RINGS WITH FINITE MAXIMAL INVARIANT SUBRINGS

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ABSTRACT. We prove that if \( \varphi \) is an (anti-) automorphism of a ring \( R \) with finite orbits on \( R \), or integral over the integers, and if \( R \) contains a finite maximal \( \varphi \)-invariant subring, then \( R \) must be finite. Special cases are when \( \varphi \) has finite order or is an involution. Two corollaries are that \( R \) must be finite when \( R \) contains only finitely many \( \varphi \)-invariant subrings or has both ascending and descending chain conditions on \( \varphi \)-invariant subrings. These generalize results in the literature for the special case when \( \varphi = \text{id}_R \).

This paper is motivated by an interesting result of T. J. Laffey [8], obtained also by A. A. Klein [7], which proves that a ring with a finite maximal subring must be finite. Special cases of this result had been proven for commutative rings [2; Theorem 8, p. 542] and for rings satisfying a polynomial identity [3]. Also, Laffey's result implies related ones on finite subrings appearing in the literature ([5] and [13]). Our purpose here is to extend [8] to rings with a fixed (anti-) automorphism. The main theorem of the paper is that if \( \varphi \) is an (anti-) automorphism of a ring \( R \) having finite orbits on \( R \), and if \( R \) contains a finite maximal \( \varphi \)-invariant subring, then \( R \) must be finite. We do not use [8], so Laffey's result is a consequence of ours, as is the case when \( \varphi \) is an involution. Results for invariant subrings corresponding to those in [5] and [13] are also consequences of our main theorem, so these papers are special cases of our result as well.

Throughout the paper \( R \) will be an associative ring, \( Z(R) = Z \) is the center of \( R \), \( \text{Aut}(R) \) is the group of automorphisms of \( R \), and \( \text{Aut}^*(R) \) is the set of anti-automorphisms of \( R \). Recall that \( \varphi \in \text{Aut}^*(R) \) means that \( \varphi \in \text{Aut}((R, +)) \) and that \( \varphi(xy) = \varphi(y)\varphi(x) \) for all \( x, y \in R \). Observe that \( G = \text{Aut}(R) \cup \text{Aut}^*(R) \) is a group under composition and fix \( \varphi \in G \). For any nonempty subset \( B \subseteq R \), let \( \langle B \rangle \) be the subring generated by \( B \), and call \( B \) \( \varphi \)-invariant if \( \varphi(B) = B \). Finally, \( S \) will henceforth denote a finite and maximal \( \varphi \)-invariant subring of \( R \).

Our general approach, like that in [8], is to study the structure of \( S \) and \( R \). We aim for a situation where \( S = F \) or \( S = M_n(F) \) for \( F \) a finite field, and try to find an element \( x \in R - S \) with \( \varphi(x) = x \) and with \( x \in C_R(S), \) the centralizer of \( S \). Then, when \( \varphi \in \text{Aut}(R) \), \( R = S[x] \) has no invariant subring properly containing \( S \), and it follows that \( x \) must be algebraic over \( F \), so \( S[x] = R \) is finite. In order to solve the problem we need to assume that every orbit of \( \varphi \) on \( R \) is finite, and until near the end of the paper, we will usually assume the special case that \( \varphi \) has finite order. As expected, our proofs are more involved than would be the case when \( \varphi = \text{id}_R \), the identity map on \( R \).
is easy when \( \varphi = \text{id}_R \) is that \( R \) must have nonzero proper (invariant) subrings unless \( \text{card}(R) = p \), a prime, and either \( R = F_p \), a field, or \( R^2 = 0 \). This is a basic but important observation which is needed in considering rings with a finite maximal subring. Our first step is to see that the corresponding statement that \( R \) must contain nontrivial invariant subrings or be finite, is true for (anti-) automorphisms of finite order, but this is not so obvious. Our first theorem does this for a generalization of the finite order case. We use an argument which has probably appeared in the literature, but we are unaware of a reference. The computation in the middle of our proof can be essentially eliminated when \( \varphi \) has finite order by applying a well known and seminal result of G. Bergman and M. Isaacs [4, Proposition 2.4, p. 76] on fixed points of finite group actions. We note that our first theorem, and some of our later results are complicated a bit by the possibility which would be true if \( \varphi \in \text{Aut}(R) \). Also, our proof does not extend to the general case when \( \varphi \) has infinite order. Finally, we let \( J \) denote the ring of integers and \( Q \) the rational numbers. If one considers \( \varphi \in G \) to be in \( \text{Hom}_R(R, R) \), then it makes sense to consider polynomials in \( \varphi \) with coefficients in \( J \).

**Theorem 1.** Let \( \varphi \in G \) be integral over \( J \). If \( R \) has no nonzero proper \( \varphi \)-invariant subring, then \( R \) is finite.

**Proof.** For any prime \( p \), both \( pR \) and \( \{ r \in R \mid pr = 0 \} \) are \( \varphi \)-invariant subrings of \( R \), so either \( pR = 0 \) for some prime, or \( pR = R \) has no \( p \)-torsion for any prime. Consequently, \( R \) is an algebra over \( F \), for \( F = F_p \) the field of \( p \) elements, or \( F = Q \). Consider \( \varphi \in \text{Hom}_F(R, R) \) and let \( X^m + a_{m-1}X^{m-1} + \cdots + a_1X + a_0 \in F[X] \) be the minimal polynomial for \( \varphi \) over \( F \), where \( a_0 \neq 0 \), and if \( F = Q \) then \( a_i = b_i/a \) for \( b_i, a \in J \) and set \( b_0 = b \). Note that \( \varphi^{-1} \) is a polynomial in \( \varphi \) of degree \( m - 1 \), and if \( F = Q \) then \( \varphi^{-1} = -(a/b)\varphi^{m-1} + (b_{m-1}/b)\varphi^{m-2} + \cdots + (b_1/b)\text{id}_R \). Thus if \( F = Q \) and \( A = J[1/ab] = J_{ab} \), the localization at the powers of \( ab \), then for all \( j \in J \) we have \( \varphi^j \in A\varphi^{m-1} + \cdots + A\varphi + \text{id}_R \). We claim that \( R \) is finite or is not nilpotent. If \( R \) is nilpotent then \( R^2 \neq R \), and since \( R^2 \) is \( \varphi \)-invariant, we must have \( R^2 = 0 \). But now, for any nonzero \( x \in R \), if \( B = F_p \) or \( B = J_{ab} \) as appropriate, then \( \langle Bx, B\varphi(x), \ldots, B\varphi^{m-1}(x) \rangle \) is \( \varphi \)-invariant, which means that \( (R, +) = Bx + B\varphi(x) + \cdots + B\varphi^{m-1}(x) \). Clearly, \( R \) is finite when \( B = F_p \). If \( B = J_{ab} \), then \( R \) is a finitely generated torsion free module over the PID \( B \neq QF(B) = Q \), so as is well known and easy to see, \( R \) cannot be a Q algebra. Therefore, if \( R \) is nilpotent it must be finite, so we may assume that \( R \) is not nilpotent.

Observe that \( \eta = \varphi^2 \in \text{Aut}(R) \) and is still algebraic over \( F \). If \( R^n = \{ r \in R \mid \eta(r) = r \} \) then \( R^n \) is a \( \varphi \)-invariant subring of \( R \), so either \( R^n = 0 \) or \( R^n = R \). Assuming that \( R^n = 0 \) we will show that \( R \) is nilpotent, a contradiction. We start by showing that we may extend \( F \) to an algebraic closure, and to this end consider \( R_K = R \otimes_F K \) for \( K \) an algebraic closure of \( F \). We may assume that \( \eta \in \text{Aut}(R_K) \) via \( \eta(r \otimes k) = \eta(r) \otimes k \), and of course, \( \eta \) is algebraic over \( K \). Since \( R \) embeds in \( R_K \) by \( r \rightarrow r \otimes 1 \), and \( R \) is not nilpotent, neither is \( R_K \). Finally, if \( y = \sum r_i \otimes k_i \in (R_K)^n \) with \( \{ k_i \} \) independent over \( F \), then \( \sum r_i \otimes k_i = \eta(y) = \sum \eta(r_i) \otimes k_i \), so each \( r_i \in R^n = 0 \) forcing \( y = 0 \) and \( (R_K)^n = 0 \).

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Therefore, since it suffices to show that $R_K$ is nilpotent, there is no loss of generality in replacing $R$ with $R_K$, so in assuming that $R$ is an algebra over the algebraically closed field $K$, and that $\eta$ is a $K$-algebra automorphism.

Now that $K$ is algebraically closed, the minimal polynomial for $\eta$ splits over $K$ and $R$ is the direct sum of its eigenspaces $R(\lambda_i) = \{ r \in R \mid (\eta - \lambda_i)r = 0 \}$ for some $i \geq 1$, where $\lambda_1, \ldots, \lambda_s$ are all the distinct eigenvalues of $\eta$. Note that all $\lambda_i \neq 0$ since $\eta$ is invertible. Since $\eta$ satisfies a polynomial of degree $m$, for each $1 \leq j \leq s$, $(\eta - \lambda_j)^m R(\lambda_j) = 0$. Using the identity $(\eta - \lambda_j)(\eta x) = \lambda_j x((\eta - \lambda_j)x) + \left((\eta - \lambda_j)x\right)\lambda_j y + \left((\eta - \lambda_j)x\right)(\eta - \lambda_j)y$ and induction, it follows that $(\eta - \lambda_j)_{2m}(R(\lambda_j)R(\lambda_j)) = 0$, forcing $R(\lambda_j)R(\lambda_j) \subseteq R(\lambda_j)$. If $R$ is not nilpotent $R^{n+1} = 0$, so for some choice of $\mu_1 \in \{\lambda_1, \ldots, \lambda_s\}$, $R(\mu_1) \cdots R(\mu_{n+1}) \neq 0$. Now since $\{\lambda_i\}$ has $s$ elements and $\mu_1 \mu_2 \cdots \mu_k \in \{\lambda_i\}$ for all $1 \leq k \leq s+1$, we must have $\mu_1 \cdots \mu_k = \mu_1 \cdots \mu_{k+r}$ for some $k \geq 1$ and $r \geq 1$. Consequently $\mu_{k+1} \cdots \mu_{k+r} = 1 \in \{\lambda_i\}$, contracting $R^n = 0$. This shows that $R^n = 0$ forces $R$ to be nilpotent, and so we may now assume that $R^n = 0$.

From $R^n = 0$ it follows that $\varphi^2 = \text{id}_R$. Should $\varphi = \text{id}_R$, then $R$ is finite or contains nonzero proper (invariant) subrings, as we mentioned earlier. Therefore, we may assume that $\varphi \neq \text{id}_R$. For any $x \in R$, $(x + \varphi(x))$ is $\varphi$-invariant, so if some $x + \varphi(x) \neq 0$ then $R = \langle x + \varphi(x) \rangle$ contradicting $\varphi \neq \text{id}_R$. We are left with the assumption that $x + \varphi(x) = 0$ for all $x \in R$, so $\varphi(x) = -x$ and it follows that $x\varphi(x) = -x^2 = \varphi(x\varphi(x))$. Now if some $x^2 \neq 0$, then $\langle x\varphi(x) \rangle = R$, and again $\varphi \neq \text{id}_R$ is contradicted. Thus $x^2 = 0$ for all $x \in R$, and since $\varphi(x) = -x$, $\langle x \rangle$ is $\varphi$-invariant so $R = \langle x \rangle$, resulting in $R^2 = 0$. This contradiction forces us to conclude that $R$ is finite and the proof of the theorem is complete.

The special case when $\varphi$ has finite order and $R$ has no $\varphi$-invariant subring is now done by Theorem 1, and it would be interesting to know if this result holds for any $\varphi \in G$. Our next result puts together two useful observations. The first is a consequence of Theorem 1, and the second is essentially [3; Lemma 2ii, p. 352]. Note that the proof is the same for either $\varphi \in \text{Aut}(R)$ or $\varphi \in \text{Aut}^*(R)$. Recall that $S$ denotes a finite maximal $\varphi$-invariant subring of $R$.

**Lemma 1.** Let $\varphi \in G$ be integral over $J$. Either $R$ is finite or $S$ contains no nonzero ideal of $R$ and $\text{card}(S)R = 0$.

**Proof.** Assume that $R$ is infinite and note that $S \neq 0$ by Theorem 1. Should $I \subseteq S$ be a nonzero ideal of $R$, then the sum $T$ of all such ideals of $R$ is a finite $\varphi$-invariant ideal of $R$ contained in $S$. It is immediate that $\varphi$ induces an (anti-) automorphism $\eta$ of $R/T$, integral over $J$, and that $S/T$ is a finite, maximal $\eta$-invariant subring of $R/T$. If $S = T$ then $R/T$ is finite by Theorem 1, and if $S \neq T$ then $\text{card}(S/T) < \text{card}(S)$, so $R/T$ is finite by induction on $\text{card}(S)$. Since $T$ is finite, $R$ must be also, and this contradiction shows that $S$ cannot contain a nonzero ideal of $R$. For the second statement, observe that $\{r \in R \mid \text{card}(S)r = 0\}$ is a $\varphi$-invariant ideal of $R$ containing $S$. We have just seen that $S$ is not an ideal of $R$, so the maximality of $S$ forces $\text{card}(S)R = 0$. 

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It is now easy to show that we may assume that $R$ has no nilpotent ideals. The end of the argument uses a computation which will arise again later. Until Theorem 7, we will assume the special case when $\varphi$ has finite order.

**Lemma 2.** If $\varphi \in G$ has finite order $m$, then $R$ is finite or semi-prime.

**Proof.** If $I$ is a nonzero nilpotent ideal of $R$, then $T = I + \varphi(I) + \cdots + \varphi^{m-1}(I)$ is a $\varphi$-invariant nilpotent ideal of $R$, and $T + S$ is a $\varphi$-invariant subring of $R$ containing $S$. We may assume that $T \not\subset S$ by Lemma 1, so $R = T + S$. Since $T$ is nilpotent, there is a maximal integer $k \geq 1$ with $T^k \not\subset S$. Of course $N = T^k$ is $\varphi$-invariant, so $R = N + S$ and $N^2 \subset S$ by the choice of $k$. Choose $x \in N - S$ and note that $R = \langle S, x, \varphi(x), \ldots, \varphi^{m-1}(x) \rangle$, since this $\varphi$-invariant subring properly contains $S$. Now for any $i, j \geq 0$, $\varphi(x)\varphi'(x) \in S$ and $\varphi'(x)S\varphi'(x) \subset S$, so we may conclude that $R = S + \sum_i (\varphi'(x)(x^i) + \varphi'(x)S + \varphi'(x)S)$, where $J$ is the ring of integers. By Lemma 1 $R$ is a torsion ring, so $J\varphi'(x)$ is finite forcing $R$ to be finite and proving the lemma.

The next step in the argument is to show that $S$ is semi-simple. Let $J(S)$ be the Jacobson radical of $S$; $J(S)$ is the unique maximal nilpotent ideal of $S$ since $S$ is finite. Observe that $\varphi(S) = S$ means that the restriction of $\varphi$ to $S$ is an (anti-) automorphism of $S$. In the next theorem, the initial computation is based on [2, p. 542].

**Theorem 2.** If $\varphi \in G$ has finite order $m$, then $R$ is finite or $S$ is semi-simple.

**Proof.** Assume that $J = J(S) \neq 0$ and consider the $\varphi$-invariant subring $JRJ + S$. Note that this is a subring because $J$ is an ideal of $S$, and is $\varphi$-invariant because $J$ is the unique maximal nilpotent ideal of $S$. If $JRJ \not\subset S$, then the maximality of $S$ shows that $R = JRJ + S$, and it follows that $R = J(RRJ + S)J + S = J^2RJ + S$. Continuing with this substitution for $R$ yields $R = J^2RJ + S$ for any $k \geq 1$, and so $R = S$ is finite since $J$ is nilpotent. Therefore we may assume that $JRJ \subset S$. For any integer $k \geq 1$, $(J^kR) = J^k((JRJ)^k)J^kR \subset J^kR$, using that $JRJ \subset S$ and $J$ is an ideal of $S$. The nilpotence of $J$ forces $J^kR$ to be nilpotent, and since we may assume that $R$ is semi-prime by Lemma 2, $J^3 = 0$ results. It follows that $J^2R + RJ^2 + S$ is a $\varphi$-invariant subring, again using $JRJ \subset S$, so either $J^2R + RJ^2 \subset S$ or $R = J^2R + RJ^2 + S$. In the latter case we have $R = J^2(J^2R + RJ^2 + S) + (J^2R + RJ^2 + S)^2 + S \subset S$, because $J^2RJ^2 \subset S$, so $R$ is finite. Thus, we may take $J^2R + RJ^2 \subset S$.

If $J^2 \neq 0$, pick $x \in J^2$, define $D_x(r) = xr - rx$, and use $J^2R + RJ^2 \subset S$ to see that $D_x: R \rightarrow S$ and is an additive map with a finite image. Hence $\text{Ker} D_x = C_p(x)$, the centralizer of $x$ in $R$, has finite index in $(R, +)$. Set $K = \cap \{C_p(x) \mid x \in J^2\}$ and observe that $K$ is a subring of $R$ of finite index in $(R, +)$, so $K$ is infinite if $R$ is. Furthermore, it is clear that $K = C_{p(J^2)}$, so $K$ is $\varphi$-invariant, since $\varphi([a, b]) = \varphi(ab - ba) = \pm[\varphi(a), \varphi(b)]$. Consequently, if $R$ is infinite, $K \not\subset S$, and so, $R = \langle K, S \rangle$. Consider $r = k_1s_1 \ldots k_sn$ for $k_i \in K$ and $s_i \in S$, let $y \in J^2$, and note that $yr = yk_1s_1 \ldots k_sn = k_1(yk_1s_1)k_2 \ldots k_sn = k_1k_2(yk_1s_2)k_3 \ldots s_n = \ldots = k_1 \ldots k_n(yk_1s_2 \ldots s_n) = (ys_1) \ldots s_nk_1 \ldots k_n$. It follows from similar computations with the other possible forms for $r \in R$ that $J^2R \subset J^2 + J^2K$, so $(J^2R)^i \subset J^{2i} + J^{2i}K$, and the nilpotence of $J$ forces $J^2R$ to be nilpotent. As above, by Lemma 2, we may assume that $J^2 = 0$. 

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The argument above, that $J^3 = 0$ implies $J^2R + RJ^2 \subseteq S$, now shows that $J^2 = 0$ leads to $R$ finite or $JR + RJ \subseteq S$ by considering the $\varphi$-invariant subring $JR + RJ + S$. Using the argument of the last paragraph, with $J$ replacing $J^2$ and now considering $D_x$ and $C_R(x)$ for $x \in J$, shows that either $R$ is finite or $J = 0$, completing the proof of the theorem.

In view of Theorem 2, if $R$ is infinite we may assume that $S$ is the direct sum of finite simple rings by Wedderburn’s Theorems, with each simple component either a finite field $F$, or $M_n(F)$. We will show that $S$ is in fact a simple ring, and to do so we need to consider idempotents $e^2 = e \in S$. Recall that for any $e^2 = e \in R$, one has the Pierce decomposition of $(R, +)$ into a direct sum of subgroups, $(R, +) = eRe \oplus eR(1-e)Re \oplus (1-e)R(1-e)$, where $R(1-e) = \{r - re \mid r \in R\}$, $(1-e)R = \{r - er \mid r \in R\}$, and $(1-e)R(1-e) = \{r - er - re + ere \mid r \in R\}$. It is immediate that $Re(1-e)R = R(1-e)eR = 0$. Finally, for any $x \in R$ one has the corresponding representation $x = exe + ex(1-e) + (1-e)xe + (1-e)x(1-e)$.

**Theorem 3.** Let $\varphi \in G$ have finite order $m$. If $e^2 = e = \varphi(e) \in Z(S)$, then either $e \in Z(R)$ or $R$ is finite.

**Proof.** Assume that $R$ is infinite. Using $\varphi(e) = e \in Z(S)$, it is clear that $eR(1-e)Re + S$ is a $\varphi$-invariant subring of $R$, so the maximality of $S$ shows that either $eR(1-e)Re \subseteq S$ or else $R = eR(1-e)Re + S$. If the second possibility holds, $R = eR(1-e)Re + eS + (1-e)S$, so $eR(1-e) = 0$ and $R = S$ is finite, a contradiction. Thus we may assume that $eR(1-e)Re \subseteq S$. Should both $eR(1-e)$, $(1-e)Re \subseteq S$, then $R = eR + eR(1-e) + (1-e)Re + (1-e)R(1-e) = eR + (1-e)R(1-e) + S$, and it follows easily that $e \in Z(R)$. We may proceed with the assumption that $eR(1-e) \not\subseteq S$, the case that $(1-e)Re \not\subseteq S$ being similar.

Choose $x \in eR(1-e) - S$ and observe that $R = S = \langle S, x, \varphi(x), \ldots, \varphi^{m-1}(x) \rangle$ since $B$ is a $\varphi$-invariant subring of $R$ which properly contains $S$. We argue that $R$ is finite much as we did in Lemma 2. If $x = er(1-e)$, then $\varphi(x) = e\varphi(r)(1-e)$ if $\varphi \in \text{Aut}(R)$ and $\varphi(x) = (1-e)\varphi(r)e$ if $\varphi \in \text{Aut}^*(R)$. In the first case, since $e \in Z(S)$, $\varphi^i(x)\varphi^j(x) = \varphi^i(x)S\varphi^j(x) = 0$, and in the second case $\varphi^{2i}(x) \in eR(1-e)$ and $\varphi^{2i+1}(x) \in (1-e)Re$. From $eR(1-e)Re \subseteq S$, it now follows that any product of three elements from $\{\varphi^i(x)\}$ and elements of $S$ is equal to a product involving only one $\varphi^i(x)$ and $S$. For example, $\varphi^{2i+1}(x)\varphi^j(x)\varphi^k(x) \in \varphi^{2i+1}(x)S$ since $\varphi^{2i}(x)\varphi^j(x) = 0$ if $k$ is even, and in $eR(1-e)Re \subseteq S$ if $k$ is odd. Consequently, for $J$ the rings of integers, $R = S + \sum_{j=0}^{m-1} ((S + J)\varphi^j(x)(S + J) + (S + J)\varphi^j(x)(S + J))\varphi^j(x)(S + J)$. But by Lemma 1 we may assume that every element of $R$ is a torsion element, so $R$ is finite. With this contradiction, the proof of the theorem is complete.

**Corollary.** If $\varphi \in G$ has finite order, then either $R$ is finite or $1_S \in S$ and $1_S = 1_R$.

**Proof.** By Theorem 2 we may assume that $S$ is semi-simple, so $S$ has an identity element $1_S$. Certainly $1_S^2 = 1_S = \varphi(1_S)$, so $1_S \in Z(R)$ by Theorem 3, unless $R$ is finite. Now $1_S R$ is an ideal of $R$, is $\varphi$-invariant, and $1_S R \supseteq 1_S S = S$. Using Lemma 1 we may assume that $1_S R \neq S$, so $1_S R = R = R1_S$. A straightforward computation shows that $1_S = 1_R$.

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Our next theorem is a key result which restricts further the structure of $S$.

**Theorem 4.** If $\varphi \in G$ has finite order $m$, then either $R$ is finite or $S$ is a simple ring.

**Proof.** Assume that $R$ is infinite, so $S$ is semi-simple by Theorem 2, and $S = S_1 \oplus \cdots \oplus S_t$, the direct sum of its simple components, which are the minimal ideals in $S$. Clearly, $\varphi^t(S_i)$ is a minimal ideal of $S$, so $\varphi^t(S_i) = S_j$ for some $j$. It is straightforward and easy to show that $\{S_1, \varphi(S_1), \ldots, \varphi^{m-1}(S_1)\}$ has $t$ distinct elements where $t \geq 1$ is minimal with $\varphi^t(S_i) = S_j$. By re-ordering $\{S_j\}$ we may assume that $S_i = \varphi^{t-1}(S_j)$ if $1 \leq i \leq t$. Let $e_i = \varphi^{t-1}(e_j)$ for $1 \leq i \leq t$ be the identity element of $S_i$, then $e = e_1 + \cdots + e_t$ is the identity of $S_1 + \cdots + S_t = eS$, and $e^2 = e = \varphi(e) \in Z(S)$.

By Theorem 3, $e \in Z(R)$, so $eR$ and $(1 - e)R$ are $\varphi$-invariant ideals and $R = eR + (1 - e)R$ is their direct sum. Should $eR = eS \subseteq S$, then $e$ contains a nonzero ideal of $R$ in contradiction to Lemma 1, so $eS \not\subseteq eR$. Hence $R \not\subseteq eS + (1 - e)R = B$, a $\varphi$-invariant subring of $R$ with $B \supseteq eS + (1 - e)S = S$. The maximality of $S$ forces $(1 - e)R \subseteq S$, so again $S$ would contain an ideal of $R$ unless $(1 - e)R = 0$. Therefore we may conclude that $e = 1_R = 1_S$. Since any ideal of $S$ is a direct sum of a subcollection of $\{S_i\}$ and $S_i = \varphi^{t-1}(S_j)$, $S$ has no nonzero proper $\varphi$-invariant ideal.

We have $1_R = 1_S = e_1 + \cdots + e_t$ is a sum of orthogonal idempotents, so $S = e_1S \subseteq e_1Re_i$. Clearly, $e_1Re_i$ is a $\varphi$-invariant subring of $R$ containing $S$, so either $e_1Re_i = R$ or $e_1Re_i = S$. In the latter case, because $R$ is infinite and $R = \sum_{i,j} e_iRe_j$, some $e_iRe_j \not= 0$ with $i \not= j$. Choose $x \in e_iRe_j$, observe that $x \notin S$, and that $B = \langle S, x, \varphi(x), \ldots, \varphi^{m-1}(x) \rangle$ is a $\varphi$-invariant subring properly containing $S$, so $B = e_1R$.

Now $\varphi^t(x) = x$ unless $\varphi^t(x) \in e_1R$, so $\varphi^t(x) = e_1Re_w$. Since $\{e_i\}$ has only $t$ distinct subscripts, any word $y_1S_1 \cdots y_tS_t$ with $y_j \in S$ and $y_j \not= x, \varphi(x), \ldots, \varphi^{m-1}(x)$ must be zero or have a subword $y_1S_1 \cdots y_{t-1}S_{t-1}y_t \in e_1Re_w \subseteq S$. It follows that $R = B = S + \sum \{S_{y_1}S_{y_2} \cdots S_{y_t} | 1 \leq t \leq m - 1 \}$ is finite. Therefore, we must have $R = e_1Re_i$, and $e_iR \subseteq Z(R)$ follows easily, so $e_iRe_i = e_iR$.

Recall that $t$ is minimal with $\varphi^t(e_iR) = e_iR$. By restriction, $\varphi^t$ induces an (anti-) automorphism of finite order on $e_iR$. If $A_1$ is a $\varphi^t$-invariant subring of $e_iR$ containing $e_iS = S_1$, then $A = \sum_{j=0}^{t-1} \varphi^j(A_1)$ is a $\varphi$-invariant subring of $R$ containing $S$. Consequently, if $A_1$ contains $S_1$ properly, then $A = R$, so $A_1 = e_iR$ and $S_1$ is a finite maximal $\varphi$-invariant subring of $e_iR$. But when $t > 1$, card$(S_1) <$ card$(S)$ and by induction on card$(S)$, $e_iR$ is finite forcing $R = \sum \varphi^j(e_iR)$ to be finite. This proves that $t = 1$, so $S = S_1$ is a simple ring.

Our next goal is to show that $Z(S) = Z(R)$. We need two lemmas to do this, the first of which is one of the inclusions and the other gives additional structural information on $R$.

**Lemma 3.** If $\varphi \in G$ has finite order $m$, then either $R$ is finite or $Z(R) \subseteq Z(S)$.

**Proof.** From Theorem 4 we may assume that $S = M_n(F)$ for $F$ a finite field and $n \geq 1$. Suppose that there is $z \in Z(R) - Z(S)$, in which case $B = \langle S, z, \varphi(z), \ldots, \varphi^{m-1}(z) \rangle$ is a $\varphi$-invariant subring of $R$ properly containing $S$. Since $S$ is maximal and $z \in Z(R) = \cdots
\( \varphi(Z(R)) \) we may write \( R = \langle S, z, \varphi(z), \ldots, \varphi^{m-1}(z) \rangle = S[z, \varphi(z), \ldots, \varphi^{m-1}(z)] \). Let 
\[ q(X) = \prod_{j=0}^{m-1} (X - \varphi^j(z)) \in Z(R)[X], \]
and note that the coefficients of \( q(X) \) are symmetric functions in \( \{z, \varphi(z), \ldots, \varphi^{m-1}(z)\} \), so each is fixed by \( \varphi \). If any of these coefficients, say \( y \), is not in \( S \), then \( R = \langle S, y \rangle = S[y] \), by the maximality of \( S \). Set \( A = S[y^2] \), a \( \varphi \)-invariant subring of \( R \) containing \( S \), so \( A = S \) or \( A = R \). Should \( A = S \) then \( y^2 \notin S \), so \( R = S[y] = S' + Sy \) is finite. If \( A = R \), then \( \bar{y} \in S^2 \), so \( y = \sum s_jy^{2j} \) with \( s_j \in S = M_n(F) \). Now \( R = S[y] = M_n(F)[y] = M_n(F[y]) \), so if \( \{e_j\} \) are the usual matrix units we may regard \( y = \sum e_jy^j \) as a diagonal matrix. Therefore, using the diagonal entries of the equation \( y = \sum s_jy^{2j} \) shows that each \( e_jy \), \( y \), is algebraic over \( F \). Hence \( F[y] \) is finite and it follows that \( R = M_n(F[y]) \) is finite. Consequently, we may assume that all the coefficients of \( q(X) \) are in \( S \). But this implies that each \( \varphi^i(z) \) is integral over \( S \), so \( R = S[z, \varphi(z), \ldots, \varphi^{m-1}(z)] \) is a finitely generated \( S \)-module, so is finite, proving the lemma.

**Lemma 4.** Let \( \varphi \in G \) have finite order \( m \). Then \( R \) is finite or every proper ideal \( I \) of \( R \) satisfies \( I \cap S = 0 \) and either:

i) \( R \) is a simple ring with \( I \);

ii) \( R \) is a sum of proper ideals; or

iii) \( R \) contains a proper \( \varphi \)-invariant ideal \( I \), \( R = I + S \), and \( I \) properly contains a prime ideal \( P \) of \( R \).

**Proof.** By Theorem 4 we may assume that \( S \) is a simple ring, by the Corollary to Theorem 3 that \( I = 1 \), by Lemma 1 that \( S \) contains no ideal of \( R \), and by Lemma 2 that \( R \) is semi-prime. If \( I \neq 0 \) is any ideal of \( R \), then \( I \cap S \) is an ideal of \( S \) so \( I \cap S = 0 \) or \( S \subseteq I \). In the latter case \( 1 \in S \subseteq I \) so \( I = R \), and indeed \( I \cap S = 0 \) for any proper ideal \( I \) of \( R \). Assume next that \( R \) is not a simple ring and has no proper \( \varphi \)-invariant ideal. Then since \( 1 \in R \), there is a proper maximal ideal \( I \) of \( R \) and \( I \not\supseteq \varphi(I) \). But now \( I = 1 + \varphi(I) \), so \( R \) is a sum of these proper ideals.

Finally assume that \( R \) is not simple and is not the sum of proper ideals. If \( I \) is the sum of all the proper ideals of \( R \), then \( I \) is a proper \( \varphi \)-invariant ideal of \( R \) and \( R = I + S \) by the maximality of \( S \). Since \( R \) is a semi-prime ring, the intersection of all its prime ideals is zero, so there is a prime ideal \( P \) of \( R \) with \( I \not\subset P \). By definition of \( I \), \( P \) is properly contained in \( I \).

We come now to our next to last preliminary result which is essential in proving our main theorem.

**Theorem 5.** If \( \varphi \in G \) has finite order \( m \), then either \( R \) is finite or \( Z(S) = Z(R) \).

**Proof.** We may assume that \( S = M_n(F) \) for \( F \) a finite field of \( p^n \) elements and \( n \geq 1 \) by Theorem 4, that \( I = 1 \) by the Corollary of Theorem 3, and that \( Z(R) \) is a subfield of \( Z(S) = F \) by Lemma 3. Suppose that there is \( z \in F - Z(R) \) and let \( z^k = 1 \) for \( k \) the order of \( z \in F - (0) \), so of course \( k \mid p^n - 1 \). Consider the expression 
\[ g(X) = z^{k-1}X + z^{k-2}X^2 + \cdots + X^{z^{k-1}} \in F \ast Z(R) Z(R)[X], \]
the free product over \( Z(R) \). It is straightforward to verify that for any \( r \in R \), \( g(r)z = zg(r) \); that is, \( g(R) \subseteq C(z) \),
the centralizer of $z$ in $R$. If some $y = g(r) \notin S$, then the maximality of $S$ implies that $R = \langle S, y, \varphi(y), \ldots, \varphi^{m-1}(y) \rangle$. Since $F = Z(S)$ and $\varphi(S) = S$, it follows that $\varphi$ restricts to an automorphism of the finite field $F$ over its prime field, so by elementary Galois theory $\varphi(z) = z^v$ for $v = p^b$ with $b \geq 0$. Now the order of $z \in F - (0)$ is $k \mid p^b - 1$, so $k$ is relatively prime to $p$ and the cyclic subgroups $(z), (\varphi(z)), \ldots, (\varphi^{m-1}(z))$ in $F - (0)$ are all equal. Consequently, $C(z) = C(\varphi(z)) = \cdots = C(\varphi^{m-1}(z))$ and $\varphi'(y) \in C(\varphi'(z)) = C(z)$. Since $z \in Z(S)$, $S \subseteq C(z)$, so $R = \langle S, y, \varphi(y), \ldots, \varphi^{m-1}(y) \rangle \subseteq C(z)$, which forces the contradiction $z \in Z(R)$. Therefore, we may assume that $g(R) \subseteq S$.

If $I$ is any proper ideal of $R$, then $g(I) \subseteq I \cap S = 0$ by Lemma 4. Since $g(X) : R \to S$ is additive, $g(R) = 0$ if $R$ is the sum of proper ideals. But $g(z) = k \neq 0$ since $p \not| \ k$, so by Lemma 4 again, either $R$ is a simple ring or for some proper ideal $P$ of $R$. In the latter case, note that $(F + P)/P \cong F$, and up to isomorphism $Z(R)$ is in $(R/P) \ast Z(R/P)$, so we may now consider $g(X) \in (R/P) \ast Z(R/P)[X]$. Certainly, $g(I/P) = 0$ in $R/P$, and since $I/P$ is a nonzero ideal in the prime ring $R/P$ we may conclude that $g(R/P) = 0$ [9; Lemma 1, p. 766]. Once again $g(z + P) \neq 0$ shows that this situation cannot occur, so we may assume that $R$ is a simple ring with 1.

Recall that $Z(R) \subseteq Z(S) = F$ is a finite field and set $\dim_{Z(R)} S = \eta$. It is well known that $S$ satisfies the standard polynomial identity $S_{q+1}[x_1, \ldots, x_{q+1}] = \sum_{\sigma} (-1)^{\sigma} x_{\sigma(1)} \cdots x_{\sigma(q+1)}$ over all permutations $\sigma$ of $\{1, 2, \ldots, q + 1\}$ [6; Lemma 6.2.2, p. 154]. Setting $h(x_1, \ldots, x_{q+1}) = S_{q+1}[g(x_1), \ldots, g(x_{q+1})] \in R \ast Z(R)[x_1, \ldots, x_{q+1}]$, where $Z(R)\{x_1, \ldots, x_{q+1}\}$ is the free algebra over $Z(R)$, we have that $h(x_1, \ldots, x_{q+1})$ is a generalized polynomial identity for $R$. Should $h \equiv 0$, then the sum of all its monomials with the variables appearing in the same order must be zero as well. In particular, $g(x_1) \cdots g(x_{q+1}) \equiv 0$, using the substitution of all $x_i \equiv z$ gives the contradiction $0 = g(z)^{q+1} = k^{q+1} \neq 0$ since $\text{char}(R) \not| \ k$. Hence $h(x_1, \ldots, x_{q+1})$ is a nontrivial generalized polynomial identity for $R$. By Martindale’s Theorem [11; Theorem 3, p. 579], since $1 \in R$ and $R$ is simple, one must have $R = \text{soc}(R) \cong M_n(D)$ for $D$ a division ring finite dimensional over $Z(R)$. It follows that $R$ is finite, proving that $Z(S) = Z(R)$, when $R$ is infinite.

In the proof of our main theorem we will be able to assume that $S = M_n(F)$ and will want to choose an element which centralizes $S$ and is fixed by $\varphi$. This is possible since $\varphi$ has infinitely many fixed points, which we prove next. Note that $\varphi$ having infinitely many fixed points does not by itself contradict $S$ finite. After all, $S$ finite does not preclude the existence of some infinite $\varphi$-invariant subring not containing $S$.

**Theorem 6.** Let $A$ be a semi-prime ring, $pA = 0$ for $p$ a prime, $\eta \in \text{Aut}(A) \cup \text{Aut}^*(A)$ of finite order, $A^n = \{x \in A \mid \eta(x) = x\}$, and $A^{-n} = \{x \in A \mid \eta(x) = -x\}$. If $A$ is infinite, then either $A^n$ or $A^{-n}$ is infinite.

**Proof.** Assume first that $\eta \in \text{Aut}(A)$ and write the order of $\eta$ as $o(\eta) = p^t$ with $p \not| \ t$. If $\sigma$ is the $p^t$-th power of $\eta$, then $\sigma(x) = x$ and $p \not| \ t$, so $A^t = \{x \in A \mid \sigma(x) = x\}$ is a semi-prime ring [12; Corollary 1.5, p. 9] and is infinite when $A$ is [10; Theorem 3, p. 364].
Now η induces an automorphism of $A^n$, so to prove the theorem when $η \in \text{Aut}(A)$, it is enough to assume that $o(η) = p^n > 1$. Consider $A$ to be a vector space over the field $F$ of $p$ elements and $η \in \text{Hom}_F(A, A)$. Note that $A^n = \text{Ker}(η - \text{id}_A)$, so it suffices to let $T = η - \text{id}_A$ and to show that $\text{Ker} T$ is infinite. Since $o(η) = p^n$ and $\text{char} F = p$, the minimal polynomial of $T$ is $X^n$. Using the cyclic decomposition, any finite dimensional $T$-invariant subspace $V$ of $A$ is the direct sum of a finite number of $T$-cyclic subspaces, say $υ_i$ each of dimension at most $c$. Clearly, $T$ acting on any $T$-invariant subspace has a nonzero kernel, so $\text{card}(\text{Ker}(T|_υ)) \geq p^n$. It follows that if $\text{card}(\text{Ker} T) = q$ is finite, then any finite dimensional $T$-invariant subspace $M$ of $A$ satisfies $\dim M \leq qc$. But if $V$ is any finite dimensional $T$-invariant subspace, say $V = \text{Ker} T$, then $T$ induces a nilpotent transformation $Y$ on $A/V$, so has a nonzero kernel when $A$ is infinite. If $x + V \in \text{Ker} Y(0)$, then $F(x) + V$ is a $T$-invariant subspace properly containing $V$. Thus there exist $T$-invariant subspaces of arbitrarily large dimension when $A$ is infinite, so $\text{Ker} T$ must be infinite and $A^n$ is infinite also.

When $η \in \text{Aut}^n(A)$, then $η^2 \in \text{Aut}(A)$, so by the case above its fixed point ring $B$ is infinite when $A$ is. Clearly $B$ is $η$-invariant, $B^n = \{b \in B \mid η(b) = b\} \subseteq A^n$, and $B^{-η} \subseteq A^{-η}$, so it suffices to replace $A$ with $B$ and assume that $η^2 = \text{id}_A$. Thus we may assume that $η$ is an involution, but cannot assume now that $A$ is semi-prime. If $p > 2$, then $A$ is the direct sum of the characteristic subspaces $A^n$ and $A^{-η}$, so one of these is infinite when $A$ is infinite. Finally, if $p = 2$ then $A$ infinite forces $A^n$ to be infinite [10; Lemma 5, p. 371]. To see this suppose that $A^n$ is finite and let $A = A^n \oplus M$ for $M$ an infinite subspace of $A$. If $Y$ is any basis of $M$ and $y \in Y$, then $y + η(y) \in A^n$ so the finiteness of $A^n$ shows that $y + η(y) = x + η(x)$ for $x, y \in Y$ and $x \neq y$. Hence $x + y = η(x + y) \in M \cap A^n = 0$ gives a contradiction. Therefore $A^n$ must be infinite, proving the theorem.

**THEOREM 7.** If $φ \in \text{Aut}(R) \cup \text{Aut}^n(R)$ has finite order, and if $S$ is a finite maximal $φ$-invariant subring of $R$, then $R$ is finite.

**PROOF.** From Theorem 4 we may assume that $S$ is a simple ring, and from Theorem 5 that $Z(R) = Z(S)$, a finite field, so applying Theorem 6 yields an element $x \in R - S$ so that $φ(x) = ±x$, unless $R$ is finite. Clearly, the maximality of $S = F$ shows that $R = \langle S, x \rangle = F[x]$. But $F[x^2]$ contains $F$ and is $φ$-invariant, so $F[x^2] = F = S$ or $F[x^2] = R = F[x]$. Therefore, $x$ is algebraic over $F$, so $R = F[x]$ is finite. Next assume that $S = M_n(F)$ with $n > 1$ and $F = Z(S) = Z(R)$. Using a theorem of Wedderburn [1; Theorem 17, p. 19] shows that $R = SA$, where $A = C_R(S)$, the centralizer of $S$ in $R$. Briefly, if $\{e_{ij}\}$ are the usual matrix units in $S$, then for $r \in R$ set $r_{ij} = \sum_k e_{ik}re_{jk}$, and note that all $r_{ij} \in A$ and $r = \sum e_{ij}r_{ij}$. Consequently, since we can take $R$ to be semi-prime by Lemma 2, we may assume that $A$ is also semi-prime because for any ideal $B$ of $A$, $SB$ is an ideal of $R$. Observe that $SB \neq 0$ if $B \neq 0$ since $1 \in S$ by the Corollary of Theorem 3. Finally, since $S$ is $φ$-invariant and $A = C_R(S)$, $A$ is also $φ$-invariant. Now unless $R$ is finite, $A$ is infinite and Theorem 6 shows that there is $x \in A - S$ with $φ(x) = ±x$. As above $R = \langle S, x \rangle = S[x] = M_n(F)[x] = M_n(F[x])$, and $S[x^2] = M_n(F[x^2])$ is a $φ$-invariant subring of $R$ containing $S$. Therefore, $x^2 \in S \cap F$, so
x is algebraic over $F$, or $M_n(F[x^2]) = R = M_n(F[x])$, and as in the proof of Lemma 3, $x$ is algebraic over $F$. In either case $R = M_n(F[x])$ is finite.

We immediately extend Theorem 7 to $\varphi \in G$ which is locally finite, that is, for all $x \in R$, and some $i = i(x) \geq 1$, $\varphi^i(x) = x$, or which is integral over $J$.

**Theorem 8.** Let $\varphi \in \text{Aut}(R) \cup \text{Aut}^*(R)$ and $S$ a finite maximal $\varphi$-invariant subring of $R$. If either $\varphi$ is locally finite or integral over $J$, then $R$ is finite.

**Proof.** Assume first that $\varphi$ is locally finite and for $i \geq 1$ set $R(i) = \{x \in R \mid \varphi^i(x) = x\}$. Clearly, each $R(i)$ is a $\varphi$-invariant subring and $R = \cup R(i)$ by the local finiteness of $\varphi$. Also, $(R(i), R(j)) \subseteq R(ij)$, so $S \subseteq R(n)$ for some $n$ because $S$ is finite. Thus $R = R(n)$ and $\varphi$ has order at most $n$, or $S = R(n)$. But $S = R(n) \neq R$ implies that some $R(t) \not\subseteq R(n)$, so $R(tn)$ is a $\varphi$-invariant subring properly containing $S$. This forces $R = R(tn)$, and $\varphi$ has order at most $tn$. Consequently, $\varphi$ must have finite order, and now Theorem 7 shows that $R$ is finite.

When $\varphi$ is integral over $J$, we may assume that $S \neq 0$ by Theorem 1, so card$(S)R = 0$ by Lemma 1. Thus $R$ is a torsion ring and so is the direct sum of its $p$-torsion components $R(p) = \{r \in R \mid pr = 0 \text{ for some } k \geq 1\}$, over those primes with $p \mid$ card$(S)$. Now each $R(p)$ is $\varphi$-invariant, and the restriction $\varphi_p$ of $\varphi$ to $R(p)$ is integral over $J$. Clearly $R(p) \cap S$ is a finite $\varphi_p$-invariant subring of $R(p)$, if $T$ is a proper $\varphi_p$-invariant subring of $R(p)$ properly containing $R(p) \cap S$, then $R \neq T + S$ and $T + S$ is a $\varphi$-invariant subring properly containing $S$, a contradiction. Hence $R(p) \cap S$ is a finite $\varphi_p$-invariant subring of $R(p)$, so either card$(R(p) \cap S) <$ card$(S)$ and $(R(p)$ is finite by induction on card$(S)$, or $S \subseteq R(p)$. Since this holds for each $R(p)$, we may assume that $R$ is finite unless $R = R(p)$ is $p$-torsion. Let $W = \{r \in R \mid pr = 0\}$, note that $W$ is a $\varphi$-invariant ideal of $R$, and that $R$ is finite or $W \not\subseteq S$ by Lemma 1. Therefore, because $W + S$ is a $\varphi$-invariant subring, $R = W + S$ and so $pR = pS \subseteq S$, again contradicting Lemma 1 unless $R$ is finite or $pR = 0$. But if $pR = 0$, then $R$ is an algebra over $F = F[x]$, the field of $p$ elements. It follows that $\varphi$ is algebraic over $F$, forcing $\varphi$ to have finite order. To see this let $m(X) = \prod q_i(X)^{a(i)}$ be the prime factorization of the minimal polynomial of $\varphi$ over $F$. If $F_t$ is the splitting field of $\prod q_i(X)$ over $F$, then $F_t$ contains $t = p^e$ elements, and each $y \in F_t$ satisfies $X^t - X$. Since each $q_i(X)$ has a root in $F_t$, it follows that $\prod q_i(X) \mid (X^t - 1)$. For $k = \max\{a(i)\}$ and $n = p^k$, clearly $m(X) \mid (X^{p^k} - 1)^n = X^{m(n-1)} - 1$, so $\varphi$ has finite order dividing $n(t - 1)$. Consequently, Theorem 7 may be applied to show that $R$ is finite.

To conclude the paper we give two special cases of Theorem 8 and two other consequences which extend results of Gilmer [5] and of Szele [13].

**Corollary 1 (Laffey [8]).** If $R$ contains a finite maximal subring, then $R$ is finite.

**Corollary 2.** If $R$ is a ring with involution $*$ and $S$ is a finite maximal $^*$-invariant subring, then $R$ is finite.

**Corollary 3.** If $\varphi \in \text{Aut}(R) \cup \text{Aut}^*(R)$ is either locally finite or integral over $J$, and if $R$ has only finitely many $\varphi$-invariant subrings, then $R$ is finite.
PROOF. Let $R = R_0 \supseteq R_1 \supseteq \cdots \supseteq R_{n+1} = 0$ be a maximal chain of $\varphi$-invariant subrings, with all inclusions proper. For each $i \leq n$, if $\varphi_i$ is the restriction of $\varphi$ to $R_i$, then $R_{i+1}$ is a maximal $\varphi_i$-invariant subring of $R_i$, and of course each $\varphi_i$ is locally finite or integral over $J$. Thus $R_n$ is finite by Theorem 1, so $R_{n-1}$ is finite by Theorem 8, and using induction together with Theorem 8 shows that $R$ is finite.

COROLLARY 4. If $\varphi \in \text{Aut}(R) \cup \text{Aut}^*(R)$ is either locally finite or integral over $J$, and if $R$ satisfies the ascending and descending chain conditions on $\varphi$-invariant subrings, then $R$ is finite.

PROOF. Using the ascending chain condition there is a proper maximal $\varphi$-invariant subring $S$. By Theorem 8, it suffices to show that $S$ is finite. By Zorn’s Lemma and the descending chain condition there is a finite maximal descending chain of $\varphi$-invariant subrings $S = S_0 \supseteq S_1 \supseteq \cdots \supseteq S_{n+1} = 0$, with all inclusions proper. The argument in Corollary 3 now shows that $S$ must be finite.

We do not know if Theorem 8 holds without any additional condition on $\varphi \in \text{Aut}(R) \cup \text{Aut}^*(R)$. Indeed, as we mentioned earlier, it would be interesting to know even if Theorem 1 holds in this case. That is, when $R$ is infinite, must there always be a nonzero proper $\varphi$-invariant subring for any given (anti-) automorphism $\varphi$?

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