ON CHARACTERIZATIONS OF CONDITIONAL EXPECTATION

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In the following $(\Omega, \alpha, \mu)$ is a totally $\sigma$-finite measure space except where noted. For a sub-$\sigma$-algebra $\beta \subset \alpha$, the conditional expectation $E(f \mid \beta)$ of $f$ given $\beta$ is a function measurable relative to $\beta$, such that

$$\int_B E(f \mid \beta) \, d\mu = \int_B f \, d\mu, \quad \text{all } B \in \beta.$$

In [5] R. G. Douglas proved, among other things the following, in the finite case:

**Theorem 1.** Suppose $\mu(\Omega) = 1$. Then a linear operator $T$ on $L^1(\Omega, \alpha, \mu)$ is a conditional expectation if and only if

1. $\|T\| \leq 1$
2. $T^2 = T$
3. $T1 = 1$.

The point of this note is to characterize conditional expectation in the $\sigma$-finite case (Theorems 2, 3).

As will be shown below we reduce Theorem 2 to Theorem 1. However to prove Theorem 3 we make use of the identification of the limit of the Chacon-Ornstein ergodic theorem ([1], [2], [4], [6]). Finally we prove Theorem 1 as a Corollary to Theorem 3.

**Theorem 2.** A linear operator $T$ on $L^1(\Omega, \alpha, \mu)$ is a conditional expectation relative to some $\sigma$-finite sub-$\sigma$-algebra $\beta \subset \alpha$ if and only if:

1. $\|T\| \leq 1$
2. $T^2 = T$
3. $T \geq 0$ i.e. $Tf \geq 0$ if $f \geq 0$
4. There is $g \in L^1(\Omega, \alpha, \mu)$ such that $Tg \geq 0$ almost everywhere and $T(Tg \cdot Tg) = Tg \cdot Tg$. 

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Theorem 3. A linear operator $T$ on $L_1(\Omega, \alpha, \mu)$ is a conditional expectation relative to some $\sigma$-finite sub-$\sigma$-algebra $\beta \subseteq \alpha$ if and only if:

1. $\|T\| \leq 1$
2. $T^2 = T$
3. $T \geq 0$ i.e. $Tf \geq 0$ if $f \geq 0$
4. There is $g \in L_1(\Omega, \alpha, \mu)$ such that $Tg > 0$ almost everywhere and $T^*Tg = Tg$.

Here $T^*$ denotes the adjoint of $T$ i.e.

$$\int Tf \cdot h \, d\mu = \int f \cdot T^*h \, d\mu, \quad f \in L_1(\Omega, \alpha, \mu), \quad h \in L_\infty(\Omega, \alpha, \mu).$$

We give the if parts of the proofs only. The only if parts are rather trivial.

Proof of Theorem 2. Let $\nu(A) = \int_A Tg \cdot d\mu$. For $f \in L_1(\Omega, \alpha, \nu)$ we define

$$Pf = \frac{T(f \cdot Tg)}{Tg}.$$ 

Clearly

1. $\|P\| \leq 1$
2. $P^2 = P$
3. $P1 = 1$

Thus by Theorem 1, there exists $\beta$,

$$Pf = E(f \mid \beta) \cdots (\nu).$$

Now let $f \in L_1(\Omega, \alpha, \mu)$, then by substitution $(f/Tg) \in L_1(\Omega, \alpha, \nu)$. Hence

$$P\left(\frac{f}{Tg}\right) = E\left(\frac{f}{Tg} \mid \beta\right) \cdots (\nu)$$

and consequently

$$\frac{Tf}{Tg} = E\left(\frac{f}{Tg} \mid \beta\right) \cdots (\nu)$$

But $Tg$ is a $\beta$-measurable function by virtue of assumption (2.4) of the hypothesis. Therefore $Tf$ is $\beta$-measurable. This together with the above equation, and definition of $\nu$ implies

$$Tf = E(f \mid \beta) \cdots (\mu).$$

Finally $\sigma$-finiteness of $\beta$ follows from the positivity of $Tg$. 

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Proof of Theorem 3. By $B([1], [2])$, ([4, pp. 26–29] and [6, pp. 194–211]) $T$ is a conservative operator, and satisfies:

$$
\lim_{N \to \infty} \frac{\sum_{n=0}^{N} T^{n} f}{\sum_{n=0}^{N} T^{n+1} g} = \frac{Tf}{Tg} = \frac{E\{f \mid \beta\}}{E\{Tg \mid \beta\}}.
$$

Where $\beta$ is the $\sigma$-finite, $\sigma$-algebra of sets invariant under $T^*$. Since $Tg$ is $\beta$-measurable by assumption (3.4) of the hypothesis, $Tf = E\{f \mid \beta\}$.

Proof of Theorem 1. Trivially $\|T\| = 1$, $T\ast 1 = 1$, and $T 1$ satisfies assumption (3.4) of the hypothesis of Theorem 3. We shall show that $T \geq 0$.

By [3], it is easy to show that there is a linear operator $|T|$ the modulus of $T$ satisfying:

(a) $\| |T| \| \leq 1$

(b) $|Tf| \leq |T| \|f\| \in L_1(\Omega, \alpha, \mu)$

(c) $|T|f(\omega) = \sup_{|\theta| \leq f} |Tg| (\omega)$, where $f \geq 0$,

$|T| 1 = 1$ follows from (a) and (c), which together with (b) imply that $|T| = T$ i.e. $T \geq 0$. Theorem 3 completes the proof.

References


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