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LINEAR TRANSFORMATIONS ON SYMMETRIC SPACES II

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ABSTRACT Let U be a finite dimensional vector space over an infinite field F Let $U^{(r)}$ denote the r-th symmetric product space over U Let T $U^{(r)} \longrightarrow U^{(s)}$ be a linear transformation which sends nonzero decomposable elements to nonzero decomposable elements Let dim $U \geq s+1$ Then we obtain the structure of T for the following cases (i) F is algebraically closed, (ii) F is the real field, and (iii) T is injective

1. **Introduction.** Let U be a finite dimensional vector space over an infinite field F. For each positive integer $r \geq 2$, let $U^{(r)}$ denote the r-th symmetric product space over U. A linear mapping T from $U^{(r)}$ to $U^{(s)}$ is called a *decomposable mapping* if it maps nonzero decomposable elements to nonzero decomposable elements. If dim $U \geq s+1$, we obtain the structure of T when F is algebraically closed or the real field or T is injective. When $r = s$ and F is algebraically closed of characteristic either 0 or exceeding r, the structure of T was studied in [3, 6, 8].

A subspace of $U^{(r)}$ is called a *decomposable subspace* if it consists entirely of decomposable elements. A decomposable subspace is called *maximal* if it is not contained in any other decomposable subspaces. We first determine the form of an infinite family of certain maximal decomposable subspaces such that any two of them have a non-zero intersection and then use these results to determine the structure of decomposable mappings.

Throughout this paper we assume that dim $U \geq 2$.

2. **Maximal decomposable subspaces.** For any r vectors $x_1, \ldots, x_r$ in U, we shall use $x_1 \cdots x_r$ to denote a decomposable element of $U^{(r)}$. For convenience we call $x_1 \cdots x_r$ a *decomposable element* of length r, any vector in U a decomposable element of length 1 and any scalar in F a decomposable element of length 0. Let x and y be nonzero decomposable elements of length r and s respectively where $r \geq s$. Then y is called a *factor* of x if $x = y \cdot z$ for some decomposable element of length $r - s$.

Let x be a nonzero decomposable element of length $r-1$. Then the set $\{x \cdot u : u \in U\}$, denoted by $x \cdot U$, is a decomposable subspace and is called a *type 1 subspace* of $U^{(r)}$. Two distinct type 1 subspaces $x \cdot U$ and $y \cdot U$ in $U^{(r)}$ are called *adjacent* if x and y have a common factor of length $r-2$.

A *k-field* is a field over which every polynomial of degree less than or equal to k splits completely. Let W be a 2-dimensional subspace of U. Then it is shown in [2, Proposition 10] that $W^{(r)}$ is a decomposable subspace if and only if F is an r-field. The decomposable subspace $W^{(r)}$ is called a *type r subspace* of $U^{(r)}$. Let $y_1, \ldots, y_{r-k}$ be vectors in $U - W$,

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$1 < k < r$, then the subspace $\{y_1 \cdots y_{r-k} \cdot z : z \in W^{(k)}\}$ is denoted by $y_1 \cdots y_{r-k} \cdot W^{(k)}$. If F is a k-field, then $y_1 \cdots y_{r-k} \cdot W^{(k)}$ is a decomposable subspace and is called a *type k subspace* of $U^{(r)}$. If F is an r-field with char $F = 0$ or char $F > r$ then every maximal decomposable subspace of $U^{(r)}$ is of type i for some $1 \leq i \leq r$ when dim $U \geq 3$ (see [2]).

Let $u \in U$. Then u^r denotes the decomposable element $\overbrace{u \cdots u}^{r \text{ times}}$ in $U^{(r)}$. Proposition 9 in [2] can be improved as follows:

PROPOSITION 1. *$\{u^r : u \in U\}$ is a decomposable subspace of $U^{(r)}$ if and only if F is a perfect field of characteristic p and $r = p^t$ for some positive integer t.*

PROOF. Suppose $\{u^r : u \in U\}$ is decomposable. Then Proposition 9 in [2] implies that char $F = p > 0$ and $r = p^t$ for some t. Let $\lambda \in F - \{0\}$, $u \in U - \{0\}$. Then $\lambda u^r = v^r$ for some $v \in U$, $v \neq 0$. Hence $v = au$ for some nonzero a in F. Hence $a^r = \lambda$. This shows that F is a perfect field. The sufficiency follows from the fact that $\lambda u^r = (au)^r$, $a^r = \lambda$ and $u_1^r + u_2^r = (u_1 + u_2)^r$ (see [2]).

Suppose F is a perfect field of characteristic $p > 0$ and $r \geq p^t$ for some positive integer t. Let x be a nonzero decomposable element of length $r - p^t$ over U. Then the decomposable subspace $\{x \cdot u^{p^t} : u \in U\}$ is denoted by $x \cdot U^{p^t}$ and is called a *power type subspace of degree t*.

The following result is obtained by modifying the proof of the theorem in [7]:

PROPOSITION 2. *Let M be a maximal decomposable subspace of $U^{(r)}$ over the infinite field F. Then one of the following holds:*

(i) M is a type 1 subspace;

(ii) $M \subseteq x \cdot W^{(k)}$ for some 2-dimensional subspace W of U and some nonzero decomposable element x of length $r - k$ where $1 < k \leq r$;

(iii) M is a power type subspace and F is a perfect field of prime characteristic $p \leq r$.

We remark that equality holds in (ii) only if F is a k-field.

3. **Intersections of maximal decomposable subspaces.** Throughout this section U will denote a finite dimensional vector space over an infinite perfect field F of characteristic $p > 0$. We study the intersection properties of maximal decomposable subspaces and determine the form of an infinite family of maximal decomposable subspaces of type 1 or power type such that any two members of the family have a nonzero intersection. These results will be used in Section 4.

LEMMA 1. *Two power type decomposable subspaces $M = x \cdot U^{p^t}$ and $N = y \cdot U^{p^t}$ of $U^{(r)}$, $r > p^t$, are equal if and only if $x = \lambda y$ for some $\lambda \in F$.*

PROOF. The sufficiency is clear. We prove the necessity. Choose a vector $z \notin \langle y_1 \rangle \cup \cdots \cup \langle y_{r-k} \rangle$ where $y = y_1 \cdots y_{r-k}$, $k = p^t$. Then $M = N$ implies that

$$x \cdot z^k = y \cdot w^k$$

for some $w \in U$. Since $\langle z \rangle \neq \langle y_i \rangle$ for all i, it follows that $\langle z \rangle = \langle w \rangle$. Therefore $x = \lambda y$ for some $\lambda \in F$.

LEMMA 2. *Let $M = x \cdot U^k$, $k = p^t$ and $N = y \cdot U^m$, $m = p^\ell$, be two distinct decomposable subspaces of $U^{(r)}$ where $t \geq \ell \geq 0$ and $r = p^t$. Then* $\dim(M \cap N) = 1$ *if and only if either*

(i) $t > \ell$ and $y = \lambda x \cdot f^{k-m}$ for some $f \in U$, $\lambda \in F$ or

(ii) $r \geq m + k$ and $x = z \cdot a^m$, $y = z \cdot f^k$ for some $a, f \in U$ and nonzero decomposable element z of length $r - k - m$.

Otherwise $M \cap N = 0$.

PROOF. Suppose that $M \cap N \neq 0$. Then there are nonzero vectors f and $a \in U$ such that $x \cdot f^k = y \cdot a^m \neq 0$. Either $\langle f \rangle = \langle a \rangle$ or $\langle f \rangle \neq \langle a \rangle$. If $\langle f \rangle = \langle a \rangle$, then $t \neq \ell$, otherwise $M = N$, a contradiction. Hence $t > \ell$ and $y = \lambda x \cdot f^{k-m}$ for some $\lambda \in F$. If $\langle f \rangle \neq \langle a \rangle$, then clearly $r \geq k + m$ and $x = z \cdot a^m$, $y = z \cdot f^k$ for some nonzero decomposable element z of length $r - k - m$.

Conversely, if (i) holds, then $M \cap N = \langle x \cdot f^k \rangle$ and if (ii) holds, then $M \cap N = \langle z \cdot a^m \cdot f^k \rangle$.

Let $z \cdot U^{p^t}$ be a decomposable subspace of $U^{(r)}$ where $r > p^t$, $t \geq 0$. Then every factor of z is also called a *factor* of $z \cdot U^{p^t}$.

Let $\mathcal{P}_0(U^{(r)})$ denote the collection of all type 1 subspaces of $U^{(r)}$. For each positive integer t, let $\mathcal{P}_t(U^{(r)})$ denote the collection of all power type decomposable subspaces of degree t in $U^{(r)}$. The following result was proved in [3, Proposition 6] for $t = 0$.

PROPOSITION 3. *Let $\mathcal{C} \subseteq \mathcal{P}_t(U^{(r)})$ be an infinite family such that $M_1, M_2 \in \mathcal{C}$ implies that $M_1 \cap M_2 \neq 0$. Then $r \geq 2k$, $k = p^t$ and there exists a nonzero decomposable element y of length $r - 2k$ such that for any $M \in \mathcal{C}$,*

$$M = y \cdot a_M^k \cdot U^k$$

for some $a_M \in U$.

PROOF. It follows from Lemma 2 that $r \geq 2k$. If $r = 2k$, the assertion is clear from Lemma 2. Hence we assume that $r > 2k$. Let $M = x \cdot U^k$ be a fixed decomposable subspace in $\mathcal{C}$. By Lemma 2, each $N \in \mathcal{C}$ has a common factor of length $r - 2k$ with M. Since $\mathcal{C}$ is infinite, it follows from Lemma 2 that there exist an infinite subset $\mathcal{D}$ of $\mathcal{C}$ and a nonzero decomposable element y of length $r - 2k$ such that

$$\mathcal{D} = \{y \cdot v^k \cdot U^k : v \in V \subseteq U\}$$

for some infinite subset V of U where $\langle v_1 \rangle \neq \langle v_2 \rangle$ for distinct $v_1, v_2 \in V$. Let $z \cdot U^k$ be any member of $\mathcal{C}$. Since V is infinite, there exists $v \in V$ such that v is not a factor of z. Hence by Lemma 2, $z = y \cdot w^k$ for some $w \in U$. This completes our proof.

PROPOSITION 4. *Let $\mathcal{C}$ be an infinite collection of decomposable subspaces of type 1 or power type in $U^{(r)}$ such that for every M_1, M_2 in $\mathcal{C}$, $M_1 \cap M_2 \neq 0$. Then $\mathcal{C} \subseteq \mathcal{P}_t(U^{(r)})$ for some integer $t \geq 0$, except possibly when* $\operatorname{char} F = 2$, *in which case, there exist a* *nonzero decomposable element x of length $r - 2^{s+1}$ for some non-negative integer s and a subset W of U such that*

$$\mathcal{C} = \{x \cdot U^{2^{s+1}}\} \cup \{x \cdot w^{2^s} \cdot U^{2^s} : w \in W\}. \tag{1}$$

PROOF. Let $\mathcal{D}_i = \mathcal{C} \cap \mathcal{P}_i(U^{(r)})$. Then $\mathcal{D}_t$ is infinite for some non-negative integer t. By Proposition 3, there exist a nonzero decomposable element x of length $r - 2k$ where $k = p^t$ and a subset W of U such that

$$\mathcal{D}_t = \{x \cdot a^k \cdot U^k : a \in W\}.$$

Suppose that some decomposable subspace $y \cdot U^m \in \mathcal{C}$ where $y = y_1 \cdots y_{r-m}$, $m = p^\ell$ and $\ell < t$. Choose $c \in W$ such that $\langle c \rangle \notin \{\langle y_1 \rangle, \ldots, \langle y_{r-m} \rangle\}$. Since $x \cdot c^k \cdot U^k$ and $y \cdot U^m$ have a nonzero intersection, it follows from Lemma 2 that

$$x \cdot c^k \cdot u^k = y \cdot v^m \tag{2}$$

for some u, v in U. Since c is not a factor y, it follows from (2) that c^k is a factor of v^m, a contradiction since $m < k$. Hence

$$\mathcal{P}_\ell(U^{(r)}) \cap \mathcal{C} = \emptyset$$

for $\ell < t$.

Suppose now some $y \cdot U^m \in \mathcal{C}$ where $m = p^\ell$, $\ell > t$ and y is a nonzero decomposable element of length $r - m$. Choose $d \in W$ such that d is not a factor of x or y. We obtain from Lemma 2 that

$$x \cdot d^k = \lambda y \cdot f^{m-k} \quad \text{or} \tag{3}$$

$$x \cdot d^k = z \cdot f^m \tag{4}$$

where $\lambda \in F, f \in U$ and z is a factor of length $r - m - k$ of y. If (4) holds, then $\langle d \rangle = \langle f \rangle$ and $k = m$ because of our choice of d. This yields a contradiction. Hence (3) holds. Thus $\langle d \rangle = \langle f \rangle$, $k = m - k$, and $x = by$ for some $b \in F$. Since $m = p^\ell = 2k = 2p^t$, we get $p = 2$ and $t + 1 = \ell$. Therefore $\mathcal{C}$ is the form (1).

4. **t-regular decomposable mappings.** Throughout this section U will denote a finite dimensional vector space over a perfect field F of characteristic $p > 0$.

A decomposable mapping T from $U^{(r)}$ to $U^{(s)}$ is called *t-regular* if the images of any two adjacent type 1 subspaces of $U^{(r)}$ under T are distinct power type decomposable subspaces of degree t.

Let f be any injective semi-linear mapping on U with respect to the automorphism $\lambda \longrightarrow \lambda^{\frac{1}{k}}$, $k = p^t$. Let z be any fixed nonzero decomposable element of $U^{(s-rp^t)}$. Then there exists a linear mapping S from $U^{(r)} \longrightarrow U^{(s)}$ such that

$$S(x_1 \cdots x_r) = z \cdot \big(f(x_1)\big)^k \cdots \big(f(x_r)\big)^k \tag{5}$$

where $k = p^t$. Clearly S is a t-regular decomposable mapping. We shall show in this section that every t-regular decomposable mapping is of the form (5).

LEMMA 3. *Let W be the subspace of $U^{(s)}$ spanned by the vectors $z \cdot u_1^k \cdot u_2^k \cdots u_r^k$ where $u_i \in U$, z is a fixed nonzero decomposable element in $U^{(s-rk)}$ and $k = p^t$. Then every linear mapping S of the form (5) maps the set of all nonzero decomposable elements of $U^{(r)}$ onto the set of all nonzero decomposable elements of W. Moreover, $S\colon U^{(r)} \longrightarrow W$ is bijective.*

PROOF. We first show that $S\colon U^{(r)} \longrightarrow W$ is bijective. Clearly $S(U^{(r)}) = W$. Let $v_1, \ldots, v_n$ be any basis of U. For any $w_1, \ldots, w_r \in U$, we have $w_i = \sum_{j=1}^n a_{ij} v_j$. Since $\operatorname{char} F = p$, it follows that

$$
\begin{aligned}
w_1^k \cdot w_2^k \cdots w_r^k &= \Big(\sum_{j=1}^n a_{1j} v_j\Big)^k \cdots \Big(\sum_{j=1}^n a_{rj} v_j\Big)^k \\
&= \Big(\sum_{j=1}^n a_{1j}^k v_j^k\Big) \cdots \Big(\sum_{j=1}^n a_{rj}^k v_j^k\Big) \\
&\in \langle \{ v_{i_1}^k \cdot v_{i_2}^k \cdots v_{i_r}^k : 1 \le i_1 \le i_2 \le \cdots \le i_r \le n \} \rangle .
\end{aligned}
$$

Since $\{ v_{i_1}^k \cdot v_{i_2}^k \cdots v_{i_r}^k : 1 \le i_1 \le i_2 \le \cdots \le i_r \le n \}$ is a linearly independent set, it follows that $\dim W = \dim U^{(r)}$ and hence S is injective.

The proof of the lemma will be complete if we can show that every decomposable element B in W is of the form $A \cdot d_1^k \cdots d_r^k$ for some $d_i \in U$. Let $B = z_1 \cdots z_s$. Using Lemma 1 in [7], we get

$$B = A \cdot y_1 \cdots y_{rk}$$

for some $y_i \in U$. Suppose that

$$y_1 \cdots y_{rk} \neq d_1^k \cdot d_2^k \cdots d_r^k$$

for all $d_i \in U$. Then

$$y_1 \cdots y_{rk} = g_1^m \cdot h_1 \cdots h_q$$

for some $g_1 \in U$, $h_i \in U$ such that $k \not| m$ and $\langle g_1 \rangle \neq \langle h_i \rangle$ for all i. Extend g_1 to a basis $g_1, \ldots, g_n$ of U. Write $c = rk$. Let $Q_{c,n}$ denote the set of all increasing sequences of c integers from $1, \ldots, n$. For each $\alpha = (\alpha_1, \ldots, \alpha_c) \in Q_{c,n}$, let

$$g_\alpha = g_{\alpha_1} \cdot g_{\alpha_2} \cdots g_{\alpha_c}.$$

Then $\{ g_\alpha : \alpha \in Q_{c,n} \}$ is a basis of $U^{(c)}$. Now let

$$h_i = \sum_{j=1}^n b_{ij} g_j, \quad b_{ij} \in F.$$

For each $1 \le i \le q$, let j_i denote the largest integer such that $b_{ij_i} \neq 0$. We have $j_i > 1$ for all i. It is not hard to see that

$$y_1 \cdots y_c = \Big(\prod_{i=1}^q b_{ij_i}\Big) g_1^m \cdot g_{j_1} \cdots g_{j_q} + \sum_{\alpha \in Q_{c\,n} \setminus \beta} b_\alpha g_\alpha$$

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where $b_\alpha \in F, \beta \in Q_{c,n}$ and $(\beta_{\sigma(1)}, \ldots, \beta_{\sigma(c)}) = (1, \ldots, 1, j_1, \ldots, j_q)$ for some permutation σ of $\{1, \ldots, c\}$. Hence

$$y_1 \cdots y_c \notin \langle \{g_{i_1}^k \cdots g_{i_r}^k : 1 \leq i_1 \leq \cdots \leq i_r \leq n\} \rangle.$$

This implies that

$$A \cdot y_1 \cdots y_c \notin \langle \{A \cdot g_{i_1}^k \cdots g_{i_r}^k : 1 \leq i_1 \leq \cdots \leq i_r \leq n\} \rangle = W.$$

Hence we obtain a contradiction and the proof is complete.

LEMMA 4. *Let V be a vector space over an infinite field. Let D be the set of all decomposable elements of $V^{(r)}$. If $D = \bigcup_{i=1}^m D_i$, then $\langle D_j \rangle = V^{(r)}$ for some j.*

PROOF. Let $\theta\colon \otimes^r V \longrightarrow V^{(r)}$ be the canonical mapping such that $\theta(x_1 \otimes \cdots \otimes x_r) = x_1 \cdots x_r$. Let

$$E_i = \{x_1 \otimes \cdots \otimes x_r : \theta(x_1 \otimes \cdots \otimes x_r) \in D_i\}.$$

Then $\bigcup_{i=1}^m E_i$ is the set of all decomposable elements of $\otimes^r V$. By Proposition 1 in [8], $\langle E_j \rangle = \otimes^r V$ for some j. Since θ is surjective, it follows that $\langle D_j \rangle = V^{(r)}$.

LEMMA 5. *Let $T\colon U^{(r)} \longrightarrow U^{(s)}$ be a t-regular mapping where $s > rp^t$. Then the images of all type 1 subspaces have a common factor of length $s - rp^t$.*

PROOF. Let $M_1 = x_1 \cdots x_{r-1} \cdot U$, and $M_r = y_1 \cdots y_{r-1} \cdot U$ be any two type 1 subspaces. Let

$$M_i = y_1 \cdots y_{i-1} \cdot x_i \cdots x_{r-1} \cdot U, \quad i = 1, \ldots, r-1.$$

Since $M_i \cap M_{i+1} \neq 0$, we have $T(M_i) \cap T(M_{i+1}) \neq 0$. Hence $T(M_i)$ and $T(M_{i+1})$ have a common factor of length $s - 2p^t$. Consequently $T(M_1)$ and $T(M_r)$ have a common factor of length $s - rp^t$.

Suppose $T(M_1) = Z \cdot U^{p^t}$ where $Z = z_1 \cdots z_{s-p^t}$, $z_i \in U$. Let $\{Z_1, \ldots, Z_m\}$ be a maximal set of factors of Z of length $s - rp^t$ such that $\langle Z_i \rangle \neq \langle Z_j \rangle$ for $i \neq j$. Then the image of any type 1 subspace has a factor Z_i for some i. Let D_i be the set of all nonzero decomposable elements $v_1 \cdots v_{r-1}$ such that $T(v_1 \cdots v_{r-1} \cdot U)$ has Z_i as a factor. Then $(\bigcup_{i=1}^m D_i) \cup \{0\}$ is the set of all decomposable elements of $U^{(r-1)}$. In view of Lemma 4 $\langle D_j \rangle = U^{(r-1)}$ for some j. By Lemma 1 in [7] we see that Z_j is a factor of the image of any type 1 subspace of $U^{(r)}$.

LEMMA 6. *Let $T\colon U^{(r)} \longrightarrow U^{(s)}$ be a t-regular mapping. Let $x = a \cdot b$, $y = a \cdot c$ where a is a nonzero decomposable element of length $r - 3$, and b, c are linearly independent vectors in U. Let $m = p^t$. If z is a common factor of length $s - 2m$ for all $T(x \cdot u \cdot U)$, $u \in U - \{0\}$, then z cannot be a common factor for all $T(y \cdot u \cdot U)$, $u \in U - \{0\}$.*

PROOF. For each nonzero g in U, let $M_g = x \cdot g \cdot U$ and $N_g = y \cdot g \cdot U$. Suppose that z a common factor for all $T(M_g)$ and $T(N_g)$, $g \in U - \{0\}$. Let d be a nonzero vector in U such that $\langle d \rangle \neq \langle c \rangle$ and $\langle d \rangle \neq \langle b \rangle$.

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Since M_c, M_d are adjacent and M_d, N_d are adjacent, it follows from Lemma 2 that

$$T(M_c) = z \cdot f^m \cdot U^m \tag{6}$$

$$T(M_d) = z \cdot d_1^m \cdot U^m \tag{7}$$

$$T(N_d) = z \cdot d_2^m \cdot U^m \tag{8}$$

for some $f, d_1, d_2 \in U$. Since T is t-regular, $\langle f \rangle \neq \langle d_1 \rangle$ and $\langle d_1 \rangle \neq \langle d_2 \rangle$. Note that M_c and N_d are adjacent and hence $\langle f \rangle \neq \langle d_2 \rangle$.

Now in view of (6), (7) and (8),

$$\begin{aligned} T(a \cdot b \cdot c \cdot d) &= z \cdot f^m \cdot u^m \\ &= z \cdot d_1^m \cdot v^m \\ &= z \cdot d_2^m \cdot w^m \end{aligned}$$

for some $u, v, w \in U$. Since $\langle d_1 \rangle \neq \langle d_2 \rangle$, we have $\langle d_1 \rangle = \langle w \rangle$, $\langle d_2 \rangle = \langle v \rangle$. It then follow that either

$$\langle f \rangle = \langle d_1 \rangle = \langle w \rangle \text{ or } \langle f \rangle = \langle d_2 \rangle = \langle v \rangle,$$

a contradiction. Hence the lemma is proved.

The following result follows from Theorem 1 in [8] and the proof of Theorem 6 in [8].

PROPOSITION 5. *Let V be a finite dimensional vector space over an infinite field. If L: $V^{(r)} \longrightarrow V^{(r)}$ is a linear mapping such that the images of any two adjacent type 1 subspaces of $U^{(r)}$ are distinct type 1 subspaces, then $L = \lambda P_r(g)$ where λ is a nonzero scalar and $P_r(g)$ is the r-th induced power of a nonsingular linear mapping g on V.*

THEOREM 1. *Let T: $U^{(r)} \longrightarrow U^{(s)}$ be a t-regular mapping. Then $s \geq rp^t$ and T is of the form (5).*

PROOF. If $r = 2$, by Proposition 3, $s \geq 2p^t$. Let $r \geq 3$. Let $k = p^t$. Let $y_1, \ldots, y_{r-1}$ be nonzero vectors of U such that $\langle y_i \rangle \neq \langle y_j \rangle$ for $i \neq j$. Let $M = y_1 \cdots y_{r-1} \cdot U$ and $T(M) = B \cdot U^k$ where B is a nonzero decomposable vector in $U^{(s-k)}$. In view of Proposition 3 and Lemma 5 we have for each $i = 1, \ldots, r-1$,

$$\big\{T(y_1 \cdots \hat{y}_i' \cdots y_{r-1} \cdot u \cdot U) : u \in U - \{0\}\big\} \subseteq \big\{B_i \cdot u^k \cdot U^k : u \in U - \{0\}\big\} \tag{9}$$

for some nonzero decomposable element B_i of length $s - 2k$. Since

$$T(M) \in \big\{T(y_1 \cdots \hat{y}_i \cdots y_{r-1} \cdot u \cdot U) : u \in U \setminus \{0\}\big\}$$

it follows that

$$B \cdot U^k = B_i \cdot u_i^k \cdot U^k \tag{10}$$

for some nonzero u_i in U. Hence u_i^k is a factor of B by Lemma 1. Since $\langle y_m \rangle \neq \langle y_j \rangle$ for $m \neq j$, by Lemma 6 we have $\langle B_m \rangle \neq \langle B_j \rangle$ for $m \neq j$. This shows that $\langle u_m \rangle \neq \langle u_j \rangle$ for

$m \neq j$. Consequently B is a decomposable element of length $\geq (r-1)k$. This implies that $s-k \geq (r-1)k$ and thus $s \geq rk$. Suppose $s > rk$. Then Lemma 5 implies that there exists a decomposable element A of length $s - rk$ such that A is a factor of the image of any type 1 subspace of $U^{(r)}$. Hence from (9) we obtain A is a factor of B_i for each i. Hence (10) implies that $A \cdot u_1^k \cdots u_{r-1}^k$ is a factor of B.

Let $W = \langle\{A \cdot v_1^k \cdots v_r^k : v_i \in U\}\rangle$. Then $T(M) \subseteq W$. Using induction on $r-1$, it is easily shown that $\{y_1 \cdots y_{r-1} : y_i \in U, \langle y_i\rangle \neq \langle y_j\rangle \text{ for } i \neq j\}$ spans $U^{(r-1)}$ and hence we have $T(U^{(r)}) \subseteq W$. Let S be any linear mapping of the form (5). Then Lemma 3 implies that $S^{-1}\colon W \longrightarrow U^{(r)}$ sends any two distinct power type decomposable subspaces in W of degree t to distinct type 1 subspaces of $U^{(r)}$. Hence $S^{-1} \circ T\colon U^{(r)} \longrightarrow U^{(r)}$ sends any two adjacent type 1 subspaces to distinct type 1 subspaces. In view of Proposition 5, $S^{-1} \circ T = \lambda P_r(g)$ for some nonsingular linear map g on U and $\lambda \in F - \{0\}$. Thus $T = \lambda S \circ P_r(g)$ is of the form (5).

The following result is obtained by modifying the proof of Theorem 1 and putting $t = 0$.

THEOREM 2. *Let V be a finite dimensional vector space over an infinite field. If L is a linear mapping from $V^{(r)}$ to $V^{(s)}$ such that the images of any two adjacent type 1 subspaces of $V^{(r)}$ are distinct type 1 subspaces, then $r \leq s$ and*

$$L(v_1 \cdots v_r) = A \cdot g(v_1) \cdots g(v_r) \tag{11}$$

for some nonzero decomposable element A of length $s - r$ and some nonsingular linear mapping g on U.

5. **Main results.** In this section U is a finite dimensional vector space over an arbitrary infinite field F.

We shall need the following:

LEMMA 7. *Let $T\colon U^{(r)} \longrightarrow U^{(s)}$ be a decomposable mapping such that the images of any two adjacent type 1 subspaces are distinct. Let* $\dim U > 2$. *Let* $\operatorname{char} F = 2$. *Let x be a fixed nonzero decomposable element of length $r-2$ over U. Let $m = 2^k$ where k is a non-negative integer. Then it is impossible that*

$$\{T(x \cdot y \cdot U) : y \in U - \{0\}\} = \{z \cdot U^{2m}\} \cup \{z \cdot w^m \cdot U^m : w \in W\} \tag{12}$$

for some nonzero decomposable element z of length $s - 2m$ over U and some $W \subseteq U$.

PROOF. Suppose that (12) holds. Let $M_y = x \cdot y \cdot U$, $y \in U - \{0\}$. Assume that

$$T(M_e) = z \cdot U^{2m}.$$

Let u, v be linearly independent vectors of U such that $e = u + v$. Then

$$T(M_u) = z \cdot u_1^m \cdot U^m, \tag{13}$$

$$T(M_v) = z \cdot v_1^m \cdot U^m, \tag{14}$$

for some $u_1, v_1 \in U$ where $\langle u_1 \rangle \neq \langle v_1 \rangle$. Let a, b be two linearly independent vectors of U such that $a, b \notin \langle u_1, v_1 \rangle$. Since $T(M_e) = z \cdot U^{2m}$, it follows that

$$T(x \cdot e \cdot f) = z \cdot a^{2m}$$
$$T(x \cdot e \cdot g) = z \cdot b^{2m}$$

for some f, g in U. Clearly $\langle a \rangle \neq \langle b \rangle$ implies that $\langle f \rangle \neq \langle g \rangle$. We have either $\langle f \rangle \neq \langle e \rangle$ or $\langle g \rangle \neq \langle e \rangle$. We may assume that $\langle e \rangle \neq \langle f \rangle$. Let $A = x \cdot e \cdot f$. In view of (13) and (14), we have

$$B = x \cdot u \cdot f \longrightarrow z \cdot f_1^m \cdot u_1^m$$
$$C = x \cdot v \cdot f \longrightarrow z \cdot f_2^m \cdot v_1^m$$

for some f_1, f_2 in U. Since $\langle f \rangle \neq \langle e \rangle$, we have

$$T(M_f) = z \cdot f_3^m \cdot U^m$$

for some f_3 in U. Since $T(A)$, $T(B)$, $T(C) \in z \cdot f_3^m \cdot U^m$, it follows that

$$\langle f_3 \rangle = \langle a \rangle = \langle f_1 \rangle = \langle f_2 \rangle$$

by our choice of a.

Now, let $f_1 = \alpha a, f_2 = \beta a$, $\alpha, \beta \in F$. We obtain

$$\begin{aligned} T(B + C) &= z \cdot (\alpha a)^m \cdot u_1^m + z \cdot (\beta a)^m \cdot v_1^m \\ &= z \cdot a^m \cdot (\alpha u_1 + \beta v_1)^m \text{ since char } F = 2 \\ &= T(A) \\ &= z \cdot a^{2m}. \end{aligned}$$

This implies that $a = \alpha u_1 + \beta v_1$, contradicting our choice of a. This completes the proof.

THEOREM 3. *Let $T\colon U^{(r)} \longrightarrow U^{(s)}$ be a decomposable mapping such that the images of any two adjacent type 1 subspaces are distinct. If* $\dim U \geq s + 1$, *then T is of the form (11), except possibly when F is a perfect field of prime characteristic p and $s \geq rp^t$, $t > 0$, in which case T may be of the form (5).*

PROOF: CASE 1. F is a perfect field of prime characteristic $p \leq s$. Let M be a type 1 subspace of $U^{(r)}$. Then $T(M)$ is a decomposable subspace of $U^{(s)}$ and

$$\dim U = \dim M = \dim T(M) \geq s + 1.$$

Suppose that $T(M) \subseteq x \cdot W^{(k)}$ for some 2-dimensional subspace W and some nonzero decomposable element x of length $s - k$ where $1 < k \leq s$. We shall show that this leads to a contradiction. First note that

$$\dim T(M) \leq \dim W^{(k)} = k + 1 \leq s + 1.$$

This implies that $k = s$ and $T(M) = W^{(s)}$, $\dim U = s+1$ and F is an s-field. Let $M = x \cdot y \cdot U$ where x is a nonzero decomposable element of length $r-2$ and $y \in U - \{0\}$. Let $M_u = x \cdot u \cdot U$ and $\mathcal{C} = \{T(M_u) : u \in U, u \neq 0\}$. We shall show that $W^{(s)}$ is the only type s subspace $\mathcal{C}$. Suppose there is another type s subspace $V^{(s)}$ in $\mathcal{C}$. Then $W^{(s)} \cap V^{(s)} \neq \{0\}$ implies that $W \cap V$ is 1-dimensional. Choose a nonzero vector z in U such that

$$T(x \cdot y \cdot z) = w_1 \cdots w_s$$

where $\langle w_i \rangle \neq \langle w_j \rangle$ for $i \neq j$, $\langle y \rangle \neq \langle z \rangle$ and $W \cap V \neq \langle w_i \rangle$ for all $i = 1, \ldots, s$. Since $w_1 \cdots w_s \in T(M_z)$, clearly $T(M_z)$ is not a power type decomposable subspace. If

$$T(M_z) = z_1 \cdots z_{s-1} \cdot U$$

for some z_i in U, then

$$T(M_z) \cap W^{(s)} \neq 0$$

and

$$T(M_z) \cap V^{(s)} \neq 0$$

imply that $z_1 \cdots z_{s-1} \in W \cap V$ and hence

$$\langle z_1 \rangle = \cdots = \langle z_{s-1} \rangle = W \cap V.$$

Since $w_1 \cdots w_s \in z_1 \cdots z_{s-1} \cdot U$, it follows that $\langle w_i \rangle = W \cap V$ for some i, a contradiction. Hence $T(M_z) = N^{(s)}$ for some 2-dimensional subspace N of U because of Proposition 2. Note that

$$x \cdot y \cdot z \in M_z \cap M_y$$

implies that $w_1, \ldots, w_s \in W \cap N$. Hence

$$\langle w_1, \ldots, w_s \rangle = W = N,$$

a contradiction to our hypothesis on T since M_z and M_y are adjacent. This shows that $W^{(s)}$ is the only type s subspace in $\mathcal{C}$. Let $\mathcal{C} - \{T(M)\} = \mathcal{D}$. Then $\mathcal{D}$ consists of decomposable subspaces of type 1 or power type in $U^{(s)}$. In view of Proposition 4,

$$\mathcal{D} - T(M_g) \subseteq \mathcal{P}_t(U^{(s)})$$

for some integer $t \geq 0$ and some $g \in U$. By Proposition 3,

$$\mathcal{D} - T(M_g) = \{e \cdot u^m \cdot U^m : u \in Z\}$$

where $m = p^t$, $Z \subseteq U$ and e is a nonzero decomposable element of length $s - 2m$. Let v and w be linearly independent vectors in U such that

$$T(M_v) = e \cdot v_1^m \cdot U^m,$$
$$T(M_w) = e \cdot w_1^m \cdot U^m.$$

Then there exists $h \in U$ such that $h \notin \langle g\rangle \cup \langle y\rangle \cup \langle v\rangle \cup \langle w\rangle$ and

$$T(x \cdot v \cdot h) = e \cdot v_1^m \cdot d_1^m$$

where $d_1 \notin \langle v_1, w_1\rangle$. Then

$$T(M_h) = e \cdot h_1^m \cdot U^m$$

for some h_1 in U. Clearly h_1^m is a factor of $v_1^m \cdot d_1^m$. But $\langle h_1\rangle \neq \langle v_1\rangle$, hence $\langle d_1\rangle = \langle h_1\rangle$. Since $T(M_h)$, $T(M_v)$, $T(M_w)$ all have a nonzero intersection with $T(M_y)$, it follows that $\langle d_1, v_1, w_1\rangle \subseteq W$, a contradiction since $\dim W = 2$, $\dim\langle d_1, v_1, w_1\rangle = 3$. This contradiction shows that $T(M)$ is not contained in $x \cdot W^{(k)}$ for any 2-dimensional subspace W of U and any nonzero decomposable element x. Hence by Proposition 2 the image of any type 1 subspace is either a type 1 or a power type subspace.

Suppose that $T(Q) \in \mathcal{P}_t(U^{(s)})$ for some type 1 subspace Q and some $t \geq 0$. Let R be any type 1 subspace of $U^{(r)}$. Clearly there exist type 1 subspaces $Q_1, \ldots, Q_r$ such that $Q_1 = Q$, $Q_r = R$ and Q_i, Q_{i+1} are adjacent. In view of Proposition 4 and Lemma 7, $T(Q_i) \in \mathcal{P}_t(U^{(s)})$ implies that $T(Q_{i+1}) \in \mathcal{P}_t(U^{(s)})$. Hence $T(R) \in \mathcal{P}_t(U^{(s)})$. This shows that T sends type 1 subspaces to type 1 subspaces or T is a t-regular mapping. Hence the theorem follows from Theorems 1 and 2.

CASE 2. F is not a perfect field of characteristic $p \leq s$.

Note that there are no power type decomposable subspaces in $U^{(s)}$. It is easily seen from the proof of Case 1 that T sends type 1 subspaces to type 1 subspaces. Hence the theorem follows from Theorem 2.

THEOREM 4. *Let $T\colon U^{(r)} \longrightarrow U^{(s)}$ be a decomposable mapping and* $\dim U \geq s+1$. *If (i) F is algebraically closed or (ii) F is the field of real numbers or (iii) T is injective, then T is of the form (11), except possibly when F is a perfect field of prime characteristic p, $s \geq rp^t$, $t > 0$, in which case T may be of the form (5).*

PROOF. We shall show that the images of any 2 adjacent type 1 subspaces under T are distinct. When F is algebraically closed, this can be shown by using exactly the same argument as in the proof of Proposition 5 in [3]. When T is injective, clearly the images of any 2 adjacent type 1 subspaces are distinct.

We now consider the case when $F = \mathbb{R}$, the real field. Suppose that there exist 2 adjacent type 1 subspaces $M_1 = x \cdot y_1 \cdot U$, $M_2 = x \cdot y_2 \cdot U$ such that $T(M_1) = T(M_2)$ where x is a decomposable element of length $r-2$. Let y be any nonzero vector in U. Let $M = x \cdot y \cdot U$. Then $T(M)$ is a decomposable subspace of dimension $\geq s+1$. Note that $T(M) \neq W^{(s)}$ for any 2-dimensional subspace W of U since $\mathbb{R}$ is not a s-field. Using this we see from Proposition 2 that $T(M)$ is a type 1 subspace since $\dim T(M) \geq s+1$. Since $\langle x \cdot y_1 \cdot y, x \cdot y_2 \cdot y\rangle \subseteq M$, it follows that $T(M)$ contains a 2-dimensional subspace of $T(M_1)$. This implies that $T(M) = T(M_1)$ since both $T(M)$ and $T(M_1)$ are type 1 subspaces. Let $H = \langle\{x \cdot y \cdot u : y \in U, u \in U\}\rangle$. Then $T(H)$ is a type 1 subspace. Hence $T|_H$ induces a linear mapping T_1 from $U^{(2)}$ to U sending every nonzero decomposable element to a nonzero vector in U. Let $\theta\colon \otimes^2 U \longrightarrow U^{(2)}$ be the linear mapping such that $\theta(u \otimes v) = u \cdot v$.

Then $T_1 \circ \theta$ is a linear mapping sending every nonzero decomposable element to a nonzero vector in U. By Theorem 4 in [1], we see that $\dim U = 4$ or 8. Now the mapping T_1 induces a symmetric bilinear mapping f from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^{n+1}$, $n = \dim U$, such that $f(a,b) = 0$ implies $a = 0$ or $b = 0$. The existence of f implies by a theorem of Hopf [4,5] that the real projective space of dimension $n-1$ can be differentiably embedded into $\mathbb{R}^n$, $n = 4$ or 8. But it is known that such an embedding is not possible (see [9], [12]). Hence we obtain a contradiction. This proves that the images of any 2 adjacent type 1 subspaces under T are distinct.

Hence the theorem follows from Theorem 3.

The following example shows that Theorem 4 is false if neither of the conditions (i) to (iii) hold:

EXAMPLE. Let Q_n be a field extension of the rational field Q of degree $n \geq 2$. Let e be a fixed nonzero decomposable element in $Q_n^{(s-1)}$ where $s > 2$. Then there exists a linear mapping $\theta\colon Q_n^{(r)} \longrightarrow Q_n^{(s)}$ such that

$$\theta(x_1 \cdots x_r) = \left(\prod_{i=1}^{r} x_i\right) \cdot e$$

where $\prod_{i=1}^{r} x_i$ denotes the product of the x_i's in the field Q_n. Clearly θ is a decomposable mapping and $\theta(Q_n^{(r)}) = e \cdot Q_n$. Hence θ is not of the form (11).

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