PERIODIC QUEUES IN HEAVY TRAFFIC

G. I. FALIN,* Moscow State University

Abstract

An analytic approach to the diffusion approximation in queueing due to Burman (1979) is applied to the $M(t)/G/1/\infty$ queueing system with periodic Poisson arrivals. We show that under heavy traffic the virtual waiting time process can be approximated by a certain Wiener process with reflecting barrier at 0.

Introduction

An important property of periodic queues is the absence of a steady state in its usual sense. As a matter of fact, a special ‘periodic stationary regime’ exists. Consider for example the virtual waiting time process $W(t)$ in the $M(t)/G/1/\infty$ queue with FIFO discipline, Poisson arrival process with periodic intensity $\lambda_t$ (without loss of generality we can assume that the period equals 1) and general service time distribution function $B(x)$ with finite mean $\beta_1$ and variance $\sigma^2 = \beta_2 - \beta_1^2$. For example, $\lambda_t$ could be $\lambda + \beta \sin 2\pi t$ with $\lambda > 0$. If $\Lambda_1 = \int_0^1 \lambda_u \, du$, $\lambda = \Lambda_1$, $\rho = \lambda \beta_1 < 1$, then there exists the family $H_t(x)$ of distribution functions, periodic functions of $t$ (i.e. $H_{t+1}(x) = H_t(x)$), such that $\lim_{t \to \infty} [P(W(t) < x) - H_t(x)] = 0$. This fact is clearly seen from Figure 1 where the dependence of $E(W(t))$ on $t$ is given in the case $\lambda_t = 0.5(1 + \sin 2\pi t)$, and $B(x)$ is the uniform distribution $[0, 2]$ (results were obtained using an Atari 130 XE). This ‘periodic stationarity’ imposes essential mathematical difficulties on the analysis of queues. Determination of the functions $H_t(x)$ and even determination of the mean value $\int_0^1 x \, dH_t(x)$ can be made only with the help of a computer. To simplify the problem in practice, the ‘principle of the mean’ is usually used: a periodic queue is approximated by the corresponding stationary queue with arrival intensity $\lambda = \lim_{t \to \infty} \frac{1}{t} \int_0^t \lambda_u \, du = \Lambda_1$. Of course we have to find conditions when this approximation is reasonable.

In this note we shall show that this principle holds in heavy traffic in the sense of convergence of a scaled non-stationary process to a stationary process (which is in fact a reflected Brownian motion). To prove this we turn time heterogeneity into space heterogeneity (via a supplementary variable) which allows us to use the methodology of Burman [1].

Main result

Let us consider a sequence of $M(t)/G/1/\infty$ queues (indexed by a parameter $n$, although we shall usually omit it) with arrival rates $\lambda^{(n)}$. As a matter of convenience we suppose that the distribution of service time $B(x)$ does not depend on $n$.

* Postal address: Department of Probability, Mechanics and Mathematics Faculty, Moscow State University, Moscow 119899, USSR.

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Theorem. If as $n \to \infty$, $\lambda^{(n)} = \int_0^1 \lambda^{(n)} \, dt$ tends to $1/\beta_1 - 0$ so that $\sqrt{n}(\rho^{(n)} - 1)$, where $\rho^{(n)} = \lambda^{(n)} \beta_1$, tends to $-m$, then the scaled processes $W(nt)/\sqrt{n}$ converge (in the sense of convergence of finite-dimensional distributions) to a Brownian motion with reflecting barrier at the origin with infinitesimal mean $m$ and infinitesimal variance $\beta_2/(2\beta_1)$.

Proof. The main idea of the proof consists in turning time heterogeneity into space heterogeneity. To do this let us denote by $\tau(t)$ the fractional part of $t$. The process $(W(t), \tau(t))$ is a time-homogeneous Markov process with state space $R_+ \times [0, 1]$, but with transition characteristics which depend on the second coordinate $\tau$ of a point $(x, \tau)$ of the state space.

Another problem now arises. Under heavy traffic only the first coordinate of the process $(W(t), \tau(t))$ will converge to a diffusion process. But this difficulty can be removed by using the method of proving functional limit theorems in queueing theory due to Burman [1].

The infinitesimal generator of the process $(W(t), \tau(t))$ is expressed as follows:

$$ Af(x, \tau) = \lambda \int_0^x [f(x + u, \tau) - f(x, \tau)] \, dB(u) + f'(x, \tau) - f'(x, \tau), \quad \text{if} \quad x > 0, \quad 0 < \tau < 1, $$

$$ Af(0, \tau) = \lambda \int_0^1 [f(u, \tau) - f(0, \tau)] \, dB(u) + f'(0, \tau), \quad \text{if} \quad 0 < \tau < 1, $$

and operates on functions $f(x, \tau)$ satisfying the boundary condition $f(x, 1) = f(x, 0)$.

For the generator $A_n$ of the scaled process $(W(nt)/\sqrt{n}, \tau(nt))$ these formulas become:

$$ A_n f(x, \tau) = n\lambda \int_0^x [f\left(x + \frac{1}{\sqrt{n}} u, \tau\right) - f(x, \tau)] \, dB(u) + nf'(x, \tau) - \sqrt{n}f'_x(x, \tau) $$

$$ = \sqrt{n}\lambda f'_x(x, \tau)\beta_1 + \frac{1}{2}\lambda \beta_2 f''_{xx}(x, \tau) + nf'_x(x, \tau) - \sqrt{n}f'_x(x, \tau) + o(1), $$

if $x > 0, \ 0 < \tau < 1$; and

$$ A_n f(0, \tau) = n\lambda \int_0^1 \left[f\left(\frac{u}{\sqrt{n}}, \tau\right) - f(0, \tau)\right] \, dB(u) + nf'(0, \tau) $$

$$ = \sqrt{n}\lambda f'_x(0, \tau)\beta_1 + \frac{1}{2}\lambda \beta_2 f''_{xx}(0, \tau) + nf'_x(0, \tau). $$

The boundary condition does not change.
For any twice continuously differentiable function \( f(x) \) define
\[
f_n(x, \tau) = f(x) + \frac{1}{\sqrt{n}} f'(x) g(\tau) + \frac{1}{n} f''(x) h(\tau),
\]
where the functions \( g(\tau) \), \( h(\tau) \) will be defined below. For such functions we have;
\[
A_n f_n(x, \tau) = \sqrt{n} f'(x)[\lambda_n \beta_1 - 1 + g'(\tau)] + f''(x)[(\lambda_n \beta_1 - 1)g(\tau) + \frac{1}{2} \lambda_n \beta_2 + h'(\tau)] + o(1),
\]
\[
A_n f_n(0, \tau) = \sqrt{n} f'(0)[\lambda_n \beta_1 + g'(\tau)] + f''(0)[\lambda_n \beta_1 g(\tau) + \frac{1}{2} \lambda_n \beta_2 + h'(\tau)] + o(1).
\]
The boundary condition becomes;
\[
g(1) = g(0), \quad h(1) = h(0).
\]
From the above it is easy to see that \( A_n f_n \) can converge to a limit which does not depend on \( \tau \) only if the functions \( g(\tau), h(\tau) \) satisfy the equations:
\[
\lambda_n \beta_1 - 1 + g'(\tau) = \frac{1}{\sqrt{n}} c_1, \quad (\lambda_n \beta_1 - 1)g(\tau) + \frac{\lambda_n \beta_2}{2} + h'(\tau) = c_2,
\]
where \( c_1, c_2 \) are to be determined from the boundary conditions.
The first equation yields
\[
g(\tau) = (\tau - \lambda_n \beta_1) + \frac{1}{\sqrt{n}} c_1 \tau + g(0).
\]
By the boundary condition \( g(1) = g(0) \) we get
\[
c_1 = -\sqrt{n}(1 - \rho^{(\omega)}) = -m + o(1),
\]
so \( g(\tau) = \rho \tau - \lambda_n \beta_1 + g(0) \). This allows us to obtain
\[
h(\tau) = c_2 \tau + \frac{1}{2} \lambda_n \beta_2 + \int_0^\tau (1 - \lambda_n \beta_1)g(u) \, du + h(0).
\]
The boundary condition \( h(1) = h(0) \) gives
\[
C_2 = \beta_2/(2\beta_1) + o(1),
\]
Using the functions \( g(\tau), h(\tau) \) we get
\[
\lim_{n \to \infty} A_n f_n(x, \tau) = -mf'(x) + \frac{\beta_2}{2\beta_1} f''(x) = A_0 f(x).
\]
The generator \( A_0 \) corresponds to a diffusion process with infinitesimal mean \( m \) and infinitesimal variance \( \beta_2/(2\beta_1) \).

Similar analysis of behaviour at the boundary \( x = 0 \) implies that functions \( f(x) \) from the domain of the generator \( A_0 \) have to satisfy the condition \( f(0) = 0 \). This means that the generator \( A_0 \) corresponds to diffusion with a reflecting barrier at the origin.

To complete the proof it is sufficient to refer to results in [1] and [2] which imply that the above allow us to guarantee the desired convergence.

References
