquad y = -1 + \varepsilon\ f(x)$$

where ε is a small parameter, that $f(x)$ and its derivative are of order unity, and that an interval $(X_1\ ,\ X_2)$ exists that contains the support of $f(x)$.

Suppose that any one of the waves given by (5) is incident on the non-uniformities from $x = -\ \infty$. Then ψ must satisfy (2) and (3), but (4) must be replaced by the condition

$$(11) \qquad \psi = 0\ ,\quad y = -1 + \varepsilon\ f(x)\ .$$

This condition may be changed into a set of conditions applied at $y = -1$ by expanding ψ in a Taylor series about $y = -1$,

https://doi.org/10.1017/S0004972700041277 Published online by Cambridge University Press

(12)
$$\psi(x\,,\,-1+\varepsilon f)=\psi(x\,,\,-1)+\varepsilon f\,\frac{\partial\psi}{\partial y}\,(x\,,\,-1)+\frac{(\varepsilon f)^2}{2!}\,\frac{\partial^2\psi}{\partial y^2}\,(x\,,\,-1)+\ldots$$

and assuming that ψ may be expressed in the form

$$\psi=\psi^{(0)}+\varepsilon\,\psi^{(1)}+\varepsilon^2\,\psi^{(2)}+\ldots \tag{13}$$

where $\psi^{(0)}$ is given by (5) with $n=n_i$, i being the mode of the incident wave. In this way the single problem of determining ψ that satisfies (2), (3) and (11) may be reduced to a series of similar problems (one for each order of ε), namely

$$\left(1-\frac{B}{\Omega^2}\right)\frac{\partial^2\psi^{(i)}}{\partial x^2}+\frac{\partial^2\psi^{(i)}}{\partial y^2}-B\,\frac{\partial\psi^{(i)}}{\partial y}=0\,, \tag{14}$$

$$\frac{\partial^2\psi^{(i)}}{\partial x^2}+\Omega^2\,\frac{\partial\psi^{(i)}}{\partial y}=0\,,\quad y=0\,, \tag{15}$$

$$\psi^{(i)}=-\,g^{(i)}\,,\quad y=-1\,, \tag{16}$$

where

(17)
$$g^{(0)}=0\,,\ g^{(1)}=f\,\frac{\partial\psi^{(0)}}{\partial y}\,(x\,,\,-1)\,,\ldots\,,$$
$$g^{(i)}=\sum_{s=1}^{i}\frac{f^s}{s!}\,\frac{\partial^s}{\partial y^s}\,\psi^{(i-s)}\,(x\,,\,-1)\,.$$

Equations (14) and (16) show that each $\psi^{(i)}$ $(i\geq 1)$ represents the motion in an infinite strip of unit depth produced by sources on the bottom of strength $2\,\frac{dg^{(i)}}{dx}\,e^{i\omega t}$. The function $g^{(i)}(x)$ depends only on $\psi^{(0)},\,\psi^{(1)},\ldots,\psi^{(i-1)}$ and this enables each of the $\psi^{(i)}$ to be determined successively. As well as satisfying (14), (15) and (16), $\psi^{(i)}$ must satisfy radiation conditions at $x=\pm\infty$. These are that $\psi^{(0)}$ is the only incident wave so that each of the $\psi^{(i)}$ $(i\geq 1)$ must represent an outgoing wave at both $x=\pm\infty$. Now it can be shown, [3], that if the phase velocity of an internal wave has a positive component in the direction of

Ox^* then so has its group velocity. Thus $\psi^{(i)}$ has an outgoing group velocity as required if its phase velocity is outgoing.

The obvious way of determining $\psi^{(i)}$ is to take a Fourier transform with respect to x . If it is assumed that $\psi^{(i)}$ and its derivatives are no more than algebraically large as $x \to \pm\infty$, then a theory based on generalised functions [4] may be used.

Thus suppose that $\psi(x\,,\,y)$ belongs to the space S' , of distributions defined by the continuous linear functional $\langle\psi\,,\,g\rangle$, where $g \in S$, the space of good functions[1]. If the Fourier transform[2] $G(w)$, of $g(x)$, is defined by

$$G(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{-iwx}\, dx \,,$$

then the generalised Fourier transform may be defined by invoking Parseval's Relation,

$$\langle\Psi\,,\,G\rangle = \langle\psi\,,\,g\rangle \,.$$

Hence a set of equations equivalent to (14), (15) and (16) is

(18) $$\frac{d^2\Psi^{(i)}}{dy^2} - B\,\frac{d\Psi^{(i)}}{dy} + \left(\frac{B}{\Omega^2} - 1\right) w^2\Psi^{(i)} = 0 \,,$$

(19) $$\Omega^2\,\frac{d\Psi^{(i)}}{dy} = w^2\Psi^{(i)} \,,\quad y = 0 \,,$$

(20) $$\Psi^{(i)} = -G^{(i)}(w) \,,\quad y = -1 \,.$$

It may readily be shown that the solution of equations (18) to (20) is[3]

---

1 A good function is one which is everywhere infinitely differentiable and such that it and all its derivatives are $O(|x|^{-N})$ as $|x| \to \infty$ for all N .

2 Capital letters will be used throughout to denote Fourier transforms.

3 The delta functions on the right hand side of (21) arise because in generalized function theory, they are solutions of the equation

$$\frac{\Psi^{(i)}(w\,,\,y)}{k(w\,,\,y)} = 0 \,.$$

$$\psi^{(i)}(w,y) = k(w,y)\, G^{(i)}(w) + \sum_{s=1}^{N} \left\{\left[C_{1s} G^{i}(n_s)\, \delta(w - n_s) + C_{2s}\, G^{(i)}(-n_s)\, \delta(w + n_s)\right] e^{B/2\, y} \sinh \zeta_s (y+1)\right\} \tag{21}$$

where

$$k(w,y) = \frac{e^{B(y+1)/2}\{[\zeta^2 - B/4(2\Omega^2-B)]\sinh \zeta y + (\Omega^2-B)\zeta\cosh \zeta y\}}{[\zeta^2 - B/4(2\Omega^2-B)]\sinh \zeta - (\Omega^2-B)\zeta\cosh \zeta}, \tag{22}$$

$$\zeta = \left\{\frac{B^2}{4} - w^2 \left(\frac{B}{\Omega^2} - 1\right)\right\}^{\frac{1}{2}} \tag{23}$$

and ζ_s are the roots of the equation

$$\tanh \zeta = \frac{\zeta(\Omega^2-B)}{\zeta^2 - B/4(2\Omega^2-B)}, \tag{24}$$

and C_{1s} and C_{2s} are constants.

Now equation (7) and (24) are the same, and (23) is obtained from (6) by replacing n^2 by w^2. Thus, in accordance with Figure 1, (24) will have only two real roots

$$w = \pm\, n_0\,, \quad if \quad \Omega^2 > B$$

but infinitely many

$$w = \pm\, n_s \qquad s = 0,1,2,\ldots, \quad if \quad \Omega^2 < B\,.$$

The asymptotic behaviour of $\psi^{(i)}(x,y)$ for large distances from the mound, may now be determined by Lighthill's techniques [4]. Thus

$$\begin{aligned}\psi^{(i)}(x,y) \sim -\sqrt{2\pi}\, i \sum_{s=0}^{N} R(n_s,y)\Bigg\{&\frac{G^{(i)}(n_s)e^{in_s x}}{2}\Bigg(-\operatorname{sgn} x + \frac{i\, C_{1s} n_s}{\pi\Omega^2\, \zeta_s{}^2}\left[2\zeta_s \sinh^2\zeta_s + (\Omega^2-B)(\zeta_s - \sinh \zeta_s \cosh \zeta_s)\right] e^{-B/2}\Bigg) \\ &+ \frac{G^{(i)}(-n_s)e^{-in_s x}}{2}\Bigg(\operatorname{sgn} x + \frac{i\, C_{2s} n_s}{\pi\Omega^2\zeta_s{}^2}\left[2\zeta_s \sinh^2\zeta_s + (\Omega^2-B)(\zeta_s - \sinh \zeta_s \cosh \zeta_s)\right] e^{-B/2}\Bigg)\Bigg\},\end{aligned} \tag{25}$$

where

$$R(n_s,y) = \frac{e^{B(y+1)/2}\,\Omega^2\zeta_s{}^2\sinh\zeta_s(y+1)}{n_s\{2\zeta_s\sinh^2\zeta_s + (\Omega^2-B)(\zeta_s - \sinh\zeta_s\cosh\zeta_s\}}\,, \tag{26}$$

and N , the upper limit of the summation in (25), is zero if $\Omega^2 > B$ and infinity if $\Omega^2 < B$

The radiation conditions are satisfied if

$$C_{1s} = C_{2s} = \frac{-\pi i\,e^{B/2}\,\Omega^2\,\zeta_s{}^2}{n_s\{2\,\zeta_s\sinh^2\zeta_s + (\Omega^2-B)(\zeta_s - \sinh\zeta_s\cosh\zeta_s)}\,, \tag{27}$$

whence

$$\psi^{(i)} \sim -i\sqrt{2\pi}\sum_{s=0}^{\infty} R(n_s,y)e^{-in_sx}\,G^{(i)}(-n_s)\,,\ x\to\infty\,,\ \Omega^2 < B \tag{28}$$

$$\sim -i\sqrt{2\pi}\sum_{s=0}^{\infty} R(n_s,y)e^{in_sx}\,G^{(i)}(n_s)\,,\ x\to-\infty\,,\ \Omega^2 < B \tag{29}$$

and

$$\psi^{(i)} \sim -i\sqrt{2\pi}\,R(n_0,y)\,e^{-in_0x}G^{(i)}(-n_0)\,,\ x\to\infty\,,\ \Omega^2 > B \tag{30}$$

$$\sim -i\sqrt{2\pi}\,R(n_0,y)\,e^{in_0x}G^{(i)}(n_0)\,,\ x\to-\infty\quad \Omega^2 > B\,.$$

In point of fact (28) is not merely asymptotic but is exact for $x > X_2$ and (29) is exact for $x < X_1$ where, as previously remarked, (X_1,X_2) is any interval that contains the support of $f(x)$ and hence of $g^{(i)}(x)$.

Results for a homogeneous liquid may be obtained by taking $B = 0$ in those equations above that hold for the case $\Omega^2 > B$.

4. Reflection and transmission coefficients

If $\Omega^2 < B$ it follows from (13), (28) and (29) that

https://doi.org/10.1017/S0004972700041277 Published online by Cambridge University Press

$$\psi = e^{-in_i x} Y_i(y) + \sum_{s=0}^{\infty} a_s e^{in_s x} Y_s(y) , \quad x < X_1 ,$$

(32)

$$= \sum_{s=0}^{\infty} b_s e^{-in_s x} Y_s(y) , \quad x > X_2 ,$$

where

(33)
$$\begin{cases} Y_s(y) = e^{By/2} \sinh \zeta_s (y+1) , \\ a_s = - \dfrac{\sqrt{2\pi}\, i\, e^{B/2}\, \Omega^2 \zeta_s{}^2 \{\varepsilon G^{(1)}(n_s) + \varepsilon^2 G^{(2)}(n_s) + \dots\}}{n_s [2 \zeta_s \sinh^2 \zeta_s + (\Omega^2 - B)(\zeta_s - \sinh \zeta_s \cosh \zeta_s)]} , \\ b_s = \delta_{is} - \dfrac{\sqrt{2\pi}\, i\, e^{B/2}\, \Omega^2 \zeta_s{}^2 \{\varepsilon G^{(1)}(-n_s) + \varepsilon^2 G^{(2)}(-n_s) + \dots\}}{n_s [2 \zeta_s \sinh^2 \zeta_s + (\Omega^2 - B)(\zeta_s - \sinh \zeta_s \cosh \zeta_s)]} . \end{cases}$$

Now if $\overline{P}(x)$ denotes the mean rate at which energy is being transferred across unit width of a plane normal to Ox , then

$$\overline{P}(x) = \frac{1}{T} \int_0^T \int_{-h}^0 p\, u\, dy^*\, dt \tag{34}$$

where T is the period of the motion and p is the pressure perturbation and is given in terms of ψ by, [2],

$$\frac{\partial p}{\partial x^*} = i\, \omega\, \rho \frac{\partial \psi}{\partial y^*} \quad \text{and} \quad \frac{\partial p}{\partial y^*} = - i\, \omega\, \rho \left\{1 - \frac{B}{\Omega^2}\right\} \frac{\partial \psi}{\partial x^*} . \tag{35}$$

It follows from (28), (29), (34) and (35) that

$$\frac{\overline{P}(x)}{\rho_0 \omega} = \frac{C_i}{2n_i} - \sum_{s=0}^{\infty} \frac{|a_s|^2 C_s}{2n_s} \qquad x < X_1 \tag{36}$$

$$= \sum_{s=0}^{\infty} \frac{|b_s|^2 C_s}{2n_s} \qquad x > X_2 \tag{37}$$

where

$$C_s = \int_{-1}^{0} \left|\frac{dY_s}{dy}\right|^2 e^{-By}\, dy$$

(38)

$$= \left|\frac{B^2}{8}\left(\frac{\sinh 2\zeta_s}{2\,\zeta_s} - 1\right) + \frac{B}{4}\left(\cosh 2\zeta_s - 1\right) + \frac{\zeta_s{}^2}{2}\left(1 + \frac{\sinh 2\zeta_s}{2\,\zeta_s}\right)\right| .$$

(36) gives the mean rate at which energy is being carried by the incident and reflected waves and (37) the mean rate for the transmitted waves. Thus if

$$R_s{}^i = \frac{\text{mean power carried by mode } s \text{ of reflected wave}}{\text{mean power carried by incident wave}}$$

and

$$T_s{}^i = \frac{\text{mean power carried by mode } s \text{ of transmitted wave}}{\text{mean power carried by incident wave}}$$

then

$$R_s{}^i = \frac{|a_s|^2\, C_s n_i}{c_i\, n_s} \tag{39}$$

and

$$T_s{}^i = \frac{|b_s|^2\, C_s n_i}{C_i\, n_s} . \tag{40}$$

If $\Omega^2 > B$ the incident, reflected and transmitted waves must all be surface ones so that there is a single reflection coefficient, R , and a single transmission coefficient, T , given by

$$R = |a_0|^2 , \qquad T = |b_0|^2 . \tag{41}$$

The expressions (33) for a_s and b_s that must be substituted into (39) and (40) to give $R_s{}^i$ and $T_s{}^i$ involve $G^{(i)}(\pm\, n_s)$, and a convenient method of calculating the functions $G^{(i)}(w)$ is as follows.

Let $f_s = \dfrac{f^s}{s!}$ so that equation (17) can be written

$$g^{(i)} = \sum_{s=1}^{i-1} f_s \frac{\partial^s}{\partial y^s} \psi^{(i-s)}(x\,,\, -1) + f_i \frac{\partial^i \psi^{(0)}}{\partial y^i}(x\,,\, -1) .$$

Then by (5) and the convolution theorem

https://doi.org/10.1017/S0004972700041277 Published online by Cambridge University Press

$$G^{(i)}(w) = \frac{1}{\sqrt{2\pi}} \sum_{s=1}^{i-1} \int_{-\infty}^{\infty} F_s(w-v) \frac{d^s\psi^{(i-s)}}{dy^s}(v,-1)dv \tag{42}$$

$$+ \frac{\partial^i}{\partial y^i}\left\{e^{By/2}\sinh \zeta_i(y+1)\right\} F_i(u+n_i) .$$

Equations (21), (27) and (42) enable $G^i(w)$ $(i = 0,1,2,\ldots)$ to be determined successively and the first few are

$$\begin{cases} G^{(0)}(w) = 0 , \\ G^{(1)}(w) = e^{-B/2} \zeta_i F_1 (w+n_i) , \\ G^{(2)}(w) = Be^{-B/2} \zeta_i F_2(w+n_i) + \dfrac{e^{-B/2}\zeta_i}{\sqrt{2\pi}} \displaystyle\int_\Gamma \frac{\partial k}{\partial y}(u,-1)F(u+n_i)F(w-u)du , \end{cases} \tag{43}$$

where $k(w,y)$ is given by (22) and the path of integration Γ is the real axis with indentations below the points $-n_s$ and above the points $+n_s$.

5. An example

As an example of the application of the previous results consider the case

$$f(x) = e^{-x^2/l^2} .$$

Then

$$F_1(w) = \frac{l}{\sqrt{2}} e^{-l^2w^2} ,$$

$$F_2(w) = \frac{l}{2} e^{-l^2w^2/2} ,$$

$$G_1(w) = \frac{le^{-B/2}\zeta_i}{\sqrt{2}} e^{-l^2(w+n_i)^2} ,$$

$$G_2(w) = le^{-B/2}\zeta_i e^{-B/2-l^2(w+n_i)^2/2} \left\{\frac{B}{4} + \frac{il\sqrt{\pi}}{2\sqrt{2}} \sum_{s=0}^{\infty} \frac{\partial R(n_s,-1)}{\partial y}\right.$$

$$\left.\left[w(a_s) - w(b_s) + 2e^{-b_s^2} - i\sqrt{\frac{2}{\pi}}\frac{1}{ln_s}\right]\right\} ,$$

https://doi.org/10.1017/S0004972700041277 Published online by Cambridge University Press

where

$$w(z) = e^{-z^2}\left\{1 + \frac{2i}{\sqrt{2\pi}}\int_0^z e^{t^2}dt\right\}$$

is the probability integral, [5],

$$a_s = \frac{l}{\sqrt{2}}(n_i + 2n_s - w), \qquad b_s = \frac{l}{\sqrt{2}}(n_i - 2n_s - w) ,$$

and $R(n_s , y)$ is defined by (26).

6. Approximate results for small stratification

If B is small and if

$$\Omega^2 < \frac{B(1 + B/4)}{1 + B/2}$$

so that all the possible modes are internal ones (Figure 1), then, to the first order in B , the following simplified results are obtained:

(44)
$$\begin{aligned} \eta_0 &= (B - \Omega^2)^{\frac{1}{2}} , \\ n_0 &= \Omega , \\ C_0 &= B - \Omega^2 , \\ \eta_s &= s\,\pi \\ n_s &= \frac{s\,\pi\,\Omega}{(B - \Omega^2)^{\frac{1}{2}}} \\ C_s &= \frac{s^2\pi^2}{2} \end{aligned} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\}, \quad s \geqq 1 .$$

If these values are used to calculate $R_s{}^i$ and $T_s{}^i$ and only the leading terms in ε are retained then it is found that:

(a) if the incident wave is the zeroth mode $(i = 0)$

(45)
$$R_0{}^0 = \frac{\pi\ \varepsilon^2\Omega^2F^2(2n_0)}{2}, \qquad T_0{}^0 = 1 + \frac{\pi\ \varepsilon^2\Omega^2F^2(0)}{2} ,$$

$$R_s{}^0 = \frac{\pi^2\varepsilon^2 s\Omega^2F^2(n_s + n_0)}{(B - \Omega^2)^{\frac{1}{2}}}, \quad T_s{}^0 = \frac{\pi^2\varepsilon^2 s\Omega^2F^2(-n_s + n_0)}{(B - \Omega^2)^{\frac{1}{2}}}, \ s \neq 0 ,$$

https://doi.org/10.1017/S0004972700041277 Published online by Cambridge University Press

(b) if the incident wave is not the zeroth mode $(i \geqq 1)$

$$R_0^{\,i} = \frac{\pi^2\varepsilon^2 i\Omega^2 F^2(n_0 + n_i)}{(B - \Omega^2)^{\frac{1}{2}}}, \qquad T_0^{\,i} = \frac{\pi^2\varepsilon^2 i\Omega^2 F^2(-n_0 + n_i)}{(B - \Omega^2)^{\frac{1}{2}}},$$

(46)

$$R_s^{\,i} = \frac{2\pi^3\varepsilon^2 is\Omega^2 F^2(n_s + n_i)}{B - \Omega^2}, \quad T_s^{\,i} = \frac{2\pi^3\varepsilon^2 is\Omega^2 F^2(-n_s + n_i)}{B - \Omega^2}, \; s \neq i,$$

$$= 1 + \frac{2\pi^3\varepsilon^2 i^2\Omega^2 F^2(0)}{B - \Omega^2}, \quad s = i.$$

Nearly all the reflection and transmission coefficients given by (45) and (46) tend to infinity as $\Omega \to B^{\frac{1}{2}}$ so that the analysis must fail for Ω near $B^{\frac{1}{2}}$.

7. Discussion

According to the present analysis the wave system consists of the incident wave and the waves produced by sources distributed along the line $y = -1$ having the same frequency as the incident wave. If $\Omega^2 < B$ the sources produce an infinite number of waves having the lengths shown in Figure 1. The strength of the sources is such that they induce at the lower boundary a normal velocity equal and opposite to that due to the incident wave.

If only the terms that are of the first order is ε are retained this strength is, by (5) and (17),

$$\rho_1(x) = 2\,\zeta_i\, e^{-in_i x - B/2}\{f'(x) - in_i\}.$$

$\rho_1(x)$ depends only on the *incident* wave so that the first order theory only takes account of one-time reflections. The effects of multiple reflections, and of the changes in the wave length of the waves as they traverse the region of variable depth will appear in higher terms.

It is instructive to consider the case

$$f(x) = H(x - l) - H(x + l)$$

which represents a low rectangular sill. Then

$$\rho_1(x) = 2\,\zeta_i e^{-in_i x - B/2}\{\delta(x - l) - \delta(x + l) - in_i\},$$

https://doi.org/10.1017/S0004972700041277 Published online by Cambridge University Press

and

$$G^{(1)}(\omega) = e^{-B/2}\zeta_i F(\omega + n_i) = \sqrt{\frac{2}{\pi}}\,\frac{e^{-B/2}\zeta_i \sin l(n_i + \omega)}{n_i} .$$

This expression for $G^{(1)}$ together with equations (28) and (29) show that the reflected and transmitted waves have a sinusoidal dependence on l (and $(n_i \pm n_r)$), the reflected wave containing the factor $\sin l(n_i + n_r)$ and the transmitted wave the factor $\sin l(n_i - n_r)$. The occurence of these factors is consistent with the following simple physical considerations. $\rho_1(x)$ includes a point source at $x = -l$ and an equal sink at $x = + l$. During the passage of the incident wave from $x = -l$ to $x = + l$ its phase changes by $2l \times \frac{2\pi}{\lambda_i}$ and during the passage of the reflected wave from $x = l$ to $x = - l$ its phase changes by $\frac{4\pi l}{\lambda_r}$. The reflected waves from the sources will reinforce if the sum of these phase changes is π so that the factor $\sin\{l(n_i + n_r)\}$ is to be expected in the reflected wave. Similarly the factor $\sin\{l(n_i - n_r)\}$ is to be expected in the transmitted wave.

In the case of a general topography, the strengths and phases of the source distribution depend on the wave length of the incident waves. The topography may be such that the phases change with the wave length in such a way that the incremental waves always add destructively to give a much lower reflection coefficient than would be expected. This behaviour is especially noted in certain smooth depth changes. Schelkunoff [6] has called this phenomena "a better matching of incremental waves".

The above process is well illustrated by the mound

$$f(x) = \frac{1 + d}{\cosh \frac{\pi x}{\theta} + d}$$

where d and θ are real positive constants. This gives

$$G^{(1)}(\omega) \propto \frac{\sinh \Delta\omega}{\sinh \theta\omega}$$

where

https://doi.org/10.1017/S0004972700041277 Published online by Cambridge University Press

$$d = \frac{\cos \pi\Delta}{\theta} .$$

If d is small, the mound is gentle and as d is increased it approaches a rectangular sill. If $d < 1$, the radiations add destructively and the additions to the incident wave are slowly varying functions of the wave length. However, when $d > 1$, Δ becomes imaginary and a rapidly varying dependence on the wave length is obtained.

If λ_i is small compared to the depth, n_i is large and equation (17) shows that then $g^{(i)}$ is much larger than $g^{(i-1)}$. Thus the perturbation expansion will not converge and the analysis ceases to be valid.

Another important parameter is the ratio of the length of the mound to the wave length. In §5 results correct to the second order in ε are given for the case

$$f(x) = e^{-x^2/l^2} .$$

Here l is the horizontal length scale of the mound and it is noted that it only occurs in the results in the combination ln where n is a wave number. The magnitude of this parameter determines whether the reflections occuring are weak or strong as for the case when the liquid is homogeneous. $G^{(2)}(w)$ is seen to be of order $e^{-l^2w^2/2}$ and $G^{(1)}(w)$ of order $e^{-l^2w^2}$. If lw is of order unity, then $G^{(2)}(w)$ will be considerably larger than $G^{(1)}(w)$ and the results of the first order theory in ε will not hold. Equation (17) shows that the r^{th} order source distribution is a linear combination of terms of the type $f^j \times$ (function of w and y), $(j = 1,2,\dots,r)$ which suggests that for the above example $G^{(r)}(w)$ will contain terms of the type $e^{-l^2w^2/j} \times$ (function of x and y). When coupled with the r^{th} power of the expansion parameter, ε, these results suggest that as ln increases, the overall reflection will decrease and the main contribution will no longer come from the lowest order terms. The physical implication is that multiple reflections become more and more important as the process moves to weak reflection.

The expression for $G_1(w)$ in §5 in conjunction with (28) and (29) shows that most of the energy in the reflected waves will be carried by the surface or lowest mode whilst the energy in the transmitted wave will be mostly carried by the incident mode and its closest neighbours.

When the present investigation was well advanced the authors attention was drawn to the work of Cox and Sandstrom [1] which appears to be the only other investigation of the effect of depth variation on gravity waves in a continuously stratified fluid. Their analysis, however, only deals with the case of small stratification and only the first order effects of changes in the depth are considered. They obtain results that are equivalent to those given by equations (44), (45) and (46).

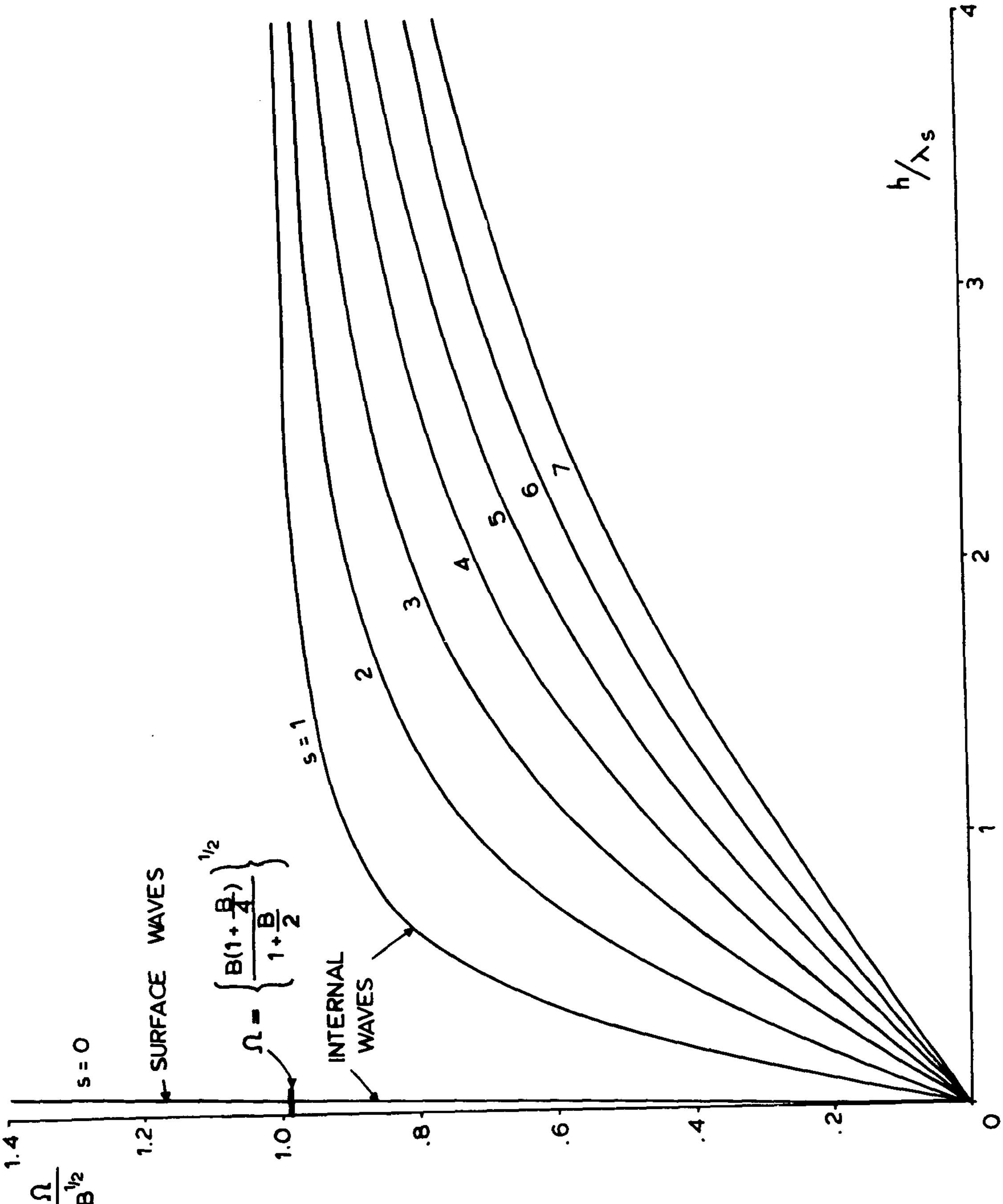

Figure 1

Horizontal wave lengths of Surface and Internal Waves in Liquid of Uniform Depth.

https://doi.org/10.1017/S0004972700041277 Published online by Cambridge University Press

References

[1] C.S. Cox and H. Sandstrom, "Coupling of internal and surface waves in water of variable depth", *J. Ocean. Soc. Japan*, 20th Annivers. Vol. (1962), 499-513.

[2] Horace Lamb, *Hydrodynamics*, (Cambridge University Press, Sixth Edition, 1932).

[3] D.E. Mowbray and B.S.H. Rarity, "A theoretical and experimental investigation of the phase configuration of internal waves of small amplitude in a density stratified liquid", *J. Fluid Mech.* 28 (1967), 1-16.

[4] M.J. Lighthill, *Fourier analysis and generalized functions*, (Cambridge University Press, 1962).

[5] V.N. Faddeyeva and N.M. Terent'ev, *Tables of the probability integral for complex argument* (Pergamon, Oxford, 1961).

[6] S.A. Schelkunoff, "Remarks concerning wave propogation in stratified media", *Comm. Pure Appl. Math.* 4 (1951), 117-128.

Department of Mathematics,
University of Western Australia,
Nedlands. Western Australia.