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HYPERTYPES OF TORSION-FREE ABELIAN GROUPS OF FINITE RANK

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Let G be a torsion-free abelian group of finite rank n and let F be a full free subgroup of G. Then G/F is isomorphic to $T_1 \oplus \ldots \oplus T_n$, where $T_1 \subseteq T_2 \subseteq \ldots \subseteq T_n \subseteq \mathbb{Q}/\mathbb{Z}$. It is well known that type $T_1 =$ inner type G and type $T_n =$ outer type G. In this note we give two characterisations of type T_i for $1 < i < n$.

In 1963 Fuchs [2] introduced the notions of the inner type (IT) and outer type (OT) of a torsion-free abelian group G of finite rank n. He showed that if F is a full free subgroup of G and $G/F = T_1 \oplus \ldots \oplus T_n$ where the T_i are subgroups of $\mathbb{Q}/\mathbb{Z}$ with $T_1 \subseteq T_2 \subseteq \ldots \subseteq T_n$, then $IT(G) = \text{type}(T_1)$ and $OT(G) = \text{type}(T_n)$. In this note we generalise the result of Warfield by characterising type(T_i), the ith hypertype of G, for $1 \leqslant i \leqslant n$.

Fundamental references are [**1**, **2**, **4**] and [**5**]. In particular, the reader is assumed to be familiar with the basic properties of height and type in torsion-free abelian groups, and with the notions of inner and outer type. We also assume familiarity with quasi-isomorphism concepts.

If A and B are groups we write $A \leqslant B$ to denote that A is isomorphic to a subgroup of B. For an integral prime p, A_p denotes the usual localisation of A at p. If G is a torsion-free abelian group and S is a subset of G, then $\langle S \rangle_*$ is the pure subgroup generated by S. If $x \in G$, then $h_p^G(x)$ is the p-height of x computed in G. If G has rank n and $0 \leqslant i \leqslant n$, we define $\underline{P_i(G)} = \{X \mid X \text{ is a pure subgroup of } G \text{ of rank } i\}$. If $T \leqslant \mathbb{Q}/\mathbb{Z}$, then type $T =$ type X, where $\mathbb{Z} \subseteq X \subseteq \mathbb{Q}$ and $T \cong X/\mathbb{Z}$.

If $T \leqslant (\mathbb{Q}/\mathbb{Z})^n$, then it is easy to see that T can be written as $T = \oplus_{i=1}^n T_i$ where $T_1 \leqslant T_2 \leqslant \ldots \leqslant T_n \leqslant \mathbb{Q}/\mathbb{Z}$. We say such a direct sum is a *standard decomposition.* Thus, with each $T \leqslant (\mathbb{Q}/\mathbb{Z})^n$ we can associate a set of types, typeset $T = \{\text{type } T_1, \ldots, \text{type } T_n\}$, where $T = \oplus T_i$ is a standard decomposition. It is easy to check that typeset T is a complete set of quasi-isomorphism invariants for subgroups T of $(\mathbb{Q}/\mathbb{Z})^n$.

Let G be a torsion-free abelian group of rank n - henceforth simply called a "group". The *Richman type* of G, $RT(G)$, is the quasi-isomorphism class of the torsion

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group G/F, where F is any full free subgroup of G. The quasi-isomorphism class of G/F, denoted $[G/F]$, is independent of the choice of F so that $RT(G)$ is an invariant of G([3]). Furthermore, since $G/F \leqslant (Q/Z)^n$, our earlier remarks imply that $RT(G)$ is determined by the set of types typeset(G/F). If $G/F = \oplus_{i=1}^n T_i$ is a standard decomposition, we call $HT_i(G) =$ type T_i the *ith hypertype* of G. As mentioned above, $HT_1(G) = IT(G)$ and $HT_n(G) = OT(G)$.

The properties of $HT_2(G)$ will be investigated in a forthcoming paper [**3**]. Our main result here, which leads to the desired characterisations of hypertypes, displays the relationship between successive hypertypes.

THEOREM 1. *Let G be a group of rank $n > 1$. Then for $2 \leqslant i \leqslant n$,*

$$HT_i(G) = \sup\{HT_{i-1}(G/X) \mid X \in P_1(G)\}.$$

PROOF: Let $\{x_1, \ldots, x_n\}$ be a maximal rationally independent subset (hereafter, basis) of G, and $F = \oplus_{i=1}^n Zx_i$, a full free subgroup of G. Let $G/F = \oplus_{i=1}^n T_i$ be a standard decomposition of G/F. For each $X \in P_1(G)$, $(F+X)/X$ is a full free subgroup of G/X. Write $(G/X)/((F+X)/X) = S_1 \oplus \ldots \oplus S_{n-1}$ as a standard decomposition. Since $\oplus_{i=1}^{n-1} S_i$ is a homomorphic image of $\oplus_{i=1}^n T_i$ it follows that $\left|(T_i)_p\right| \geqslant \left|(S_{i-1})_p\right|$ for all primes p and $2 \leqslant i \leqslant n$. Since $HT_i(G) =$ type T_i and $HT_{i-1}(G/X) =$ type S_{i-1} we have $HT_i(G) \geqslant HT_{i-1}(G/X)$ for all $X \in P_1(G)$.

For a fixed prime p, choose an element x_i of minimal p-height among the basis elements, and let $X(p) = \langle x_i \rangle_*$. Note that if $e_p = p\text{-height}(x_i) = \min\{p\text{-height}(x_j) \mid 1 \leqslant j \leqslant n\}$, then $T_1 = \oplus_p Z(p^{e_p})$. Furthermore, for each p, the minimality of e_p implies that $(X(p)+F)_p/F_p$ is a pure subgroup of G_p/F_p and hence, a summand. Therefore, $G_p/(X(p)+F)_p \simeq (T_2 \oplus \ldots \oplus T_n)_p$, so that $HT_i(G) \leq \sup\{HT_{i_1}(G/\langle x_j \rangle *) \mid 1 \leqslant j \leqslant n\}$. Thus, $HT_i(G) \leqslant \sup\{HT_{i-1}(G/X) \mid X \in P_1(G)\}$ and the theorem follows. ∎

Remark. Using induction, we may extend the results in the proof of Theorem 1 to show that for each prime p, and $1 \leqslant i \leqslant n$, there is a subset I of $\{1, 2, \ldots, n\}$ of cardinality i, such that $Y = Y_I \equiv \langle X_j \mid j \in I \rangle_*$ satisfies $Y_p/(Y \cap F)_p \simeq (T_1 \oplus \ldots \oplus T_i)_p$ and $G_p/(Y+F)_p \simeq (T_{i+1} \oplus \cdots \oplus T_n)_p$.

COROLLARY 1. *Suppose there exists $x \in G$ such that type $x = IT(G)$. Then, if $X = \langle x \rangle_*$, $RT(G) = [\oplus_{i=1}^n T_i]$ implies that $RT(G/X) = [\oplus_{i=2}^n T_i]$.*

PROOF: Choose a basis $\{x = x_1, \ldots, x_n\}$ of G such that $h_p^G(x) \leqslant h_p^G(x_i)$ for all p and $2 \leqslant i \leqslant n$. Then in the notation of Theorem 1, $X(p) = \langle x \rangle_*$ for all p, and the argument in the second half of the proof of Theorem 1 shows that $(T_i)_p \cong (S_{i-1})_p$ for all primes p and $2 \leqslant i \leqslant n$, where $G/(X+F) = S_1 \oplus \ldots \oplus S_n$ is a standard decomposition. Thus $RT(G/X) = [\oplus_{i=1}^{n-1} S_i] = [\oplus_{i=2}^n T_i]$. ∎

Note that the hypothesis of Corollary 1 holds if G is homogeneous. This corollary is proved for homogeneous groups in [**3**] using different techniques.

COROLLARY 2. *Let G be a group of rank n. Then, for $1 \leqslant i \leqslant n$, $HT_i(G) = \sup\{IT(G/X) \mid X \in P_{i-1}(G)\}$.*

PROOF: The proof is by induction on i. For $i = 1$, $HT_1(G) = IT(G) = IT(G/(0))$, and the result is true. Assume $i > 1$ and that the result holds for $i-1$. By Theorem 1 and the induction hypothesis, $HT_i(G) = \sup\{HT_{i-1}(G/X) \mid X \in P_1(G)\} = \sup\{\sup\{IT((G/X)/Y') \mid Y' \in P_{i-2}(G/X)\} \mid X \in P_1(G)\}$. However, each $Y' \in P_{i-2}(G/X)$ is of the form $Y' = Y/X$ for a unique $Y \in P_{i-1}(G)$. Thus, $HT_i(G) = \sup\{\sup\{IT(G/Y) \mid X \subset Y \in P_{i-1}(G)\} \mid X \in P_1(G)\} = \sup\{IT(G/Y) \mid Y \in P_{i-1}(G)\}$, as required. ∎

Note that for $i = n$ we have $OT(G) = HT_n(G) = \sup\{IT(G/X) \mid X \in P_{n-1}(G)\} = \sup\{\text{type}(G/X) \mid X \in P_{n-1}(G)\}$, the standard definition of the outer type of G.

Arguing as in the proof of Corollary 2, it is easy to show:

COROLLARY 3. *Let G be a group of rank n and $1 \leqslant j < i \leqslant n$. Then $HT_i(G) = \sup\{HT_{i-j}(G/X) \mid X \in P_j(G)\}$.*

If τ is a type, a group G is called a *hyper-τ group* if every proper homomorphic image of G is τ-homogeneous completely decomposable. These groups are investigated in [**3**]. One of the results in that paper may be generalised as follows (see [**3**], Theorem 3.1).

COROLLARY 4. *Let G be a hyper-τ group. Then $RT(G) = [S \oplus T^{n-1}]$, where type $S = IT(G)$ and type $T = \tau$.*

PROOF: If G is a hyper-τ group then, clearly, $HT_n(G) = OT(G) = \tau$. But by Theorem 1, $HT_2(G) = \sup\{IT(G/X) \mid X \in P_1(G)\} = \tau$. Thus, $HT_i(G) = \tau$ for $i = 2, \ldots, n$ and the result follows. ∎

THEOREM 2. *Let G be a torsion-free group of rank n. Then for $1 \leqslant i \leqslant n$, $HT_i(G) = \inf\{OT(X) \mid X \in P_i(G)\}$.*

PROOF: Fix Y in $P_i(G)$. If F is a full free subgroup of G, then $F \cap Y$ and $(F+Y)/Y$ are full free subgroups of Y and G/Y, respectively. Write $G/F = T_1 \oplus \ldots \oplus T_n$, $Y/(F \cap Y) = U_1 \oplus \ldots \oplus U_i$ and $(G/Y)/((F+Y)/Y) = S_1 \oplus \ldots \oplus S_{n-i}$ as standard decompositions. Then there is an exact sequence,

$$0 \to \oplus_{j=1}^{i} U_j \to \oplus_{j=1}^{n} T_j \to \oplus_{j=1}^{n-i} S_j \to 0.$$

It is routine to show that type $U_j \geqslant$ type T_j for $1 \leqslant j \leqslant i$. Since $OT(Y) =$ type U_i and $HT_i(G) =$ type T_i, we have shown $OT(Y) \geqslant HT_i(G)$ for all $Y \in P_i(G)$. However, by the remark following the proof of Theorem 1, for each prime p, there is a subset I of $\{1, 2, \ldots, n\}$ of cardinality i such that $Y = Y_I = \langle x_j \mid j \in I \rangle_* \in P_i(G)$ satisfies $Y_p/(Y \cap F)_p \simeq (T_1 \oplus \ldots \oplus T_i)_p$. Thus, $HT_i(G) =$ type $T_i \leq \inf\{OT(Y_I) \mid I$ is a subset of $\{1, \ldots, n\}$ of cardinality $i\}$, and the proof is complete. ∎

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