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Group Actions and Codes

V. Puppe

Abstract. A $\mathbb{Z}_2$-action with "maximal number of isolated fixed points" (*i.e.*, with only isolated fixed points such that $\dim_k(\oplus_i H^i(M;k)) = |M^{\mathbb{Z}_2}|, k = \mathbb{F}_2$) on a 3-dimensional, closed manifold determines a binary self-dual code of length $= |M^{\mathbb{Z}_2}|$. In turn this code determines the cohomology algebra $H^*(M;k)$ and the equivariant cohomology $H^*_{\mathbb{Z}_2}(M;k)$. Hence, from results on binary self-dual codes one gets information about the cohomology type of 3-manifolds which admit involutions with maximal number of isolated fixed points. In particular, "most" cohomology types of closed 3-manifolds do not admit such involutions. Generalizations of the above result are possible in several directions, *e.g.*, one gets that "most" cohomology types (over $\mathbb{F}_2$) of closed 3-manifolds do not admit a non-trivial involution.

This note is concerned with certain aspects of the following statement and question due to F. Raymond and R. Schultz: "It is generally felt that a manifold 'chosen at random' will have very little symmetry. Can this intuitive notion be made more precise? ..." (s. [Pu2] and the references given there for further comments and results in this direction). Here we mainly consider involutions on closed 3-manifolds, but—in principle—similar arguments could be applied to $\mathbb{Z}_p$- or S^1-actions on closed manifolds.

1 *m*-Involutions and Filtrations

Let X be a connected finite-dimensional space with an involution, which has only isolated fixed points. If the number of isolated fixed points is maximal, *i.e.*, if it equals the total dimension of $H^*(X;k)$ as a (graded) k-vector space, $k = \mathbb{F}_2$, we call the involution an m-involution for the purpose of this note. In this case—using the Borel construction and the localization theorem for equivariant cohomology—one gets that the inclusion of the fixed point set, $X^G \hookrightarrow X$, induces an injection of $k[t]$-algebras $H^*_G(X) \to H^*_G(X^G) \cong k^n \otimes k[t]$, $k[t] = H^*_G(pt) = H^*(BG)$, which becomes an isomorphism after inverting t. Here $G = \mathbb{Z}_2, H^*(-)$ denotes Čech-cohomology with coefficients in $k = \mathbb{F}_2$, $n = \dim_k \oplus_i H^i(X)$ (*cf.*, *e.g.*, [AP], Chapter 1, Section 3).

We therefore study the following algebraic situation. Let $\mathcal{A}^*$ (corresponding to $H^*_G(X)$) be a graded connected algebra, flat over $k[t]$, with a degree preserving injection $\iota\colon \mathcal{A}^* \to k^n \otimes k[t]$, $\deg(t) = 1$, (corresponding to $H^*_G(X) \to H^*_G(X^G)$) where k^n is concentrated in degree 0 and is equipped with the componentwise multiplication $\nu\colon k^n \times k^n \to k^n$, $\nu\big((a_1,\dots,a_n),(b_1,\dots,b_n)\big) = (a_1b_1,\dots,a_nb_n)$. Assume that after localization with respect to t the map ι becomes an isomorphism. Let $A^* := \mathcal{A}^* \bigotimes_{k[t]} k_0$ (corresponding to $H^*(X) \cong H^*_G(X) \bigotimes_{k[t]} k_0 = H^*_G(X)/tH^*_G(X)$),

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where $k_0 = k[t]/tk[t]$ as $k[t]$-module, and $d := \max\{i; A^i \neq 0\}$ (corresponding to the cohomological dimension $\mathrm{cd}(X) := \max\{i; H^i(X) \neq 0\}$). We assume that d is finite. Then $F_j(k^n) := \{y \in k^n; y \otimes t^j \in \operatorname{im} \iota\}$ defines a multiplicative filtration on k^n with

$$F_{-1}(k^n) = 0 \subset F_0(k^n) = \langle \mathbf{1} \rangle \subset \cdots \subset F_d(k^n) = k^n$$

where $\langle \mathbf{1} \rangle$ denotes the k-vector space generated by the unit $\mathbf{1} := (1, \ldots, 1) \in k^n$.

The graded algebra $\bigoplus_{i=0}^{d} F_i(k^n)/F_{i-1}(k^n)$ associated to the filtration $F_*(k^n)$ is isomorphic to A^*. Moreover, $F_*(k^n)$ determines the injection $\iota\colon \mathcal{A}^* \to k^n \otimes k[t]$, since $\mathcal{A}^*$ can be considered as the subalgebra of $k^n \otimes k[t]$, which is generated by all elements of the form $y \otimes t^j$ with $y \in F_j(k^n)$ (*cf.* [Pu1], (1.15)–(1.16)).

Remark 1 For $k = \mathbb{F}_2$, every element in k^n is idempotent, *i.e.*, $v^2 = v$ for all $v \in k^n$. As a consequence, $a^2 = 0$ for all $a \in A^i$, $i > 0$.

For the topological situation this gives

Corollary 1 *If X is a connected finite-dimensional topological space of cohomological dimension d, which admits an m-involution, then x^2 vanishes for all $x \in \tilde{H}^*(X)$.*

We are mainly interested in the case where A^* is a Poincaré algebra of formal dimension d (*i.e.*, in more algebraic terms, a graded Gorenstein Artin algebra of socle degree d), corresponding to the topological situation that X is a Poincaré duality space over k, *e.g.*, a closed d-dimensional manifold. Then the orientation $\sigma\colon A^* \to k$ defined by $\sigma|_{A^i} = 0$ for $i \neq d$, $\sigma|_{A^d}\colon A^d \xrightarrow{\cong} k$ (which, together with the multiplication $A^* \times A^* \to A^*$ gives a dual pairing on A^*) can be extended to a $k[t]$-linear map $\tilde{\sigma}\colon \mathcal{A}^* \to k[t]$, which induces an orientation $\bar{\sigma}\colon k^n \to k$, by evaluating at $t = 1$. (Note that $\mathcal{A}^* \to k^n \otimes k[t]$ induces an isomorphism after evaluating at 1, because $\mathcal{A}^*[t^{-1}] \to k^n \otimes k[t, t^{-1}]$ is an isomorphism.)

It turns out that $\bar{\sigma}(e_i) = 1$ for $e_i := (0 \cdots 0, 1, 0 \cdots 0) \in k^n$, $i = 1, \ldots, n$; $\bar{\sigma}(\mathbf{1}) = 0$ and $F_{d-1}(k^n) = \ker \bar{\sigma}$. More generally: $F_j^{\perp} = F_{d-j-1}$, where the orthogonal complement is taken with respect to the non-degenerate bilinear form $s\colon k^n \times k^n \xrightarrow{\nu} k^n \xrightarrow{\bar{\sigma}} k$. (*cf.* [AP] Chapter 5, Section 1 and 2 and [Pu2] for analogous situations, coming from group actions.)

We want to use this algebraic data to get necessary conditions for the existence of m-involutions on closed manifolds or finite-dimensional Poincaré duality spaces. Taking d-fold products of elements in A^1, the Poincaré algebra A^* gives rise to a symmetric multilinear form of degree d, where $d := \max\{i, A^i \neq 0\}$ is the formal dimension of A^*. This d-form is non-degenerate if and only if A^* is generated by A^1, in which case the multiplication on A^* can be recovered from the d-form (*cf.* Remark 2 below for the case $d = 3$). In terms of the above filtration the d-form is given by

$$F_1/F_0 \times \cdots \times F_1/F_0 \xrightarrow{\bar{\nu}} F_d/F_{d-1} \xrightarrow{\bar{\sigma}} k$$

where the first map, $\bar{\nu}$, is the d-fold multiplication induced by the multiplication, ν, of k^n, and the second is induced by the orientation $\bar{\sigma}\colon k^n \to k$.

Definition A based multiplicative self-dual filtration of length d in k^n is a filtration $F_{-1} = 0 \subset F_0 = \langle \mathbf{1} \rangle \subset F_1 \cdots \subset F_{d-1} \subset F_d = k^n$, together with a basis $\bar{v}_1, \dots, \bar{v}_m$ of F_1/F_0, such that $\nu(F_i \times F_j) \subset F_{i+j}$ and $F_j^{\perp} = F_{d-j-1}$, where the orthogonal complement is taken with respect to the non-degenerate bilinear form $s\colon k^n \times k^n \xrightarrow{\nu} k^n \xrightarrow{\bar{\sigma}} k$. (Note that n must be even, since $s(\mathbf{1}, \mathbf{1}) = \sum_{i=1}^n \bar{\sigma}(e_i)$ has to vanish in $k = \mathbb{F}_2$.) Let $\mathcal{F}^d_{m,n}$ denote the space of all based multiplicative self-dual filtrations of length d in k^n. Let $\mathcal{S}^d(k^m)$ denote the space of symmetric d-forms on k^m. A based filtration $F_* \in \mathcal{F}^d_{m,n}$ gives rise to a symmetric d-form $\mathcal{M}(F_*) \in S^d(k^m)$ given on the standard basis vectors e_i, $i = 1, \dots, m$, of k^m by $\mathcal{M}(F_*)(e_{i_1}, \dots, e_{i_d}) := \bar{\sigma}\big(\bar{\nu}(\bar{v}_{i_1} \cdots \bar{v}_{i_d})\big)$, with $\bar{v}_{i_1} \cdots \bar{v}_{i_d} \in F_d/F_{d-1}$. In total, this defines a map $\mathcal{M}\colon \mathcal{F}^d_{m,n} \to \mathcal{S}^d(k^m)$.

We want to study the properties of the map $\mathcal{M}$. Clearly $\mathcal{M}$ is equivariant with respect to the $\mathrm{GL}(m; k)$-actions on $\mathcal{F}^d_{m,n}$ and $\mathcal{S}^d(k^m)$ given by base change. But $\mathcal{M}$ is also invariant with respect to the action of the symmetric group S_n on $\mathcal{F}^d_{m,n}$ given by permuting coordinates in k^n. The group S_n can be viewed as the automorphism group of the algebra k^n (equipped with the componentwise multiplication ν).

The image of $\mathcal{M}$ is contained in $\mathcal{S}^d_{\prime}(k^m) := \{\mu \in S^d(k^m);\ \mu(e_{i_1}, \dots, e_{i_\alpha}) = 0$ if $i_r = i_s$ for $s \neq r\}$, since $vv = v$ for all $v \in k^n$. In particular, $v_i v_i \in F_1$ if $v_i \in F_1$ (*cf.* Remark 1).

2 *m*-Involutions and Codes

We will be mainly interested in the case $d = 3$, which corresponds to m-involutions on closed 3-manifolds, but we start with the cases $d = 1, 2$ for illustration.

(1) The case $d = 1$ is trivial: $m = 1$, $n = 2$, since $F_1/F_0 \cong k$. On the topological side this corresponds to a reflection of the 1-dimensional sphere S^1.

(2) Let $d = 2$. Then $n = m + 2$, both even, and there is just one $\mathrm{GL}(m; k)$-orbit in $\mathcal{F}^2_{m,n}$ represented by $F_0 = \langle \mathbf{1} \rangle \subset F_1 = \ker \bar{\sigma} \subset F_2 = k^n$ and a chosen basis $\bar{v}_1, \dots, \bar{v}_m$ of F_1/F_0. The following vectors v_1, $i = 1, \dots, m$ in $\ker \bar{\sigma}$ represent a basis $\bar{v}_1, \dots, \bar{v}_m$ in $\ker \bar{\sigma}/\langle \mathbf{1} \rangle$:

$$v_i := \begin{cases} (1^i, 0, 1, 0^{n-i-2}) & \text{for } i \text{ odd} \\ (1^i, 0^{n-i}) & \text{for } i \text{ even}, \text{ where } 1^i := (1, \dots, 1) \in k^i, \textit{etc.} \end{cases}$$

The symmetric bilinear form, which is obtained by applying the map $\mathcal{M}$ to this element in $\mathcal{F}^2_{m,n}$, is represented by the following $(m \times m)$-matrix

$$\begin{pmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{pmatrix} \quad \text{(all other entries are zero)}$$

So of the two $\mathrm{GL}(m,k)$-orbits in $\mathcal{S}^2(k^m)$, consisting of non-degenerate 2-forms, only one is in the image of $\mathcal{M}\colon \mathcal{F}^2_{m,m+2} \to \mathcal{S}^2(k^m)$. On the topological level one has the classical result that all non-degenerate forms in $\mathcal{S}^2(k^m)$, ($k = \mathbb{F}_2$), can be realized as the cup-product form (with coefficients k) of compact surfaces (s. [MH], Chapter V, 1). But only connected sums of tori (including the empty sum $= S^2$) admit m-involutions.

(3) $d = 3$. It follows that $n = 2m+2$, and a based multiplicative self-dual filtration $F_0 = \langle \mathbf{1} \rangle \subset F_1 \subset F_2 = \ker \bar{\sigma} \subset F_3 = k^n$ is completely determined by the $(m+1)$-dimensional subspace $F_1 \subset k^n$, with the property $F_1^{\perp} = F_1$, and a choice of a basis in F_1/F_0.

Note that the condition $F_1^{\perp} = F_1$ already implies that $\mathbf{1} \in F_1$, for F_1 is a maximal isotropic subspace with respect to the non-degenerate bilinear form on k^n; and if V is any isotropic subspace of k^n, so is $V + \langle \mathbf{1} \rangle$.

Hence in this case the space $\mathcal{F}^3_{m,n}$ is just the space of maximal isotropic subspaces of k^n (with respect to the non-degenerate bilinear form given by the component-wise multiplication, ν, and the orientation $\bar{\sigma}\colon k^n \to k, \bar{\sigma}(e_i) = 1$ for $i = 1, \dots, n$) together with a chosen basis of F_1/F_0, or in terms of coding theory:

The orbit space, $\bar{\mathcal{F}}^3_{m,n} := \mathrm{GL}(m;k) \setminus \mathcal{F}^3_{m,n}$ with respect to the action of $\mathrm{GL}(m;k)$ by base change on F_1/F_0 is the space of all binary self-dual codes of length $n = 2m+2$, *i.e.*, the space of maximal isotropic subspaces of k^n. Its quotient $\tilde{\mathcal{F}}^3_{m,n} := \mathrm{GL}(m;k) \setminus \mathcal{F}^3_{m,n}/S_n$ with respect to the action of the symmetric group can on one hand be viewed as the space of the equivalence classes of binary self-dual codes (*cf.*, *e.g.*, [Pl]), on the other hand as the space of equivalence classes of embeddings $\mathcal{A}^* \hookrightarrow k^n \otimes k[t]$ as above, where $A^* := \mathcal{A}^* \otimes_{k[t]} k_o$ is a Poincaré algebra of formal dimension $d = 3$, $A^1 \cong k^m$, and the equivalence relation is taken with respect to automorphisms of the $k[t]$-algebra $k^n \otimes k[t]$.

Remark 2 For $d = 3$ one can identify $\mathcal{S}^3(k^m)$ with the space of all Poincaré algebras A^* of formal dimension $d = 3$ and $A^1 = k^m$. Those algebras which are generated by A^1 correspond to non-degenerate forms in $\mathcal{S}^3(k^m)$.

M. Postnikov (s.[Po]) has shown that every form in $S^3_{\prime}(k^m)$ can be realized as the cup-product form of a closed orientable 3-manifold. In fact, he showed that every $\mu \in S^3(k^m)$, which fulfills the condition $\mu(e_i, e_i, e_j) = \mu(e_i, e_j, e_j)$ (in $\mathbb{F}_2$) for all $i, j = 1, \dots, m$, can be realized by a closed orientable 3-manifold. Every form $\mu \in S^3(k^m)$ with $\mu(e_i, e_i, e_j) + \mu(e_i, e_j, e_j) = \mu(e_1, e_i, e_j)$ for all $i, j = 1, \dots, m$, can be realized by a non-orientable closed 3-manifold; here e_1 corresponds to the (non-zero) first Stiefel-Whitney class.

On the other hand, if a 3-manifold M admits an m-involution, then its cup-product form must be in the image of $\mathcal{M}$. We want to use just a rough estimate for the size of $\mathcal{M}(\mathcal{F}^3_{m,n})$ in $\mathcal{S}^3_{\prime}(k^m)$ in order to show that "most" closed 3-manifolds do not admit an m-involution.

The number of elements in $\mathcal{F}^3_{m,n}$ is given by

$$|\mathcal{F}^3_{m,n}| = \prod_{i=1}^{m} (2^{2i} - 1) 2^{m-i} < 2^{\frac{3m^2+m}{2}}.$$

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On the other hand $|\mathcal{S}^3_\prime(k^m)| = 2^{\binom{m}{3}}$.

Hence $\frac{|\mathcal{F}^3_{m,n}|}{|\mathcal{S}^3_\prime(k^m)|} \leq \frac{2^{\frac{3m^2+m}{2}}}{2^{\binom{m}{3}}} = 2^{-\frac{1}{6}(m^3-12m^2-m)}$ and $\lim_{m\to\infty} \frac{|\mathcal{F}^3_{m,n}|}{|\mathcal{S}^3_\prime(k^m)|} = 0$.

Let $\bar{S}^3_\prime(k^m)$ denote the orbit space of the $\mathrm{GL}(m;k)$-action given on $S^3_\prime(k^m)$ by base change. In [W], p. 227 the size of $\bar{S}^3_\prime(k^m)$ is given for $m \leq 6$ in the context of the classification of fake tori up to PL-homeomorphism (in somewhat different terminology). One has:

$|\bar{S}^3_\prime(k^1)| = |\bar{S}^3_\prime(k^2)| = 1$
$|\bar{S}^3_\prime(k^3)| = |\bar{S}^3_\prime(k^4)| = 2$
$|\bar{S}^3_\prime(k^5)| = 3, |\bar{S}^3_\prime(k^6)| = 6.$

To calculate $|\bar{S}^3_\prime(k^m)|$ for large m seems a difficult problem. But a rough estimate shows that most equivalence classes of $\mathcal{S}^3_\prime(k^m)$ are not in the image of the induced map

$$\bar{\mathcal{M}}\colon \bar{\mathcal{F}}^3_{m,n} := \mathrm{GL}(m;k) \setminus \mathcal{F}^3_{m,n} \to \mathrm{GL}(m;k) \setminus \mathcal{S}^3_\prime(k^m) =: \bar{\mathcal{S}}^3_\prime(k^m).$$

Recall that this map factors through the quotient $\tilde{\mathcal{F}}^3_{m,n} := \mathrm{GL}(m;k) \setminus \mathcal{F}^3_{m,n}/S_n$ with respect to the action of the symmetric group. We denote the induced map by $\tilde{\mathcal{M}}$.

Since $\mathrm{GL}(m;k)$ acts freely on $\mathcal{F}^3_{m,n}$, one has as a very rough estimate

$$\frac{|\bar{\mathcal{F}}^3_{m,n}|}{|\bar{\mathcal{S}}^3_\prime(k^m)|} \leq \frac{|\mathcal{F}^3_{m,n}|}{|\mathcal{S}^3_\prime(k^m)|} = 2^{-\frac{1}{6}(m^3-12m^2-m)},$$

in particular,

$$\lim_{m\to\infty} \frac{|\bar{\mathcal{M}}(\bar{\mathcal{F}}^3_{m,n})|}{|\bar{\mathcal{S}}^3_\prime(k^m)|} = 0.$$

Corollary 2 *For large enough m, most $\mathbb{F}_2-$cohomology types of closed 3-manifolds do not admit an m-involution.*

Remark 3 In [Pl], [PS1], [PS2], [CP] and [CPS] binary self-dual codes up to length 32 are classified. Let $F_1 \oplus F_1'$ be the direct sum of two binary self-dual codes F_1 and F_1' of length n and n', respectively. If $\bar{v}_1, \ldots, \bar{v}_m$ and $\bar{v}'_1, \ldots, \bar{v}'_{m'}$ are bases of $F_1/\langle \mathbf{1}_n \rangle$ and $F_1'/\langle \mathbf{1}_{n'} \rangle$ respectively, where $\mathbf{1}_n \in k^n$ and $\mathbf{1}_{n'} \in k^{n'}$, then we can choose $\overline{(\mathbf{1}_n, 0)}, \overline{(v_1, 0)}, \ldots, \overline{(v_m, 0)}, \overline{(0, v'_1)}, \ldots, \overline{(0, v'_{m'})}$ as a basis of $(F_1 \oplus F_1')/\langle \mathbf{1}_{n+n'} \rangle$, where the v_i and $v'_{i'}$ are representatives of the $\bar{v}_i$ and $\bar{v}'_{i'}$, and $(v_i, 0) \in k^n \oplus k^{n'}, (0, v'_i) \in k^n \oplus k^{n'}$. The 3-form corresponding to $F_1 \oplus F_1'$ (and the above basis) under the map $\mathcal{M}$ can be easily described in terms of 3-forms corresponding to F_1 and F_1'. If one defines the "connected sum" $\mu \# \mu'$ of two 3-forms μ, μ' on k^m and $k^{m'}$, by

$$(\mu\#\mu')(e_i, e_j, e_k) = \mu(e_i, e_j, e_k) \quad \text{for } e_i, e_j, e_k \in k^m$$

$$(\mu\#\mu')(e'_i, e'_j, e'_k) = \mu'(e'_i, e'_j, e'_k) \quad \text{for } e'_i, e'_j, e'_k \in k^{m'} \text{ and otherwise equal to zero}$$

then $\mathcal{M}(F_1 \oplus F_1') = \mu_0^1 \# \mathcal{M}(F_1) \# \mathcal{M}(F_1')$, where μ_0^1 is the trivial form on k^1. Note that on the other hand the "connected sum" operation on forms corresponds to the connected sum operation on manifolds, *i.e.*, $\mu(M\#M') = \mu(M)\#\mu(M')$, where μ denotes the form given by the cup-product of the manifold. If M and M' are manifolds with differentiable m-involution, then taking the connected sum at fixed points one can see that $M\#M'$ admits an m-involution.

The trivial form, μ_0^1, is realized by $S^1 \times S^2$ with an m-involution, being the diagonal operation corresponding to the obvious m-involutions on S^1 and S^2 $\big((x_0, x_1) \mapsto (x_0, -x_1)$ for $(x_0, x_1) \in S^1 \subset \mathbb{R}^2$; $(x_0, x_1, x_2) \mapsto (x_0, -x_1, -x_0)$ for $(x_0, x_1, x_2) \in S^2 \subset \mathbb{R}^3\big)$. So, if F_1 and F_1' are realized by two manifolds with m-involutions M and M', respectively, then $F_1 \oplus F_1'$ can be realized by $M\#(S^1 \times S^2)\#M'$.

It is shown in [S], Corollary 2.9 (in terms of Poincaré algebras) that any form in $S^3(k^m)$ has a unique (up to order) connected sum decomposition into indecomposable forms.

Examples Using the notation of [Pl] one gets:

$$\begin{aligned} \bar{\mathcal{M}}(C_2 \oplus D_{14}) &= \bar{\mu}_0^1 \# \bar{\mathcal{M}}(C_2) \# \bar{\mathcal{M}}(D_{14}) \\ &= \bar{\mu}_0^1 \# \bar{\mathcal{M}}(A_8) \# \bar{\mathcal{M}}(A_8) \\ &= \bar{\mathcal{M}}(A_8 \oplus A_8), \end{aligned}$$

since $\bar{\mathcal{M}}(C_2) = 0$ and $\bar{\mathcal{M}}(D_{14}) = \bar{\mathcal{M}}(A_8)\#\bar{\mathcal{M}}(A_8)$. Similarly one can see that $\bar{\mathcal{M}}(C_2 \oplus I_{18}) = \bar{\mathcal{M}}(A_8 \oplus B_{12})$. In particular, $\tilde{\mathcal{M}} \colon \tilde{\mathfrak{F}}^3_{m,n} \to \bar{\mathbb{S}}^3_{\prime}(k^m)$ is not injective for $m \geq 7$. But for $m \leq 6$ the map $\tilde{\mathcal{M}}$ is injective. This can be seen by checking that the "rank distribution" of the graded algebra A^* which corresponds to $\bar{\mathcal{M}}(c)$, *i.e.*, the map that assign to each element $a \in A^1$ the rank of the linear map $m_a \colon A^1 \to A^2$, given as multiplication by a, is different for inequivalent codes $c \in \bar{\mathfrak{F}}^3_{m,n}$ if $m \leq 6$. It is not difficult to check that $\tilde{\mathcal{M}}$ is indeed a bijection for $m \leq 5$. On the other hand $\tilde{\mathcal{M}}$ can not be surjective for $m = 6$ since $|\tilde{\mathfrak{F}}^3_{6,14}| = 4$ (s. [Pl]) and $|\bar{S}^3_{\prime}(k^6)| = 6$ (s. [W], p. 227). Alternatively, a straight forward calculation shows that the equivalence class of the form $\mu \in S^3_{\prime}(k^6)$, defined by $\mu_{123} = \mu_{145} = \mu_{246} = 1$, $\mu_{ijk} = 0$ otherwise, gives an algebra A^* with a rank distribution which differs from the rank distributions coming from the four equivalence classes in $\tilde{\mathfrak{F}}^3_{6,14}$ of the codes $C_2^7, C_2^3 \oplus A_8, C_2 \oplus B_{12}, D_{14}$.

Similar considerations (s. [Pi] for details) show that $\tilde{\mathcal{M}}$ is also not surjective for $m = 7$ and 8. Using the classification of binary self-dual codes up to length 30 in [Pl], [PS1], [PS2], [CP] and [CPS] instead of the very rough estimate $|\tilde{\mathfrak{F}}^3_{m,n}| \leq |\bar{\mathfrak{F}}^3_{m,n}|$ above, and on the other hand the estimates $|\operatorname{GL}(m;k)| \leq 2^{m^2} \cdot 0{,}29$ for $m \geq 9$ $\Big(|\operatorname{GL}(m;k)| = \prod_{i=1}^m (2^m - 2^{m-i})$, so $\frac{|\operatorname{GL}(m;k)|}{2^{m^2}} = \prod_{i=1}^m (1 - \frac{1}{2^i})$ decreases, as m increases, and $\prod_{i=1}^9 (1 - \frac{1}{2^i}) < 0{,}29\Big)$ and $|\bar{S}^3_{\prime}(k^m)| \geq \frac{|S^3_{\prime}(k^m)|}{|\operatorname{GL}(m;k)|}$, one gets the following table:

m	9	10	11	12	13	14
$\lvert\tilde{\mathfrak{F}}^3_{m,n}\rvert$	16	25	55	103	261	731
$\lvert\bar{S}^3_{\prime}(k^m)\rvert$	$> 3 \cdot 2^{3}$	$> 3 \cdot 2^{20}$	$> 3 \cdot 2^{44}$	$> 3 \cdot 2^{76}$	$> 3 \cdot 2^{117}$	$> 3 \cdot 2^{168}$

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In particular, $\frac{|\bar{\mathcal{M}}(\bar{\mathcal{F}}^3_{14,30})|}{|\bar{S}^3_{,}(k^{14})|} \leq 2^{-160}$, while the above very rough estimate only gives 2^{-63} as an upper bound for this quotient.

Summarizing, we obtain the following properties of $\tilde{\mathcal{M}}$:

$$\tilde{\mathcal{M}} \quad \text{injective} \iff m \leq 6$$

$$\tilde{\mathcal{M}} \quad \text{surjective} \iff m \leq 5.$$

We remark here that the rank distribution corresponding to $\bar{\mathcal{M}}(c)$ is not determined, though somewhat related, to the weight distribution of the code c; *e.g.*, the codes $A_8 \oplus A_8$ and E_{16} have the same weight distribution (see, *e.g.*, [Pl]) but the algebras corresponding to $\bar{\mathcal{M}}(A_8 \oplus A_8)$ and $\bar{\mathcal{M}}(E_{16})$ have different rank distributions: $\bar{\mathcal{M}}(E_{16})$ gives an element a (corresponding to line 8 of the listing of E_{16} in [Pl]) with $rk(m_a) = 6$, while for all elements $a \in A^1$ belonging to $\bar{\mathcal{M}}(A_8 \oplus A_8)$ one gets $rk(m_a) \leq 4$.

(4) The case $d = 4$.

In this case, too, the filtration $\langle \mathbf{1} \rangle = F_0 \subset F_1 \subset F_2 \subset F_3 \subset F_4 = k^n$ is completely determined by the properties of the subspace $F_1 \subset k^n$,*i.e.*,

(i) $\mathbf{1} \in F_1$
(ii) $F_1 \times F_1 \to F_1^{\perp} = F_2$
(iii) $F_3 = \ker \bar{\sigma}$

In terms of coding theory $F_1 \subset k^n$ is a self-orthogonal (*i.e.*, $F_1 \subset F_1^{\perp}$), binary code of dimension m and length n. That F_1 is self-orthogonal follows from (i) and (ii) (and the fact that $F_1 \subset F_2$), but in particular (ii) imposes an additional restriction on the code. It might be interesting to classify these kinds of self-orthogonal codes, and to see, what kind of restriction for closed 4-manifolds with m-involution can be derived from this classification. On the topological side one would have to deal with the question: Which 4-forms can be realized by cohomology algebras of closed 4-manifolds?

(5) The cases with $d > 4$ get more and more involved, but there are still connections with coding theory, *e.g.*, if $d = 2d' + 1$, then the term $F_{d'}$ in the filtration F_* is a self-dual code. But the d-form corresponding to F_* is determined by F_1 (and a chosen basis of F_1/F_0).

3 General Involutions

To show that most closed 3-manifolds cannot admit any non-trivial involution, we follow the strategy which was applied in [Pu2] (s. p. 285 and Remarks 1.3.), slightly modified. A non-trivial involution on a manifold M induces an involution on $H^*(M;k)$. If this is trivial (*i.e.*, the action is cohomologically trivial), then the first non-vanishing differential in the Serre spectral sequence of the Borel construction $M \to M_G := M \times EG \to BG$, $G = \mathbb{Z}_2$, would correspond to a non-trivial derivation on $H^*(X;k)$ of negative degree. If the involution on $H^*(M;k)$ is trivial and the Serre spectral sequence collapses at the $\mathbb{E}_2$-level, then $\dim \oplus_i H^i(M^G) = \dim \oplus_i H^i(M)$, which is equivalent to $H^*(M;k)$ being TNHZ (totally non-homologous to zero) in $H^*(M_G;k)$ (see, *e.g.*, [AP], (3.10.4)). While the classification of binary self-dual codes gives rather precise information about the algebras, which might occur as cohomology algebras of closed 3-manifolds, which admit m-involutions, the counting argument, which proves Corollary 2, can be generalized to the TNHZ case (s. (a) below). So, in view of the above strategy, we show that most algebras in question do not admit automorphisms of order 2 (s. (b) below). Therefore an involution on a corresponding manifold must be cohomologically trivial (in most cases). Finally we prove, that most algebras in question do not admit non-trivial derivations of negative degree (s. (c) below). This reduces the problem to the TNHZ case. Combining these results one gets

Theorem *Most $\mathbb{F}_2$-cohomology algebra types of closed 3-manifolds do not admit non-trivial involutions.*

(a) The* TNHZ*-case If X is a Poincaré duality space over $k = \mathbb{F}_2$ of cohomological dimension d with an involution, such that $\dim \oplus_i H^i(X^G) = \dim \oplus_i H^i(X)$, then again—applying equivariant cohomology—this leads to an embedding of $k[t]$-algebras

$$H_G^*(X) \xrightarrow{\iota} H_G^*(X^G) = H^*(X^G) \otimes k[t],$$

which becomes an isomorphism after inverting t. But the fixed point components $F_\gamma \subset X^G$ need not be points. By a result of Chang-Skjelbred and Bredon (see, *e.g.*, [AP]), one has $H^*(X^G) = \prod_\gamma H^*(F_\gamma)$, where F_γ is a Poincaré duality space over k of cohomological dimension $d_\gamma \leq d$. So the algebraic situation looks as follows:

One has an embedding of $k[t]$-algebras

$$\mathcal{A}^* \xrightarrow{\iota} \Big(\prod_\gamma B_\gamma^*\Big) \otimes k[t],$$

which becomes an isomorphism after localization with respect to t; $A^* := \mathcal{A}^* \otimes_{k[t]} k_0$ and the B_γ^* are Poincaré algebras. Most of what we have discussed in the case where $\prod_\gamma B_\gamma^* = k^n$ can be generalized.

In case $d = 3$, for example, one is lead to discuss the following generalization of binary self-dual codes. Consider an orientation $\bar\sigma\colon B^* = \prod_\gamma B_\gamma^* \to k$, where the B_γ^* are Poincaré algebras of formal dimension $d_\gamma < d = 3$ and length n_γ, and the restriction $\bar\sigma_\gamma\colon B_\gamma^* \to k$ of the orientation $\bar\sigma$ to B_γ^* gives a dual pairing $B_\gamma^* \times B_\gamma^* \to B_\gamma^* \xrightarrow{\bar\sigma_\gamma} k$ on B_γ^*. Consider all self-dual subspaces $F_1 \subset \prod B_\gamma^* = B^*$ with respect to the dual pairing $B^* \times B^* \to B^* \xrightarrow{\bar\sigma} k$.

To the knowledge of the author "codes" of this kind have not been studied systematically in coding theory. In particular there is no classification at hand for small length. But counting arguments, similar to those above, show that the 3-forms obtained from those self-dual subspaces can only make up a small part of the space of all 3-forms (*cf.* [Pu2] Proposition 3 and its proof). This implies that "most" $\mathbb{F}_2$- cohomology algebra types of closed 3-manifolds do not admit an involution such that $\dim \oplus_i H^i(X^G) = \dim \oplus_i H^i(X)$, which is the *TNHZ*-case.

(b) Automorphisms of order 2 We want to show that most Poincaré algebras of formal dimension 3 do not admit non-trivial involutions, *i.e.*, non-trivial automorphisms of order 2 (*cf.* [Pu2] p. 288–289, for similar arguments in case $k = \mathbb{Q}$).

Proposition 1 *Let $A_2(m) \subset S^3(k^m)$ be the subset of forms, such that the corresponding Poincaré algebras admit non-trivial involutions. Then there exists a $m_0 \in \mathbb{N}$ such that*

$$\frac{|A_2(m)|}{|S^3(k^m)|} \leq \left[\frac{m}{2}\right] 2^{-\frac{1}{2}(m^2-9m+12)} \quad \textit{for } m \geq m_0;$$

in particular,

$$\lim_{m\to\infty} \frac{|A_2(m)|}{|S^3(k^m)|} = 0.$$

Proof For every $r = 1, \dots, [\frac{m}{2}]$ let $(v_1, \dots, v_r, \bar{v}_1, \dots, \bar{v}_r, w_1, \dots, w_s)$ denote the canonical basis of k^m $(s = m - 2r)$; define the involution

$$\begin{aligned} \varrho_r \colon k^m \to k^m \quad \text{by } \varrho_r(v_i) &= \bar{v}_i \quad \text{for } i = 1, \dots, r \\ \varrho_r(\bar{v}_i) &= v_i \quad \text{for } i = 1, \dots, r \\ \varrho_r(w_j) &= w_j \quad \text{for } j = 1, \dots, s \end{aligned}$$

Then $\{\varrho_r; r = 1, \dots, [\frac{m}{2}]\}$ gives a complete set of representatives of isomorphism classes of non-trivial $\mathbb{Z}_2$-representations on k^m $(m > 1)$. A Poincaré algebra $A^* = \bigoplus_{i=0}^3 A^i$ (over $k = \mathbb{F}_2$) admits a non-trivial involution if and only if the corresponding form $\mu \in S^3(k^m)$ is invariant under a non-trivial involution on k^m.

Hence $A_2(m) = \bigcup_{r=1}^{[\frac{m}{2}]} \mathrm{GL}(m;k)\big(S^3(k^m)\big)^{\varrho_r}$, where $S^3(k^m)^{\varrho_r}$ denotes the forms, which are invariant under ϱ_r, and $\mathrm{GL}(m;k) \times S^3(k^m) \to S^3(k^m)$ denotes the action given by base change.

Let $\alpha(m) := \binom{m+2}{3}$; then $\dim_k\big(S^3(k^m)\big)^{\varrho_r} = \alpha(s) + rs + \frac{1}{2}\Big(\alpha(m) - \big(\alpha(s) + rs\big)\Big) = \frac{1}{2}\big(\alpha(m) + \alpha(s) + rs\big)$.

The subspace $S^3(k^m)^{\varrho_r} \subset S^3(k^m)$ is invariant under the group $\mathrm{GL}(r;k) \times \mathrm{GL}(s;k)$, considered as a subgroup of $\mathrm{GL}(m;k)$ via the embedding

$$(A, B) \mapsto \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{pmatrix}.$$

Since $|\mathrm{GL}(m;k)| = 2^{m^2} \prod_{i=1}^m (1 - \frac{1}{2^i})$, one gets

$$\begin{aligned} &\big|\big(\mathrm{GL}(m;k)/\mathrm{GL}(r;k) \times \mathrm{GL}(s;k)\big)\big| \\ &\quad = 2^{m^2} \cdot \prod_{i=1}^m \Big(1 - \frac{1}{2^i}\Big) \Big/ 2^{r^2} \prod_{i=1}^r \Big(1 - \frac{1}{2^i}\Big) 2^{s^2} \prod_{i=1}^s \Big(1 - \frac{1}{2^i}\Big) \\ &\quad \leq 2^{m^2 - r^2 - s^2 + 2} \quad (\text{since } \textstyle\prod_{i=1}^r (1 - \frac{1}{2^i}) \geq 2^{-2} \text{ for all } r). \end{aligned}$$

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Therefore $\nu_r := \left|\mathrm{GL}(m;k)\big(S^3(k^m)\big)^{\varrho_r}\right| \le 2^{\frac{1}{2}\left(\alpha(m)+\alpha(s)+rs\right)+m^2-r^2-s^2+2}$.

Since $\nu_1 \ge \nu_r$ for $r = 1, \dots, [\frac{m}{2}]$ and m large enough ($m \ge 8$ suffices), one has

$$|A_2(m)| \le \left[\frac{m}{2}\right] 2^{\beta(m)},$$

where $\beta(m) = \frac{1}{2}\big(\alpha(m) + \alpha(m-2) + m - 2\big) + m^2 - (m-2)^2 + 1$. So $\frac{|A_2(m)|}{|S^3(k^m)|} \le [\frac{m}{2}]2^{\beta(m)-\alpha(m)} = [\frac{m}{2}]2^{-\frac{1}{2}(m^2-9m+8)}$ for $m \ge m_0$. ■

Let $\mathcal{R}^\sigma(m) \subset S^3(k^m)$ and $\mathcal{R}^u(m) \subset S^3(k^m)$ denote the subsets of forms that can be realized by the cohomology algebras of compact orientable and non-orientable 3-manifolds, respectively.

By the results of Postnikov [Po] one has:

$$\mathcal{R}^\sigma(m) = \{\mu \in S^3(k^m); \mu(v,v,w) + \mu(v,w,w) = 0 \text{ for all } v,w \in k^m\}, \quad \text{and}$$

$$\mathcal{R}^u(m) = \{\mu \in S^3(k^m); \exists v_o \in k^m, v \ne 0, \text{ such that}$$
$$\mu(v,v,w) + \mu(v,w,w) = \mu(v_o,v,w)\}.$$

We want to imitate the argument used to prove Proposition 1 to show that "most" algebras corresponding to forms in $\mathcal{R}^\sigma(m)$ and $\mathcal{R}^u(m)$ do not admit non-trivial involutions. We start with $\mathcal{R}^\sigma(m)$. Clearly

$$\dim_k \mathcal{R}^\sigma(m) = \binom{m+2}{3} - \binom{m}{2} =: \alpha'(m); \quad \text{and}$$

$$\dim_k \mathcal{R}^\sigma(m)^{\varrho_r} = \alpha'(s) + rs + \frac{1}{2}\big(\alpha'(m) - (\alpha'(s) + rs)\big) = \frac{1}{2}\big(\alpha'(m) + \alpha'(s) + rs\big)$$

Since

$$\mathcal{A}_2(m) \cap \mathcal{R}^\sigma(m) = \bigcup_{r=1}^{[\frac{m}{2}]} \mathrm{GL}(m;k)\mathcal{R}^\sigma(m)^{\varrho_r}, \quad \text{and}$$

$$\nu_1^\sigma \ge \nu_r^\sigma \quad \text{for } r = 1, \dots, \left[\frac{m}{2}\right] \text{ and } m \ge m_0,$$

where $\nu_r^\sigma := |\,\mathrm{GL}(m;k)\mathcal{R}^\sigma(m)^{\varrho_r}| \le 2^{\frac{1}{2}\left(\alpha'(m)+\alpha'(s)+rs\right)+m^2-r^2-s^2+2}$, one gets

$$|\mathcal{A}_2(m) \cap \mathcal{R}^\sigma(m)| \le \left[\frac{m}{2}\right] 2^{\beta'(m)} \quad \text{with}$$

$$\beta'(m) = \frac{1}{2}\big(\alpha'(m) + \alpha'(m-2) + (m-2)\big) + m^2 - (m-2)^2 + 1.$$

The calculation for $\mathcal{R}^u(m)$ instead of $\mathcal{R}^\sigma(m)$ is similar.

One therefore gets

Proposition 1′ *For $\mathcal{R}'(m) = \mathcal{R}^\sigma(m)$ or $\mathcal{R}^u(m)$ there exists a $m_0 \in \mathbb{N}$ such that*

$$\frac{|\mathcal{A}_2(m) \cap \mathcal{R}'(m)|}{|\mathcal{R}'(m)|} \le \left[\frac{m}{2}\right] 2^{-\frac{1}{2}(m^2-11m+11)} \quad \textit{for } m \ge m_0,$$

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in particular

$$\lim_{m\to\infty} \frac{|\mathcal{A}_2(m) \cap \mathcal{R}'(m)|}{|\mathcal{R}'(m)|} = 0. \qquad \blacksquare$$

(c) Derivations We want to show next that most Poincaré algebras over $\mathbb{F}_2$ of formal dimension 3 do not admit non-trivial derivations of negative degree. Let $A^* = \bigoplus_{i=0}^3 A^i$ denote a Poincaré algebra over $\mathbb{F}_2$ of formal dimension 3. Let $a_1, \dots, a_m$ denote a basis of A^1; $b_1, \dots, b_m$ the dual basis of A^2, and $c \in A^3$ the dual of $1 \in A^0$.

The product in A^* is completely determined by $a_i a_j = \sum_{k=1}^m \mu_{ijk} b_k$ for $i, j = 1, \dots, m$.

Remark 4 A^* is generated by A^1 if and only if the $m^2 \times m$-matrix (μ_{ijk}), $(i, j) \in \{1, \dots, m\}^2$, $k \in \{1, \dots, m\}$ has rank m.

If $d \colon A^* \to A^*$ is a derivation of degree -1 then

$$d(a_i)a_j + a_i d(a_j) = d(a_i a_j) = \sum_{k=1}^m \mu_{ijk} d(b_k)$$

If $d(a_i) := \alpha_i$ and $d(b_k) := \sum_l \beta_k^l a_l$ one gets

$$\alpha_i a_j + \alpha_j a_i = \sum_{k,l} \mu_{ijk} \beta_k^l a_l \quad \text{for all } i, j = 1, \dots, m$$

in particular, one gets

$$\sum_k \mu_{ijk} \beta_k^l = 0 \quad \text{for all } \neq l \neq j.$$

Remark 5 Assume that the $(m-1)^2 \times m$ matrices (μ_{ijk}), $(i, j) \in \{1, \dots, \hat{l}, \dots, m\}^2$, $k \in \{1, \dots, m\}$ have rank m for all l, as a consequence $\beta_k^l = 0$ for all $k, l = 1, \dots, m$. This implies first that $d(b_k) = 0$ for all b_k, and then that $d(a_i) = \alpha_i = 0$ for all i. Hence, there are no non-trivial derivations of degree (-1) under the assumption. Moreover there cannot exist non-trivial derivations of degree $< (-1)$ either, since A^* is generated by A^1 as seen in Remark 4.

Remark 6 The condition in Remark 5 is sufficient but not necessary to obtain that A^* has no non-trivial derivations of negative degree.

Proposition 2 *Let $\mathcal{D}(m) \subset S^3(k^m)$ denote those forms for which the corresponding Poincaré algebra admits a non-trivial derivation of negative degree. Then*

$$\lim_{m\to\infty} \frac{|\mathcal{D}(m)|}{|S^3(k^m)|} = 0$$

Proof Let $\bar{\mathcal{D}}(m) \subset S^3(k^m)$ denote those forms which do not fulfill the condition in Remark 5. Then, by Remark 6, $\mathcal{D}(m) \subset \bar{\mathcal{D}}(m)$ and hence it suffices to show that $\lim_{m\to\infty} \frac{|\bar{\mathcal{D}}(m)|}{|S^3(k^m)|} = 0$.

Assume that for a fixed l_o the $(m-1)^2 \times m$ matrix $M_{\hat{l}_o} := (\mu_{ijk}), (i, j) \in \{1, \dots, \hat{l}_o, \dots, m\}^2, k \in \{1, \dots, m\}$ has rank $< m$. We may assume $l_o = m$. Hence one of the columns of $M_{\hat{m}}$ can be expressed as a linear combination of the others. Assume this is the case for the p-th column. Then $\{\mu_{ijp}; i, j < m\}$ is determined by the $\{\mu_{ijk}; i, j < m, k \neq p\}$ and the coefficients of the linear combination. So the number of forms that fulfill these conditions is $\leq 2^{(m-1)} \cdot 2^{\binom{m+2}{3} - \binom{m}{2}} = 2^{\binom{m+2}{3} - \binom{m}{2} + m - 1}$.

Since l_0 and p above can be chosen independently between 1 and m, we obtain

$$|\bar{\mathcal{D}}(m)| \leq m^2 \cdot 2^{\binom{m+2}{3} - \binom{m}{2} + m - 1}.$$

So $\frac{|\bar{\mathcal{D}}(m)|}{|S^3(k^m)|} \leq m^2 2^{-\frac{1}{2}(m^2 - 3m + 2)}$ and the proposition follows. ■

Similar to the above proof one obtains

Proposition 2′ *For $\mathcal{R}'(m) = \mathcal{R}^\sigma(m)$ or $\mathcal{R}^u(m)$*

$$\lim_{m\to\infty} \frac{|\mathcal{D}(m) \cap \mathcal{R}'(m)|}{|\mathcal{R}'(m)|} = 0.$$ ■

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Fachbereich Mathematik und Statistik
Universität Konstanz
78457 Konstanz
Germany
e-mail: Volker.Puppe@uni-konstanz.de