Can. J. Math., Vol. XXXVIII, No. 5, 1986, pp. 1065-1080

THE TENSOR PRODUCT OF OPERATIONAL LOGICS

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1. Introduction. The concept of an operational logic has been developed by Randall and Foulis ([**1**]-[**4**], [**10**], [**11**]) as a part of a larger effort to obtain a formalism suitable for expressing, comparing, and evaluating various approaches to empirical science, statistics, and in particular, quantum mechanics. The structure of these logics is similar to that of an orthomodular partially ordered set which is often used as a model for quantum logics. However, the operational logic is a more general structure which, among other features, allows for the creation of a tensor product of logics to represent the coupling of physical systems. Randall and Foulis have shown that, given certain reasonable physical constraints, such a product is not possible within the category of orthomodular posets [**12**].

The operational logic is actually derived in the Randall-Foulis formalism from the more primitive notion of a collection of physical experiments known as a manual or generalized sample space. The manual provides a link between the abstraction of a logic and the physical reality of empirical science. There may be many manuals associated with a single operational logic; each representing a slightly different experimental configuration. In this way a manual may be viewed as a refinement of the logic to give a more detailed account of the relationships between the physical quantities in question.

A variety of products, including the tensor product, have been defined within the category of manuals in order to reflect different levels of influence exhibited by the states of the coupled systems. This tensor product of manuals may be extended in a natural way to a tensor product for operational logics.

2. Operational logics. An operational logic can be defined (independently of consideration for an underlying manual) as a set $\mathscr{L}$ equipped with a binary relation $\perp$, a partially defined binary operation $\oplus$, a unary operation $'$, and special elements 0 and 1 which satisfy the axioms below.

A1: (Orthogonal sum) $p \oplus q$ is defined if and only if $p \perp q$.
A2: (Commutativity) If $p \perp q$ then $q \perp p$ and $q \oplus p = p \oplus q$.

Received February 1, 1984 and in revised form January 7, 1985.

A3: (Associativity) If $p \perp q$ and $r \perp (p \oplus q)$ then $q \perp r, p \perp (q \oplus r)$ and

$$p \oplus (q \oplus r) = (p \oplus q) \oplus r.$$

A4: (Orthocomplements) For any $p \in \mathscr{L}$ there is a unique p' such that $p \perp p'$ and $p \oplus p' = 1$.

A5: (Maximality of 1) If $p \perp 1$ then $p = 0$.

A6: (Minimality of 0) For any $p \in \mathscr{L}$, $p \perp 0$ and $p \oplus 0 = p$.

A7: (Consistency) If $p \perp p$ then $p = 0$.

By no means is this a minimal set of axioms. We present them in this form to make the features of an operational logic as transparent as possible to those accustomed to working with similar logical structures. Other familiar concepts may be easily defined. For example, a partial order can be defined on $\mathscr{L}$ by $p \leqq q$ if and only if there is some $r \in \mathscr{L}$ with $p \oplus r = q$. Frazer and Hardegree, in a comparative study of quantum logics [**6**], have more formally called the operational logic an associative ortho-algebra.

3. Manuals and their logics. A *generalized sample space* (manual), $\mathscr{A}$, is a collection of nonempty sets known as *operations*. Each operation $E \in \mathscr{A}$ is thought of as the outcome set of a single physical experiment. An important feature of the manual is that these operations are allowed to overlap and contain common outcomes. Following the terminology of classical probablity, any subset of an operation is called an *event* and we may define the following relations on the set of $\mathscr{A}$-events, $\mathscr{E}(\mathscr{A})$.

Definition 1. Events A and B are

(i) *orthogonal* ($A \perp B$) if they are disjoint and contained in a common operation.

(ii) *operational complements* (A oc B) if $A \perp B$ and $A \cup B$ is an entire operation in $\mathscr{A}$.

(iii) *operationally perspective* (A op B) if there is an event C with A oc C and B oc C.

Note that (iii) implies that operationally perspective events are in some sense equivalent since both are complements of the same event. This is formalized to give the only condition imposed on generalized sample spaces.

Definition 2. (Manual Condition) A collection of nonempty sets is a *manual* or *generalized sample space* if for events A, B, and C, A op B and $A \perp C$ implies $B \perp C$.

This leads in a very natural way to the construction of a logic for a manual $\mathscr{A}$, denoted by $\pi(\mathscr{A})$. A proposition in the logic is simply an equivalence class of "op" events and may be denoted by $p(A)$ where A is any event in that equivalence class. The rest of the structure in $\pi(\mathscr{A})$ arises as follows.

(i) $1 = p(E)$ for any operation $E \in \mathscr{A}$.

(ii) $0 = p(\emptyset)$.

(iii) $p(A) \perp p(B)$ if and only if $A \perp B$.

(iv) If $p(A) \perp p(B)$ then $p(A) \oplus p(B) = P(A \cup B)$.

(v) $p(A)' = p(C)$ where C is any event with A oc C.

(vi) $p(A) \leqq p(B)$ if and only if there is some event C with $A \subseteq C$ and C op B.

The manual condition is all that is necessary to ensure that (i)-(vi) are well-defined and $(\pi(\mathscr{A}), \perp, \oplus, ', 0, 1)$ is an operational logic. The question of going the other way is of particular interest here, i.e., given an arbitrary operational logic $\mathscr{L}$, does there exist a manual $\mathscr{A}$ with $\pi(\mathscr{A}) \simeq \mathscr{L}$? Fortunately, this can be answered affirmatively and will yield a construction, the finite partition of unity manual, which is important in finding the tensor product of two logics.

Definition 3. Given an operational logic $\mathscr{L}$, $\{p, q, r\} \subseteq \mathscr{L}$ are *jointly orthogonal* if

(i) $p \perp q$, $q \perp r$, $r \perp p$ and

(ii) $p \oplus (q \oplus r)$ is defined.

In an obvious way this definition may be extended to define finite jointly orthogonal sets of propositions in $\mathscr{L}$. Such sets become maximal under the following condition.

Definition 4. A *finite partition of unity* in $\mathscr{L}$ is a finite jointly orthogonal set of nonzero propositions in $\mathscr{L}$, $\{p_1, p_2, \dots\}$, with $\oplus p_i = 1$.

If we let $\mathscr{A}$ be the collection of all finite partitions of unity in $\mathscr{L}$ one may easily show that $\mathscr{A}$ is a generalized sample space and that $\pi(\mathscr{A}) \simeq \mathscr{L}$. We sometimes denote this finite partition of unity manual associated with $\mathscr{L}$ by $\mathscr{M}(\mathscr{L})$. In this way $\mathscr{M}(\mathscr{L})$ can be thought of as a canonical manual for a given operational logic and will provide a key for the extension of the tensor product of manuals to the tensor product of operational logics.

4. The tensor product. Three levels of products have been defined for generalized sample spaces: the cross, operational, and tensor products. Since the outcome set for each of these products is the same, we use juxtaposition, ef, to denote product outcomes rather than $e \times f$, (e, f) or $e \otimes f$. The simplest of the products is the *cross product*, $\mathscr{A} \times \mathscr{B}$, which consists of all operations of the form EF where $E \in \mathscr{A}$ and $F \in \mathscr{B}$. By the notation EF we mean the set of all outcomes ef where $e \in E$ and $f \in F$.

Operations in the *operational product*, $\overrightarrow{\mathscr{A}\mathscr{B}}$, are performed as follows. Select an operation $E \in \mathscr{A}$ and execute it to obtain an outcome e. That

https://doi.org/10.4153/CJM-1986-052-3 Published online by Cambridge University Press

outcome determines precisely which operation $F_e \in \mathscr{B}$ is to be executed. The final result will be reported as ef where $e \in E$ and $f \in F_e$. By considering all possibilities for initial operations in $\mathscr{A}$ and following operations from $\mathscr{B}$ we obtain the manual

$$\overrightarrow{\mathscr{A}\mathscr{B}} = \left\{ \bigcup_{e \in E} eF_e : E \in \mathscr{A}, F_e's \in \mathscr{B} \right\}.$$

In a similar manner we may define the manual $\overleftarrow{\mathscr{A}\mathscr{B}}$ to be all such operations initiated with operations $F \in \mathscr{B}$. It is not difficult to show that each of $\mathscr{A} \times \mathscr{B}$, $\overrightarrow{\mathscr{A}\mathscr{B}}$, and $\overleftarrow{\mathscr{A}\mathscr{B}}$ are manuals, provided $\mathscr{A}$ and $\mathscr{B}$ are manuals.

To obtain the tensor product we combine the operations in both $\overrightarrow{\mathscr{A}\mathscr{B}}$ and $\overleftarrow{\mathscr{A}\mathscr{B}}$ to form $\overleftrightarrow{\mathscr{A}\mathscr{B}}$. in general, $\overleftrightarrow{\mathscr{A}\mathscr{B}}$ will not be a manual but sufficient new operations may be added to satisfy the manual condition. Thus we define the *tensor product*, $\mathscr{A} \otimes \mathscr{B}$, of two manuals to be the smallest manual containing $\overleftrightarrow{\mathscr{A}\mathscr{B}}$. It can be shown that if $\overleftrightarrow{\mathscr{A}\mathscr{B}}$ is contained in any manual at all there will be a unique smallest manual containing it and the outcome set will remain the same as that of $\overleftrightarrow{\mathscr{A}\mathscr{B}}$. There are cases for which the tensor product of two manuals fails to exist, but some mild conditions on the states of the manuals will ensure that the tensor product does exist. For example, if the manuals admit a *unital* set of states, i.e., every outcome has a state which assigns it probability one, then the tensor product will exist. See **[3]** and **[9]** for a more detailed discussion of the existence question and the physical motivations for the definition of the tensor product. Some categorical properties, including a Universal Mapping Theorem, can be found in **[7]**.

We are now in a position to quickly extend this definition to operational logics in the following straightforward manner.

Definition 5. The *tensor product* of operational logics $\mathscr{L}_1$ and $\mathscr{L}_2$ is defined to be

$$\mathscr{L}_1 \otimes \mathscr{L}_2 = \pi(\mathscr{M}(\mathscr{L}_1) \otimes \mathscr{M}(\mathscr{L}_2)).$$

Two questions are immediately suggested by this definition. How do we know that $\mathscr{M}(\mathscr{L})$ is the "right" manual to use in creating the tensor product? Since there may be many other manuals with logic $\mathscr{L}$, perhaps another representative would give a more suitable product. Secondly, how does one go through the entire procedure of finding the partition manuals, constructing their tensor product, and then obtaining its logic in any reasonable form? Theorem 1 will go a long way towards answering both questions, but let us first illustrate the second problem with an example.

Let $\mathscr{L}$ be the Boolean algebra with m atoms, denoted 2^m, and suppose we were to apply Definition 5 to find the tensor product of $\mathscr{L}$ with another operational logic. The finite partition of unity manual $\mathscr{M}(\mathscr{L})$ has 2^m outcomes and an operation corresponding to every partition of the atoms of $\mathscr{L}$. Constructing the tensor product of this manual with another finite partition of unity manual and then determining the logic of the result could be quite a formidable task, even in the simple case of Boolean logics.

A more rational approach would be to pick a "nicer" manual to represent $\mathscr{L}$. For example, a *classical* manual $\mathscr{A}$ which consists of a single operation with m outcomes would have $\pi(\mathscr{A}) \simeq 2^m$. Constructing the tensor product with another manual $\mathscr{B}$ is trivial since $\mathscr{A} \otimes \mathscr{B} = \overrightarrow{\mathscr{A}\mathscr{B}}$ whenever $\mathscr{A}$ is classical. The logic of the tensor product would then be easier to study, but would it be the same as that obtained from Definition 5? An example will show that it is possible to have

$$\pi(\mathscr{A}_1) \simeq \pi(\mathscr{A}_2) \quad \text{and} \quad \pi(\mathscr{B}_1) \simeq \pi(\mathscr{B}_2)$$

but not

$$\pi(\mathscr{A}_1 \otimes \mathscr{B}_1) \simeq \pi(\mathscr{A}_2 \otimes \mathscr{B}_2).$$

However, in the following important result we show that a large class of manuals with logic $\mathscr{L}$ may be substituted for $\mathscr{M}(\mathscr{L})$ in applying Definition 5.

5. Uniqueness theorem.

THEOREM 1. *If $\mathscr{A}_1$ and $\mathscr{A}_2$ are locally finite manuals with*

$$\pi(\mathscr{A}_1) \simeq \mathscr{L}_1 \quad \textit{and} \quad \pi(\mathscr{A}_2) \simeq \mathscr{L}_2$$

then $\mathscr{L}_1 \otimes \mathscr{L}_2 = \pi(\mathscr{A}_1 \otimes \mathscr{A}_2)$.

By a locally finite manual we mean a manual in which each operation has only a finite number of outcomes. In particular, note that a finite partition of unity manual $\mathscr{M}(\mathscr{L})$ is locally finite. Thus we may choose any convenient locally finite manual to represent the operational logics in constructing their tensor product. Before moving to the proof of this result, let us complete the discussion of the tensor product of finite Boolean algebras with the following.

COROLLARY 1. *If $\mathscr{L}_1 \simeq 2^m$ and $\mathscr{L}_2 \simeq 2^n$ then $\mathscr{L}_1 \otimes \mathscr{L}_2 \simeq 2^{mn}$.*

Using Theorem 1 the proof of this result is quite easy. Simply use classical manuals $\mathscr{A}$ and $\mathscr{B}$ to represent $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively, and use the fact that $\mathscr{A} \otimes \mathscr{B}$ is again classical, containing a single operation with mn outcomes. This is actually a special case of the more general result on Boolean algebras discussed in Section 6.

We use the following notation for the remainder of this section. Let $\mathscr{A}$ be a locally finite manual and $\tilde{\mathscr{A}}$ be the finite partition of unity manual for the logic $\pi(\mathscr{A})$. Similarly, let $\tilde{\mathscr{B}}$ be the finite partition of unity manual associated with a second locally finite manual $\mathscr{B}$. Denote the outcome sets in each manual by $X = \cup\mathscr{A}$, $Y = \cup\mathscr{B}$, $\widetilde{X} = \cup\widetilde{\mathscr{A}}$, and $\widetilde{Y} = \cup\widetilde{\mathscr{B}}$. Denote the propositions in $\pi(\mathscr{A})$, $\pi(\mathscr{B})$, $\pi(\mathscr{A} \otimes \mathscr{B})$, and $\pi(\widetilde{\mathscr{A}} \otimes \widetilde{\mathscr{B}})$ by $p(\ )$, $q(\ )$, $\hat{p}(\ )$, and $\widetilde{p}(\ )$ respectively. It will be convenient to work with a single representative of the equivalence class making up a proposition. Therefore, for each $p \in \pi(\mathscr{A})$ choose an event $A \in \mathscr{E}(\mathscr{A})$ with $p = p(A)$ and denote this event by $A = \sigma(p)$. Similarly, select a $\sigma(q) \in \mathscr{E}(\mathscr{B})$ for each $q \in \pi(\mathscr{B})$.

To prove the uniqueness theorem we establish maps, known as interpretation morphisms, between $\mathscr{A} \otimes \mathscr{B}$ and $\widetilde{\mathscr{A}} \otimes \widetilde{\mathscr{B}}$ which can be lifted to maps on their logics. An *interpretation* $\mathscr{A} \xrightarrow{\varphi} \mathscr{B}$ is a map $\varphi: X \rightarrow \mathscr{E}(\mathscr{B})$ which is operation and $\perp$-preserving. These interpretations are obtained here via the Universal Mapping Theorem [7] which states that a positive biinterpretation $\mathscr{A} \times \mathscr{B} \xrightarrow{\varphi} \mathscr{C}$ is naturally extended to an interpretation from $\mathscr{A} \otimes \mathscr{B}$ to $\mathscr{C}$. By a *positive biinterpretation* we mean a map $\varphi: XY \rightarrow \mathscr{E}(C)$ which is operation and $\perp$-preserving on $\overleftrightarrow{\mathscr{A}\mathscr{B}}$ and sends no outcome to the empty set in $\mathscr{E}(\mathscr{C})$.

LEMMA 1. *Assume* $\mathscr{A} \otimes \mathscr{B}$ *exists and define*

$$\psi: \widetilde{X}\widetilde{Y} \rightarrow \mathscr{E}(\mathscr{A} \otimes \mathscr{B})$$

by $\psi(pq) = \sigma(p)\sigma(q)$, *then*

$$\mathscr{A} \times \mathscr{B} \xrightarrow{\psi} \mathscr{A} \otimes \mathscr{B}$$

is a positive biinterpretation.

Proof. Let

$$G = \bigcup_i p_i\left(\bigcup_j q_{ij}\right) \in \overrightarrow{\widetilde{\mathscr{A}}\widetilde{\mathscr{B}}}$$

where

$$\bigcup_i p_i = \widetilde{E} \in \widetilde{\mathscr{A}} \text{ and } \bigcup_j q_{ij} = \widetilde{F}_i \in \widetilde{\mathscr{B}} \quad \text{for each } i.$$

Since each operation is a finite partition of unity we have $\cup_i \sigma(p_i)$ is an operation $E \in \mathscr{A}$ and each $\cup_j \sigma(q_{ij}) = F_i \in \mathscr{B}$. Thus

$$\psi(G) = \bigcup_i \sigma(p_i)\left(\bigcup_j \sigma(q_{ij})\right)$$

is an operation in $\overrightarrow{\mathscr{A}\mathscr{B}}$ which is contained in $\mathscr{A} \otimes \mathscr{B}$. Similarly, $G \in \overleftarrow{\widetilde{\mathscr{A}}\widetilde{\mathscr{B}}}$ implies

$$\psi(G) \in \overleftarrow{\mathscr{A}\mathscr{B}} \subseteq \mathscr{A} \otimes \mathscr{B}.$$

This verifies that ψ is operation preserving on $\overleftrightarrow{\widetilde{\mathscr{A}}\widetilde{\mathscr{B}}}$.

Now, suppose $A_1 \perp A_2$ in $\mathscr{E}(\tilde{\mathscr{A}})$ and $B_1, B_2 \in \mathscr{E}(\tilde{\mathscr{B}})$. Say $A_1 = \cup p_i$, $A_2 = \cup p_k$, $B_1 = \cup q_j$, $B_2 = \cup q_l$ and suppose that

$$\psi(A_1B_1) \cap \psi(A_2B_2)$$

is nonempty. This implies that

$$\sigma(p_i)\sigma(q_j) \cap \sigma(p_k)\sigma(q_l)$$

is nonempty for some i, j, k, l. But $A_1 \perp A_2$ implies each $p_i \perp p_k$ and thus

$$\sigma(p_i) \perp \sigma(p_k).$$

This is a contradiction, hence it must be that

$$\psi(A_1B_2) \cap \psi(A_2B_2) = \emptyset$$

and therefore ψ is $\perp$-preserving on $\overrightarrow{\tilde{\mathscr{A}}\tilde{\mathscr{B}}}$. The symmetric argument proves this for $\overleftarrow{\tilde{\mathscr{A}}\tilde{\mathscr{B}}}$ and completes the proof that ψ is a biinterpretation.

LEMMA 2. *Assume* $\tilde{\mathscr{A}} \otimes \tilde{\mathscr{B}}$ *exists and define*

$$\alpha: XY \rightarrow \mathscr{E}(\tilde{\mathscr{A}} \otimes \tilde{\mathscr{B}})$$

by $\alpha(xy) = p(x)q(y)$, *then*

$$\mathscr{A} \times \mathscr{B} \xrightarrow{\alpha} \tilde{\mathscr{A}} \otimes \tilde{\mathscr{B}}$$

is a positive biinterpretation if and only if $\mathscr{A}$ *and* $\mathscr{B}$ *are locally finite.*

Proof. Let

$$G = \bigcup_{e \in E} eF_e \in \overrightarrow{\mathscr{A}\mathscr{B}},$$

so

$$\alpha(G) = \bigcup_{e \in E} p(e)\left(\bigcup_{f \in F_e} q(f)\right).$$

This will be an operation in $\overrightarrow{\tilde{\mathscr{A}}\tilde{\mathscr{B}}}$ if and only if $\bigcup_{e \in E} p(e)$ and $\bigcup_{f \in F_e} q(f)$ are all operations in $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{B}}$ respectively. This requires that each of these unions be over finite index sets which will result if and only if both $\mathscr{A}$ and $\mathscr{B}$ are locally finite. Otherwise, $\alpha(G)$ could have an infinite number of outcomes and couldn't possibly be an operation in $\tilde{\mathscr{A}} \otimes \tilde{\mathscr{B}}$ which must be locally finite. Thus α is operation preserving if and only if both $\mathscr{A}$ and $\mathscr{B}$ are locally finite. The remainder of the proof is completed with an argument similar to that in Lemma 1.

Assuming the conditions in Lemmas 1 and 2 hold, we may apply the

Universal Mapping Theorem to conclude that

$$\psi:\tilde{\mathscr{A}} \otimes \tilde{\mathscr{B}} \to \mathscr{A} \otimes \mathscr{B} \quad \text{and} \quad \alpha:\mathscr{A} \otimes \mathscr{B} \to \tilde{\mathscr{A}} \otimes \tilde{\mathscr{B}}$$

are positive interpretations. The following lemma describes how these may be lifted to maps between the logics.

LEMMA 3. *For any manuals $\mathscr{A}$ and $\mathscr{B}$ and a map $\varphi:X \to \mathscr{P}(\mathscr{B})$, define*

$$\bar{\varphi}:\pi(\mathscr{A}) \to \pi(\mathscr{B})$$

by

$$\bar{\varphi}(p(A)) = q(\varphi(A)),$$

then $\bar{\varphi}$ is a well-defined map preserving $\perp$, $'$, and 1 if and only if φ is a positive interpretation.

Proof. Assume φ is an interpretation. For any A in $\mathscr{E}(\mathscr{A})$, $\varphi(A) \in \mathscr{E}(\mathscr{B})$ and if $p(A) = p(B)$ in $\pi(\mathscr{B})$ we must have A op B, hence

$$\varphi(A) \text{ op } \varphi(B) \quad \text{and} \quad q(\varphi(A)) = q(\varphi(B)) \text{ in } \pi(\mathscr{B}).$$

This shows that $\bar{\varphi}$ is well-defined if and only if φ is just a morphism. The preservation of $\perp$, $'$, and 1 follow immediately from the fact that positive interpretations preserve $\perp$, operational complements, and operations.

One technical point remains before proceeding to the proof of Theorem 1. Recall that the map ψ in Lemma 1 depends on the rather arbitrary choice of the event $\sigma(p)$ to represent the proposition $p \in \pi(\mathscr{A})$. However, when ψ is lifted to a map on the logics it will be independent of σ.

LEMMA 4. *Let ψ be the interpretation of Lemma 1 and $\bar{\psi}$ be defined as in Lemma 3. If*

$$\bigcup_i p(A_i)q(B_i) \in \mathscr{E}(\tilde{\mathscr{A}} \otimes \tilde{\mathscr{B}}),$$

then

$$\bar{\psi}\left(\tilde{p}\left(\bigcup_i p(A_i)q(B_i)\right)\right) = \hat{p}\left(\bigcup_i A_iB_i\right).$$

Proof. Suppose that for some finite index set I we have events $A_i, C_i \in \mathscr{E}(\mathscr{A})$ and $B_i, D_i \in \mathscr{E}(\mathscr{B})$ such that A_i op C_i and B_i op D_i for each $i \in I$. If $\bigcup_{i \in I} A_iB_i$ is an event in $\mathscr{A} \otimes \mathscr{B}$, we must have

$$\bigcup_{i \in I} C_iD_i \in \mathscr{E}(\mathscr{A} \otimes \mathscr{B})$$

and in fact

$$\bigcup_{i \in I} A_iB_i \text{ op } \bigcup_{i \in I} C_iD_i.$$

https://doi.org/10.4153/CJM-1986-052-3 Published online by Cambridge University Press

This can be verified with a finite number of applications of the manual condition on $\mathscr{A} \otimes \mathscr{B}$. Therefore, for any choice of σ,

$$\bigcup_i A_i B_i \text{ op } \bigcup_i \sigma(p(A_i)\sigma(q(B_i))),$$

provided the union is over a finite index set and either is an event in $\mathscr{A} \otimes \mathscr{B}$. Thus we have

$$\begin{aligned}\bar{\psi}\left(\tilde{p}\left(\bigcup_i p(A_i)q(B_i)\right)\right) &= \hat{p}\left(\psi\left(\bigcup_i p(A_i)q(B_i)\right)\right) \\ &= \hat{p}\left(\bigcup_i \sigma(p(A_i))\sigma(q(B_i))\right) = \hat{p}\left(\bigcup_i A_i B_i\right).\end{aligned}$$

To complete the proof of Theorem 1 we now show that

$$\pi(\mathscr{A} \otimes \mathscr{B}) \simeq \pi(\tilde{\mathscr{A}} \otimes \tilde{\mathscr{B}})$$

by verifying that $\bar{\psi} \circ \bar{\alpha} =$ identity on $\pi(\mathscr{A} \otimes \mathscr{B})$ and $\bar{\alpha} \circ \bar{\psi} =$ identity on $\pi(\tilde{\mathscr{A}} \otimes \tilde{\mathscr{B}})$. Let $\hat{p}(C) \in \pi(\mathscr{A} \otimes \mathscr{B})$ where

$$C = \cup x_i y_i \in \mathscr{E}(\mathscr{A} \otimes \mathscr{B}).$$

By Lemma 4,

$$\begin{aligned}\bar{\psi}(\bar{\alpha}(\hat{p}(C))) &= \bar{\psi}(\tilde{p}(\alpha(C)) = \bar{\psi}(\tilde{p}(\cup p(x_i)q(y_i)) \\ &= \hat{p}(\cup x_i y_i) = \hat{p}(C).\end{aligned}$$

Let

$$\tilde{p}(\tilde{C}) \in \pi(\tilde{\mathscr{A}} \otimes \tilde{\mathscr{B}})$$

where

$$\tilde{C} = \cup p(A_i)q(B_i) \in \mathscr{E}(\tilde{\mathscr{A}} \otimes \tilde{\mathscr{B}}).$$

$$\begin{aligned}\bar{\alpha}(\bar{\psi}(\tilde{p}(\tilde{C}))) &= \bar{\alpha}(\hat{p}(\cup A_i B_i)) = \tilde{p}(\alpha(\cup A_i B_i)) \\ &= \tilde{p}(\cup \sigma(A_i B_i)) = \tilde{p}(\cup p(A_i)q(B_i)) = \tilde{p}(\tilde{C}).\end{aligned}$$

The last equality follows by again applying the manual condition a finite number of times to show

$$\bigcup_{x \in A} \bigcup_{y \in B} p(x)q(y) \text{ op } p(A)q(B) \text{ in } \tilde{\mathscr{A}} \otimes \tilde{\mathscr{B}}.$$

This completes the proof that

$$\pi(\mathscr{A} \otimes \mathscr{B}) \simeq \pi(\tilde{\mathscr{A}} \otimes \tilde{\mathscr{B}})$$

for locally finite $\mathscr{A}$ and $\mathscr{B}$.

What if $\mathscr{A}$ and $\mathscr{B}$ are not locally finite? The previous argument breaks down since we are not able to apply Lemma 2 to obtain a map from $\pi(\mathscr{A} \otimes \mathscr{B})$ to $\pi(\tilde{\mathscr{A}} \otimes \tilde{\mathscr{B}})$. However, by using Lemma 1 and following the proof of Theorem 1 one may show the following.

COROLLARY 2. *Let $\mathscr{A}$ and $\mathscr{B}$ be arbitrary manuals with $\pi(\mathscr{A}) \simeq \mathscr{L}_1$ and $\pi(\mathscr{B}) \simeq \mathscr{L}_2$ and assume $\mathscr{A} \otimes \mathscr{B}$ exists, then $\mathscr{L}_1 \otimes \mathscr{L}_2$ exists and is isomorphic to a sublogic of $\pi(\mathscr{A} \otimes \mathscr{B})$.*

The following example demonstrates that it is possible for $\mathscr{L}_1 \otimes \mathscr{L}_2$ to be a proper sublogic of $\pi(\mathscr{A} \otimes \mathscr{B})$, hence the logic of the tensor product depends on more than just the logics of its factors when the local finiteness condition is dropped.

Example. Let $\mathscr{A}_1$ be a classical manual consisting of a single operation with a countably infinite set of outcomes $X = \{a_1, a_2, \dots\}$. Let $\mathscr{A}_2$ be the manual of finite partitions of X into nonempty subsets. In both cases $\mathscr{L}_1 = \pi(\mathscr{A}_1) = \pi(\mathscr{A}_2)$ is the Boolean algebra of all subsets of X. For $\mathscr{B}$ we use a semiclassical manual consisting of an infinite number of dichotomies, i.e.,

$$\mathscr{B} = \{ \{b_1, c_1\}, \{b_2, c_2\}, \dots \}.$$

$\mathscr{L}_2 = \pi(\mathscr{B})$ is a logic consisting of just the elements 0, 1, and atoms corresponding to each b_i and c_j. Since $\mathscr{A}_2$ and $\mathscr{B}$ are locally finite, Theorem 1 implies

$$\pi(\mathscr{A}_2 \otimes \mathscr{B}) = \mathscr{L}_1 \otimes \mathscr{L}_2.$$

We will show that this logic is an incomplete lattice, while the logic of $\mathscr{A}_1 \otimes \mathscr{B}$ is complete.

Since $\mathscr{A}_1$ is a classical manual, we have

$$\mathscr{A}_1 \otimes \mathscr{B} = \overrightarrow{\mathscr{A}_1\mathscr{B}}.$$

The atoms in the logic $\pi(\mathscr{A}_1 \otimes \mathscr{B})$ are all of the form $p(a_iz)$ where $a_i \in X$ and $z \in \cup\mathscr{B}$. Distinct propositions correspond to events of the form $\cup_{i\in I} a_iB_i$ where I is an arbitrary index set in $\{1, 2, \dots\}$ and each B_i may be either a $\{b_j\}$, $\{c_k\}$, or $\{b_1, c_1\}$. Note that for all i and j,

$$p(a_ib_j, a_ic_j) = p(a_ib_1, a_ic_1) \quad \text{and}$$

$$p(a_ib_j) \vee p(a_ic_k) = p(a_ib_1, a_ic_1).$$

Also for $j \neq k$,

$$p(a_ib_j) \vee p(a_ib_k) = p(a_ib_1, a_ic_1).$$

Using these and similar relations one can check that $\pi(\mathscr{A}_1 \otimes \mathscr{B})$ is a complete atomistic orthomodular lattice.

The computation of $\mathscr{A}_2 \otimes \mathscr{B}$ is a less trivial task. One may show that a general operation in $\mathscr{A}_2 \otimes \mathscr{B}$ is obtained as follows. Start with a finite index set $I \subset \{1, 2, \dots\}$. To each $i \in I$ assign a pair of nonempty events B_i and C_i in $\mathscr{E}(\mathscr{A}_2)$ such that

(i) B_i op C_i.
(ii) For $i \neq j$, $B_i \perp B_j$ (and hence $B_i \perp C_j$).
(iii) $\bigcup_{i \in I} B_i = \bigcup_{i \in I} C_i = X$.

The set

$$G = \bigcup_{i \in I} (B_i b_i \cup C_i c_i)$$

will be an operation in $\mathscr{A}_2 \otimes \mathscr{B}$.

A crucial point is that the index set I must be finite. if one considers a set of atoms $\{p_i = p(a_i b_i): i = 1, 2, \dots\}$ it may be shown that $\bigvee_{i=1}^{\infty} p_i$ fails to exist in $\pi(\mathscr{A}_2 \otimes \mathscr{B})$. The obvious candidate would be the proposition corresponding to $\{a_1 b_1, a_2 b_2, \dots\}$ but this set of outcomes is not an event in $\mathscr{A}_2 \otimes \mathscr{B}$. It is interesting to note that this same set of atoms occurs in $\pi(\mathscr{A}_1 \otimes \mathscr{B})$ but in that case $\{a_1 b_1, a_2 b_2, \dots\}$ is an event in $\mathscr{A}_1 \otimes \mathscr{B}$. As indicated in Corollary 2, the logic $\pi(\mathscr{A}_2 \otimes \mathscr{B})$ may be embedded as a sublogic in $\pi(\mathscr{A}_1 \otimes \mathscr{B})$ and in this case $\pi(\mathscr{A}_1 \otimes \mathscr{B})$ may be viewed as a completion of $\pi(\mathscr{A}_2 \otimes \mathscr{B})$.

6. Tensor product of Boolean algebras. In this section we further examine the implications of Definition 5 in the classical case of Boolean algebras. A natural product for two arbitrary Boolean algebras is produced by considering the associated Stone spaces. The product topology on the Cartesian product of these spaces yields another totally disconnected, compact Hausdorff space. Thus the clopen sets in the product, when ordered by inclusion, give a new Boolean algebra which may be regarded as the product of the original Boolean algebras [**5**]. We will show that this product is isomorphic to the logic obtained by Definition 5.

To select a convenient locally finite manual to represent an arbitrary Boolean logic we introduce the general notion of a finite partition manual. Given a set X and collection $\mathscr{M}$ of subsets of X, let $\mathscr{F}(X, \mathscr{M})$ denote the collection of all partitions of X into a finite number of disjoint $\mathscr{M}$-sets. Provided at least one such partition exists, one can show that $\mathscr{F}(X, \mathscr{M})$ satisfies the manual condition. We will give necessary and sufficient conditions on $\mathscr{M}$ for the logic of $\mathscr{F}(X, \mathscr{M})$ to be a Boolean algebra, and show how such manuals can be generated from arbitrary Boolean algebras. An advantage to this approach is that the tensor product of two such finite partition manuals can often be found explicitly in a convenient form.

Definition 6. A collection of subsets of X will be called a *prefield* if

(i) $A, B \in \mathscr{M}$ implies there exists a finite partition of $A \cap B$ into $\mathscr{M}$-sets.

(ii) $A \in \mathscr{M}$ implies there exists a finite partition of $X \backslash A$ into $\mathscr{M}$-sets.

Note that any field of subsets is a prefield. An example of a nontrivial prefield of interest here is found by considering sets X and Y with fields of subsets $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. The collection of rectangles,

$$\mathcal{M}_1 \times \mathcal{M}_2 = \{A \times B : A \in \mathcal{M}_1, B \in \mathcal{M}_2\},$$

is a prefield for $X \times Y$ but generally not a field.

If $\mathcal{M}$ is any collection of subsets of X, we let $\langle \mathcal{M} \rangle$ denote the collection obtained by forming all finite unions of disjoint $\mathcal{M}$-sets. One may then prove that $\mathcal{M}$ is a prefield if and only if $\langle \mathcal{M} \rangle$ is a field. This is particularly useful with partition manuals when one notes that $\mathcal{F}(X, \langle \mathcal{M} \rangle)$ is a coarsening of $\mathcal{F}(X, \mathcal{M})$, i.e., events in the latter become outcomes in the former. Therefore,

$$\pi(\mathcal{F}(X, \mathcal{M}) \simeq \pi(\mathcal{F}(X, \langle \mathcal{M} \rangle)).$$

We are now in a position to state conditions for a finite partition manual to be Boolean.

THEOREM 2. *Let X be a set with $\mathcal{M}$ a nontrivial collection of subsets of X, then $\mathcal{M}$ is a prefield if and only if $\pi(\mathcal{F}(X, \mathcal{M}))$ is a Boolean algebra and*

$$\cup \mathcal{F}(X, \mathcal{M}) = \mathcal{M}.$$

The condition that $\cup(\mathcal{F}(X, \mathcal{M})) = \mathcal{M}$ is fairly minor. It only ensures that each set in $\mathcal{M}$ can indeed be included in some finite partition of X. The proof of this theorem is a straightforward application of the following result from [**7**].

LEMMA 5. *An operational logic has the structure of a Boolean algebra if and only if the following two conditions hold*:

(i) *If $p \perp q$, $q \perp r$, and $p \perp r$, then $(p \oplus q) \perp r$.*

(ii) *For any $p, q \in \mathcal{L}$, there exist $p_1, q_1, r \in \mathcal{L}$ such that $p_1 \perp r$, $q_1 \perp r$, $p_1 \perp q_1$, $p = p_1 \oplus r$, and $q = q_1 \oplus r$.*

One direction is made even easier by using the comment that

$$\pi(\mathcal{F}(X, \mathcal{M})) \simeq \pi(\mathcal{F}(X, \langle \mathcal{M} \rangle))$$

to allow one to assume that $\mathcal{M}$ is a field when showing that $\pi(\mathcal{F}(X, \mathcal{M}))$ is Boolean.

If X and Y are sets with fields of subsets $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively, it has been shown [**9**] that the tensor product of the finite partition manuals can be constructed in the natural manner;

$$\mathcal{F}(X, \mathcal{M}_1) \otimes \mathcal{F}(Y, \mathcal{M}_2) = \mathcal{F}(X \times Y, \mathcal{M}_1 \times \mathcal{M}_2).$$

Since we have shown that $\mathcal{M}_1 \times \mathcal{M}_2$ is a prefield when $\mathcal{M}_1$ and $\mathcal{M}_2$ are fields, we may combine these results to conclude the following.

LEMMA 6. *Let X and Y be sets with fields of subsets $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively, then $\pi(\mathcal{F}(X, \mathcal{M}_1) \otimes \mathcal{F}(Y, \mathcal{M}_2))$ is Boolean.*

Returning to the case of arbitrary Boolean algebras, we now use the Stone space construction to prove the following.

THEOREM 3. *Let* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *be Boolean algebras, then* $\mathscr{L}_1 \otimes \mathscr{L}_2$ (*as found with Definition* 5) *is a Boolean algebra.*

Proof. By the uniqueness theorem we may choose any locally finite manuals with logics $\mathscr{L}_1$ and $\mathscr{L}_2$ in constructing the tensor product $\mathscr{L}_1 \otimes \mathscr{L}_2$. Let S_1 and S_2 be the respective Stone spaces for the Boolean algebras and $\mathscr{M}_1$ and $\mathscr{M}_2$ be the collections of clopen sets in each. One may easily show that $\pi(S_1, \mathscr{M}_1)$ and $\pi(S_2, \mathscr{M}_2)$ are locally finite manuals and have the appropriate Boolean logics. Since the clopen sets in a Stone space form a field we may apply Lemma 6 to conclude that

$$\pi(\mathscr{F}(S_1, \mathscr{M}_1) \otimes \mathscr{F}(S_2, \mathscr{M}_2)) = \pi(\mathscr{F}(S_1 \times S_2, \mathscr{M}_1 \times \mathscr{M}_2))$$

is a Boolean algebra. To see that it is the right Boolean algebra, note that the collection $\mathscr{M}_1 \times \mathscr{M}_2$ forms a base for the product topology on $S_1 \times S_2$. Including all finite disjoint unions we have the field of clopen sets $\langle \mathscr{M}_1 \times \mathscr{M}_2 \rangle$ which is a base for the compact Hausdorff space $S_1 \times S_2$. This implies that $\langle \mathscr{M}_1 \times \mathscr{M}_2 \rangle$ is a Boolean algebra (under set inclusion) and is exactly all of the clopen sets in $S_1 \times S_2$. Thus the logic

$$\pi(\mathscr{F}(S_1 \times S_2, \langle \mathscr{M}_1 \times \mathscr{M}_2 \rangle))$$

is isomorphic to the Boolean algebra of clopen sets in $S_1 \times S_2$. This finishes the proof since

$$\begin{aligned} \mathscr{L}_1 \otimes \mathscr{L}_2 &\simeq \pi(\mathscr{F}(S_1, \mathscr{M}_1) \otimes \mathscr{F}(S_2, \mathscr{M}_2)) \\ &\simeq \pi(\mathscr{F}(S_1 \times S_2, \mathscr{M}_1 \times \mathscr{M}_2)) \\ &\simeq \pi(\mathscr{F}(S_1 \times S_2, \langle \mathscr{M}_1 \times \mathscr{M}_2 \rangle)). \end{aligned}$$

7. A tensor product of orthomodular lattices. Although the previous section has demonstrated that Definition 5 gives the expected results in the essentially classical setting of Boolean algebras, some surprising developments occur when the definition is applied to other common logical systems. In this section we show by example that the tensor product of two orthomodular lattices need not be a lattice or even an orthomodular poset.

The logic used will be that of a *pentagon* manual

$$\mathscr{A} = \{ \{a, f, b\}, \{b, g, c\}, \{c, h, d\}, \{d, i, e\}, \{e, j, a\} \}.$$

Figure 1 gives the Greechie diagram for this manual where the five sides of the pentagon denote each of the operations in the manual. One can easily check that the logic $\pi(\mathscr{A})$ is a complete, atomistic, orthomodular lattice which is obtained by coupling five copies of the Boolean algebra 2^3 (one for each operation) into a loop. For the other logic we use that of another pentagon manual

https://doi.org/10.4153/CJM-1986-052-3 Published online by Cambridge University Press

$\mathscr{B} = \{ \{q, v, r\}, \{r, w, s\}, \{s, x, t\}, \{t, y, u\}, \{u, z, q\} \}.$

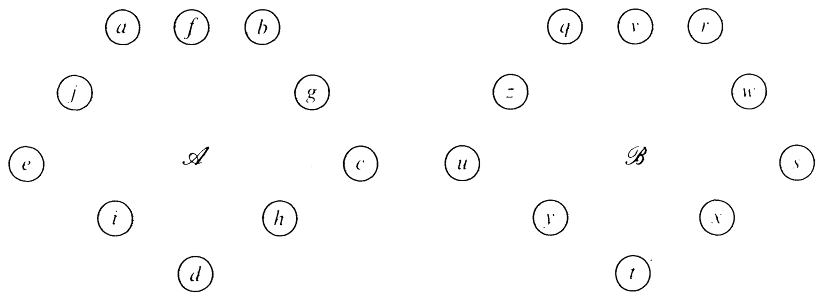

Figure 1 Pentagon Manuals $\mathscr{A}$ and $\mathscr{B}$

The tensor product of $\mathscr{A}$ and $\mathscr{B}$ can be computed explicitly (see **[9]**) and will contain 625 operations each with nine product outcomes. In particular, the following sets are operations in $\mathscr{A} \otimes \mathscr{B}$.

$$E = \{aq, bt, by, fq, cr, av, qr, fv, bu\}$$

$$F = \{aq, bt, by, fq, du, az, hu, fz, cu\}$$

$$G = \{cr, du, cq, cv, dq, dz, hq, hv, hr\}.$$

Define the events:

$$C = \{aq, bt, by, fq\} \quad B = \{cr, du\} \quad A = E \backslash C \quad D = F \backslash C.$$

Note that A op D and so $p(A) = p(D)$ in $\pi(\mathscr{A} \otimes \mathscr{B})$. We first show that $p(A) \wedge p(B)$ fails to exist in the product logic. Since $\{cr\} \in A$ and $\{du\} \in D$ we have

$$p(\{cr\}) < p(A) \quad \text{and} \quad p(\{du\}) < p(D) = p(A).$$

Obviously $p(\{cr\})$ and $p(\{du\})$ are both dominated by $p(B)$ so if $p(A) \wedge p(B)$ were to exist it must also dominate both. But this would imply that

$$p(\{cr\}) \vee p(\{du\}) \leqq p(A) \wedge p(B)$$

which in turn implies

$$p(B) \leqq p(A) \wedge p(B)$$

and thus $p(B) \leqq p(A)$. By the definition of the partial order in an operational logic we must then have $B \perp C$ in $\mathscr{A} \otimes \mathscr{B}$. However there is no operation in $\mathscr{A} \otimes \mathscr{B}$ which contains all the outcomes in both B and C so we have a contradiction and conclude that $\pi(\mathscr{A} \otimes \mathscr{B})$ is not a lattice.

A convenient condition for an operational logic to be an orthomodular poset is if and only if every finite pairwise orthogonal set is also jointly orthogonal [7]. Using the reasoning of the previous argument one may show that this fails to hold in $\pi(\mathscr{A} \otimes \mathscr{B})$ for the set $p(\{cr\})$, $p(\{du\})$, $p(C)$. Thus the logic of the tensor product of two pentagon manuals is not even an orthomodular partially ordered set. This conclusion is not so surprising when one considers a result of Randall and Foulis who showed that the following three conditions for a tensor product of orthomodular posets are not consistent.

(i) $p_1 \perp p_2$ in $\mathscr{L}_1$ or $q_1 \perp q_2$ in $\mathscr{L}_2$ implies $p_1 \otimes q_1 \perp p_2 \otimes q_2$ in $\mathscr{L}_1 \otimes \mathscr{L}_2$.

(ii) If α is a state on $\mathscr{L}_1$ and β is a state on $\mathscr{L}_2$ then $\alpha\beta$ defined by

$$\alpha\beta(p \otimes q) = \alpha(p)\beta(q)$$

is a state on $\mathscr{L}_1 \otimes \mathscr{L}_2$.

(iii) $\mathscr{L}_1 \otimes \mathscr{L}_2$ is an orthomodular poset.

This result was one of our strong motivations for considering the tensor product of a class of logics which was broader than just orthomodular posets. Conditions (i) and (ii) have a certain physically intuitive appeal while (iii) seems to be more a matter of mathematical convenience. It can be shown that Definition 5 will easily satisfy both (i) and (ii) but, as the previous example demonstrates, not (iii).

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