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Rozansky–Witten invariants via Atiyah classes

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Abstract. Recently, L. Rozansky and E. Witten associated to any hyper-Kähler manifold X a system of 'weights' (numbers, one for each trivalent graph) and used them to construct invariants of topological 3-manifolds. We give a simple cohomological definition of these weights in terms of the Atiyah class of X (the obstruction to the existence of a holomorphic connection). We show that the analogy between the tensor of curvature of a hyper-Kähler metric and the tensor of structure constants of a Lie algebra observed by Rozansky and Witten, holds in fact for any complex manifold, if we work at the level of cohomology and for any Kähler manifold, if we work at the level of Dolbeault cochains. As an outcome of our considerations, we give a formula for Rozansky–Witten classes using any Kähler metric on a holomorphic symplectic manifold.

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Recently, L. Rozansky and E. Witten [RW] associated to any hyper-Kähler manifold X an invariant of topological 3-manifolds. In fact, their construction gives a system of weights $c_\Gamma(X)$ associated to 3-valent graphs Γ and the corresponding invariant of a 3-manifold Y is obtained as the sum $\Sigma c_\Gamma(X) I_\Gamma(Y)$ where $I_\Gamma(Y)$ is the standard integral of the product of linking forms.

So the new ingredient is the system of invariants $c_\Gamma(X)$ of hyper-Kähler manifolds X, one for each trivalent graph Γ. They are obtained from the Riemannian curvature of the hyper-Kähler metric.

In this paper we give a reformulation of the $c_\Gamma(X)$ in simple cohomological terms which involve only the underlying holomorphic symplectic manifold. The idea is that we can replace the curvature by the Atiyah class [At] which is the cohomological obstruction to the existence of a global holomorphic connection. The role of what in [RW] is called 'Bianchi identities in hyper-Kähler geometry' is played here by an identity for the square of the Atyiah class expressing the existence of the fiber bundle of second order jets.

The analogy between the curvature and the structure constants of a Lie algebra observed in [RW] in fact holds even without any symplectic structure, and we study the nonsymplectic case in considerable detail so as to make the specialization to the symplectic situation easier. We show, first of all, that the Atiyah class of the tangent bundle of any complex manifold X satisfies a version of the Jacobi identity when considered as an element of an appropriate operad. In particular, we find (Theorem 2.3) that for any coherent sheaf A of $\mathcal{O}_X$-algebras the shifted cohomology space $H^{\bullet-1}(X, T_X \otimes A)$ has a natural structure of a graded Lie algebra, given by composing the cup-product with the Atiyah class. If E is any holomorphic vector bundle over X, then $H^{\bullet-1}(X, E \otimes A)$ is a representation of this Lie algebra.

Then, we unravel the Jacobi identity to make the space of cochains with coefficients in the tangent bundle into a 'Lie algebra up to higher homotopies' [S]. An algebra of this type is best described by exhibiting a complex replacing the Chevalley–Eilenberg complex for an ordinary Lie algebra. In our case this latter complex is identified with the sheaf of functions on the formal neighborhood of the diagonal in $X \times X$, the identification being given by the 'holomorphic exponential map' (the construction of this map goes back at least to the 1953 paper of E. Calabi [C] and was recently rediscovered by physicists under the name of canonical coordinates [BCOV]).

As far as the choice of cochains is concerned, we consider two versions. First, we use Dolbeault forms (and assume that X is equipped with a Kähler metric). Second, we put ourselves into the framework of formal geometry [B] [GGL] [GKF] and use relative forms on the space of formal exponential maps. The underlying algebraic result here is a 1983 theorem of D. B. Fuks [Fuk] who described the stable cohomology of the Lie algebra of formal vector fields with tensor coefficients in terms of what we can today identify as the suspension of the PROP (in the sense of [Ad]) governing weak Lie algebras. In the same fashion, we identify (Theorem 3.7.4) a certain Gilkey-type complex of natural tensors on Kähler manifolds with the suspended weak Lie PROP.

With the nonsymplectic case studied in detail, the introduction of a holomorphic symplectic structure amounts to some easy modifications, presented in Section 5. As another outcome of our considerations we obtain that the $c_\Gamma(X)$ can be calculated from the curvature of an arbitrary Kähler metric, not necessarily compatible in any way with the symplectic structure. This may be useful because the hyper-Kähler metric is rarely known explicitly.

The outline of the paper is as follows. In Section 1 we collect some general (well known) properties of the Atiyah classes of arbitrary holomorphic vector bundles. In Section 2 we specialize to the case of the tangent bundle, intepret the 'cohomological Bianchi identity' of Section 1 as the Jacobi identity and then present an unraveling of this identity on the level of Dolbeault forms on a Kähler manifold. In Section 3 we recast the properties of the Atiyah class in the language of operads and PROPs which is well suited to treat identities among operations such as the Jacobi identity, in an abstract way. At the end of Section 3 we realize the weak Lie PROP by natural differential covariants on Kähler manifolds. Section 4 is devoted to the formal geometry analog of Kähler considerations of Sections 2–3. Finally, in Section 5 we specialize to the case of holomorphic symplectic manifolds and show how the previous constructions are modified and specialized in this case, in particular, how to get the Rozansky–Witten classes $c_\Gamma(X)$ from the Atiyah class of X.

The author's thinking about this question was stimulated by the letter of M. Kontsevich [K2] where he sketched an interpretation of Rozansky–Witten invariants by applying the formalism of characteristic classes of (symplectic) foliations to the $\bar{\partial}$-foliation existing on X considered as a C^∞-manifold. By trying to understand his construction, the author arrived at the very elementary description using the Atiyah class. However, the material of Section 4 comes closer to Kontsevich's approach in that we use the formalism of tautological forms familiar in the theory of characteristic classes of foliations and Gelfand–Fuks cohomology [B].

1. Atiyah classes in general

1.1. The Atiyah class of a vector bundle

Let X be a complex analytic manifold (we can, if we want, work with smooth algebraic varieties over any field of characteristic 0). Let E be a holomorphic vector bundle on X, and $J^1(E)$ be the bundle of first jects of sections of E. It fits into an exact sequence

$$0 \to \Omega^1_X \otimes E \to J^1(E) \to E \to 0, \tag{1.1.1}$$

which therefore gives rise to the extension class

$$\alpha_E \in \mathrm{Ext}^1_X(E, \Omega^1 \otimes E) = H^1(X, \Omega^1 \otimes \mathrm{End}(E)) \tag{1.1.2}$$

known as the Atiyah class of E. An equivalent way of getting α_E is as follows. Let $\mathrm{Conn}(E)$ be the sheaf on X whose sections over $U \subset X$ are holomorphic connections in $E|_U$. As well known, the space of such connections is an affine space over $\Gamma(U, \Omega^1 \otimes \mathrm{End}(E))$, so $\mathrm{Conn}(E)$ is a sheaf of $\Omega^1 \otimes \mathrm{End}(E)$)-torsors. Sheaves of torsors over any sheaf $\mathcal{A}$ of Abelian groups are classified by elements of $H^1(X, \mathcal{A})$, and α_E is the element classifying $\mathrm{Conn}(E)$. So α_E is an obstruction to the existence of a global holomorphic connection. If E, F are two vector bundles, then, in obvious notation, we have

$$\alpha_{E\otimes F} = \alpha_E \otimes 1_F + 1_E \otimes \alpha_F, \tag{1.1.3}$$

because of the well known formula for the connection in a tensor product.

Let $\mathcal{D} = \mathcal{D}_X$ be the sheaf of rings of differential operators on X, and $\mathcal{D}^{\leqslant p} \subset \mathcal{D}$ be the subsheaf of operators of order $\leqslant p$. It has a natural structure of $\mathcal{O}_X$-bimodule, the two module structures being different. The tensor product $\mathcal{D}^{\leqslant 1} \otimes_{\mathcal{O}} E$ is dual to $J^1(E^*)$. Therefore $(-\alpha_E)$ is represented by the extension (symbol sequence)

$$0 \to E \to \mathcal{D}^{\leqslant 1} \otimes_{\mathcal{O}} E \to T \otimes E \to 0. \tag{1.1.4}$$

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1.2. The Bianchi identity

If $a, b \in H^1(X, \Omega^1 \otimes \mathrm{End}(E))$ are any elements, their cup-product $a \cup b$ is an element of $H^2(X, \Omega^1 \otimes \Omega^1 \otimes \mathrm{End}(E) \otimes \mathrm{End}(E))$. We have a natural map of vector bundles on X

$$\Omega^1 \otimes \Omega^1 \otimes \mathrm{End}(E) \otimes \mathrm{End}(E) \to S^2(\Omega^1) \otimes \mathrm{End}(E), \tag{1.2.1}$$

which is the symmetrization with respect to the first two arguments and the commutator in the second two. We denote by $[a \cup b] \in H^2(X, S^2\Omega^1 \otimes \mathrm{End}(E))$ the image of $a \cup b$ under the map induced by (1.2.1) in cohomology.

If A, B, C are three sheaves on X and $u \in \mathrm{Ext}^i(B, C)$, $v \in \mathrm{Ext}^j(A, B)$, then by $u \circ v \in \mathrm{Ext}^{i+j}(A, C)$ we will denote their Yoneda product.

If a is as before and $c \in H^1(X, \mathrm{Hom}(T \otimes T, T)) = \mathrm{Ext}^1(T \otimes T, T)$, then we denote by $a * c \in H^2(X, S^2\Omega^1 \otimes \mathrm{End}(E))$ the Yoneda product of the embedding $S^2T \otimes E \hookrightarrow T \otimes T \otimes E$ and the elements

$$a \in \mathrm{Ext}^1(T \otimes E, E), \qquad c \otimes 1 \in \mathrm{Ext}^1(T \otimes T \otimes E, T \otimes E).$$

PROPOSITION 1.2.2. *The classes α_E, α_T satisfy the following property* (*cohomological Bianchi identity*)

$$2[\alpha_E \cup \alpha_E] + \alpha_E * \alpha_T = 0 \quad \text{in } H^2(X, S^2\Omega^1 \otimes \mathrm{End}(E)).$$

Proof. Consider the two-step filtration

$$E \subset \mathcal{D}^{\leqslant 1} \otimes E \subset \mathcal{D}^{\leqslant 2} \otimes E,$$

with quotients $E, T \otimes E, S^2T \otimes E$ respectively. This filtration gives the extension classes between consecutive quotients

$$(-\alpha_E) \in \mathrm{Ext}^1(T \otimes E, E), \qquad \xi \in \mathrm{Ext}^1(S^2T \otimes E, T \otimes E),$$

whose Yoneda product is 0. Our next task is to identify ξ. In fact, we have the following lemma.

LEMMA 1.2.3. *Let $s\colon T \otimes T \to S^2T$ be the symmetrization. Then $\alpha_{T\otimes E} \in \mathrm{Ext}^1(T \otimes T \otimes E, T \otimes E)$ can be expressed as*

$$\alpha_{T\otimes E} = -\xi \circ (s \otimes 1_E) - 1_T \otimes \alpha_E.$$

The lemma implies (1.2.2) once we expand $\alpha_{T\otimes E}$ by (1.1.3).

Proof. This is a particular case of a statement from [AL], n. (4.1.2.3) applicable to any left $\mathcal{D}$-module $\mathcal{M}$ with a good filtration $(\mathcal{M}_i)$ by vector bundles. In such a situation we have the 'symbol multiplication' maps

$$\mu\colon T \otimes (\mathcal{M}_i/\mathcal{M}_{i-1}) \to \mathcal{M}_{i+1}/\mathcal{M}_i$$

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induced by the $\mathcal{D}$-action on $\mathcal{M}$. We also have natural extension classes

$$f_i \in \mathrm{Ext}^1(\mathcal{M}_{i+1}/\mathcal{M}_i, \mathcal{M}_i/\mathcal{M}_{i-1}).$$

LEMMA 1.2.4. [AL] *In the described situation the class* $(-\alpha_{\mathcal{M}_i/\mathcal{M}_{i-1}})$ *is the difference between the following two compositions* (*Yoneda pairings in which the degree of* Ext *is indicated by square brackets*)

$$T \otimes \mathcal{M}_i/\mathcal{M}_{i-1} \xrightarrow{\mu} \mathcal{M}_{i+1}/\mathcal{M}_i \xrightarrow{f_i} \mathcal{M}_i/\mathcal{M}_{i-1}[1],$$

$$T \otimes \mathcal{M}_i/\mathcal{M}_{i-1} \xrightarrow{1_T \otimes f_{i-1}} T \otimes \mathcal{M}_{i-1}/\mathcal{M}_{i-2}[1] \xrightarrow{\mu} \mathcal{M}_i/\mathcal{M}_{i-1}[1].$$

To obtain Lemma 1.2.3, we take $\mathcal{M} = \mathcal{D} \otimes E$ with $\mathcal{M}_i = \mathcal{D}^{\leqslant i} \otimes E$. Then for $i = 1$ the statement identifies $(-\alpha_{T\otimes E})$. The first composition is $\xi \circ (\sigma \otimes 1_E)$, while the second one is $-1_T \otimes \alpha_E$. This completes the proof.

1.3. ATIYAH CLASS AND CURVATURE

The class α_E can be easily calculated both in Čech and Dolbeault models for cohomology. In the Čech model, we take an open covering $X = \bigcup U_i$ and pick connections ∇_i in $E|_{U_i}$. Then the differences

$$\phi_{ij} = \nabla_i - \nabla_j \in \Gamma(U_i \cap U_j, \Omega^1 \otimes \mathrm{End}(E))$$

form a Čech cocycle representing α_E.

In the Dolbeault model, we pick a C^∞-connection in E of type $(1,0)$, i.e., a differential operator

$$\nabla \colon E \to \Omega^{1,0} \otimes E, \qquad \nabla(f \cdot s) = \partial(f) \cdot s + f \cdot (\nabla s).$$

Let $\tilde{\nabla} = \nabla + \bar{\partial}$ where $\bar{\partial}$ is the (0, 1)-connection defining the holomorphic structure. The curvature $F_{\tilde{\nabla}}$ splits into the sum $F_{\tilde{\nabla}} = F^{2,0}_{\tilde{\nabla}} + F^{1,1}_{\tilde{\nabla}}$ according to the number of antiholomorphic differentials. Then (see [At]).

PROPOSITION 1.3.1. *If* ∇ *is any smooth connection in* E *of type* $(1,\ 0)$, *then* $F^{1,1}_{\tilde{\nabla}}$ *is a Dolbeault representative of* α_E.

Remark 1.3.2. It may be worthwhile to explain why 1.3.1 is indeed a *complete* analog of the Čech construction above. Namely, holomorphic connections in E can be identified with holomorphic sections of a natural holomorphic fiber bundle $C(E)$, which is an affine bundle over $\Omega^1 \otimes \mathrm{End}(E)$. The fiber $C(E)_x$ of $C(E)$ at $x \in X$ is the space of first jets of fiberwise linear isomorphisms $E_x \times X \to E$

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defined near and identical on $E_x \times \{x\}$. Clearly, this is an affine space over $T^*_x X \otimes \mathrm{End}(E_x)$. Now, (1,0)-connections ∇ in E are in natural bijection with arbitary C^∞ sections σ of $C(E)$. Since $C(E)$ is a holomorphic affine bundle, every such σ has a well defined antiholomorphic derivative $\bar\partial\sigma$ which is a (0,1)-form with values in the corresponding vector bundle, i.e.,

$$\bar\partial\sigma \in \Omega^{0,1} \otimes \Omega^{1,0} \otimes \mathrm{End}(E) = \Omega^{1,1} \otimes \mathrm{End}(E).$$

If σ corresponds to ∇, then $\bar\partial\sigma = F^{1,1}_{\tilde\nabla}$.

Proposition 1.3.1 has a corollary for Hermitian connections. Recall [W] that a Hermitian metric in a holomorphic vector bundle E gives rise to a unique connection $\tilde\nabla = \nabla + \bar\partial$ of the above type which preserves the metric. This connection is called the canonical connection of the hermitian holomorphic bundle. It is known that $F_{\tilde\nabla}$ is in this case of type (1,1). Proposition 1.3.1 implies at once the following.

PROPOSITION 1.3.2. *If E is equipped with a Hermitian metric and $\tilde\nabla$ is its canonical connection, then $F_{\tilde\nabla}$ is a Dolbeault representative of α_E.*

1.4. Atiyah class and Chern classes

If X is Kähler, then $c_m(E) \in H^{2m}(X, \mathbf{C})$, the mth Chern class of E, can be seen as lying in $H^m(X, \Omega^m)$, and the relation between the Atiyah class and the curvature implies that $c_m(E)$ is recovered from α_E by the standard Chern–Weil construction

$$c_m(E) = \mathrm{tr}\,\pi(\alpha_E^{\otimes m}), \quad \pi\colon (\Omega^1)^{\otimes m} \otimes \mathrm{End}(E^{\otimes m}) \to \Omega^m \otimes \mathrm{End}(\Lambda^m(E)).$$

It follows that the mth component of the Chern character can be expressed as

$$\mathrm{ch}_m(E) = \frac{1}{m!}\mathrm{Alt}(\mathrm{tr}(\alpha_E^m)). \tag{1.4.1.}$$

Here α_E^m is an element of $H^m(E, (\Omega^1)^{\otimes m} \otimes \mathrm{End}(E))$ obtained using the tensor product in the tensor algebra and the associative algebra structure in $\mathrm{End}(E)$, while Alt is the antisymmetrization $(\Omega^1)^{\otimes m} \to \Omega^m$. Note that the antisymmetrization constitutes in fact an extra step which disregards a part of information: without it, we get an element

$$\hat c_m(E) = \mathrm{tr}(\alpha_E^m) \in H^m(X, (\Omega^1)^{\otimes m}). \tag{1.4.2}$$

For a vector space V let us denote by $\mathrm{Cyc}^m(V)$ the cyclic antisymmetric tensor power of V, i.e.

$$\mathrm{Cyc}^m(V) = \{a \in V^{\otimes m} : ta = (-1)^{m+1}a\}, \quad t = (12, \ldots, m), \tag{1.4.3}$$

where t is the cyclic permutation. Then, the cyclic invariance of the trace implies that

$$\hat{c}^m(E) \in H^m(X, \mathrm{Cyc}^m(\Omega^1)), \tag{1.4.4}$$

but it is not, in general, totally antisymmetric. We will call $\hat{c}_m(E)$ the *big Chern class* of E; the component of the Chern character is obtained from it by total antisymmetrization.

1.5. The Atiyah class of a principal bundle

Let G be a complex Lie group with Lie algebra **g** and $P \to G$ be a principal G-bundle on X. Let ad(P) be the vector bundle on X associated with the adjoint representation of G. By considering connections in P, we obtain, similarly to the above, its Atiyah class $\alpha_P \in H^1(X, \Omega^1 \otimes \mathrm{ad}(P))$. All the above properties of Atiyah classes are obviously generalized to this case.

2. Atiyah class of the tangent bundle and Lie brackets

2.1. Symmetry of the Atiyah class

Let X be as before and $T = TX$ be the tangent bundle of X. Specializing the considerations of (1.1) to the case when $E = T$, we get the Atiyah class α_{TX} which we can see as an element of $H^1(X, T^* \otimes T^* \otimes T)$.

PROPOSITION 2.1.1. *The element* α_{TX} *is symmetric, i.e., lies in the summand* $H^1(X, S^2(T^*) \otimes T)$.

Proof. It is enough to exhibit a sheaf of $S^2(T^*) \otimes T$-torsors from which Conn(T) (a sheaf of $T^* \otimes T^* \otimes T$-torsors) is obtained by the change of scalars. To find it, recall that any connection ∇ in TX has a natural invariant called its *torsion* τ_∇ which is a section of $\Lambda^2(T^*) \otimes T$. The sheaf $\mathrm{Conn}_{tf}(TX)$ of torsion-free connections is thus a torsor over $S^2(T^*) \otimes T$ with required properties.

2.2. Geometric meaning of torsion-free connections

It is convenient to ‘materialize’ the sheaf $\mathrm{Conn}_{tf}(TX)$ by realizing it as the sheaf of sections of a fiber bundle $\Phi(X) \to X$ whose fiber over $x \in X$ is an affine space over $S^2(T_x^* X) \otimes T_x X$. This is done as follows. For $x \in X$ let $\Phi_x(X)$ be the space of second jets of holomorphic maps ϕ: $T_x X \to X$ with the properties $\phi(0) = x, d_0\phi = \mathrm{Id}$. A similarly defined space but for self-maps $T_x X \to T_x X$ is clearly just $S^2(T_x^* X) \otimes T_x X$. Therefore $\Phi_x(X)$ is an affine space over $S^2(T_x^* X) \otimes T_x X$. The $\Phi_x(X)$ for $x \in X$ obviously unite into a fiber bundle $\Phi(X) \to X$. It is well known classically that sections of this bundle are the same as torsion-free connections.

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As a corollary of this, let us note the following interpretation of α_{TX} which can be also deduced from Lemma 1.2.3.

PROPOSITION 2.2.1. *The class* α_{TX} *is, up to a scalar factor, represented by the following extension* (*second symbol sequence*)

$$0 \to T = \mathcal{D}^{\leqslant 1}/\mathcal{D}^{\leqslant 0} \to \mathcal{D}^{\leqslant 2}/\mathcal{D}^{\leqslant 0} \to \mathcal{D}^{\leqslant 2}/\mathcal{D}^{\leqslant 1} = S^2T \to 0.$$

We now state the first main result of this section.

THEOREM 2.3. *Let* X *be any complex manifold and* A *be any quasicoherent sheaf of commutative* $\mathcal{O}_X$*-algebras. Then:*

(a) *The maps*

$$H^i(X, T \otimes A) \otimes H^j(X, T \otimes A) \to H^{i+j+1}(X, T \otimes A)$$

given by composing the cup-product with $\alpha_{TX} \in H^1(X, \mathrm{Hom}(S^2T, T))$, *make the graded vector space* $H^{\bullet-1}(X, T \otimes A)$ *into a graded Lie algebra*.

(b) *For any holomorphic vector bundle* E *on* X *the maps*

$$H^i(X, T \otimes A) \otimes H^j(X, E \otimes A) \to H^{i+j+1}(X, E \otimes A)$$

given by composing the cup-product with the Atiyah class $\alpha_E \in H^1(X, \mathrm{Hom}(T \otimes E, E))$, *make* $H^{\bullet-1}(X, E \otimes A)$ *into a graded* $H^{\bullet-1}(X, T \otimes A)$*-module*.

Remarks 2.3.1. (a) By construction, the structure of a Lie algebra on the space $H^{\bullet-1}(X, T \otimes A)$ is bilinear over the graded commutative algebra $H^\bullet(X, A)$, over which the former space is a module. Same for the module structure on $H^{\bullet-1}(X, E \otimes A)$.

(b) To see that the graded Lie algebra structure defined above is, in general, nontrivial, it suffices to take $A = S^\bullet(T^*)$ (the symmetric algebra with grading ignored), $i = j = 0$, and $a = b = 1 \in H^0(X, T \otimes T^*)$. Then the bracket $[a, b] \in H^1(X, T \otimes S^2T^*)$ is α_{TX}.

(c) Theorem 2.3 is also true for sheaves of graded commutative algebras $A^\bullet$, if we replace cohomology with the hypercohomology, i.e., consider

$$\mathbf{H}^p(X, T \otimes A^\bullet) = \bigoplus_{i+j=p} H^i(X, T \otimes A^j).$$

Proof of Theorem 2.3. (a) If $\mathbf{g}^\bullet$ is a graded vector space with an antisymmetric bracket β: $\wedge^2 \mathbf{g}^\bullet \to \mathbf{g}^\bullet$, then the left-hand side of the Jacobi identity for β is a certain element $j(\beta) \in \mathrm{Hom}(\wedge^3\mathbf{g}^\bullet, \mathbf{g}^\bullet)$. In our case $\mathbf{g}^\bullet = H^{\bullet-1}(X, T \otimes A)$ and we find that $j(\beta)$ is given by composing the cup product with a certain class

$$J \in H^2(X, \mathrm{Hom}(S^3T, T)).$$

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This class is nothing but the symmetrization of

$$[\alpha_{TX} \cup \alpha_{TX}] \in H^2(X, S^2\Omega^1 \otimes \mathrm{Hom}(T, T)),$$

so it vanishes by the 'cohomological Bianchi identity' (1.2.2) applied to $E = T$.

(b) If $\mathbf{g}^\bullet$ is a graded Lie algebra, $M^\bullet$ is a graded vector space with a map c: $\mathbf{g}^\bullet \otimes M^\bullet \to M^\bullet$, then the left-hand side of the identity

$$[g_1, g_2]m - g_1(g_2m) - (-1)^{\deg(g_1)\deg(g_2)} g_2(g_1m) = 0$$

is a certain element $\tau(c) \in \wedge^2\mathbf{g}^* \otimes \mathrm{Hom}(M, M)$, vanishing if and only if M is a $\mathbf{g}$-module. In our case $\mathbf{g}^\bullet = H^{\bullet-1}(X, T \otimes A)$, $M^\bullet = H^{\bullet-1}(X, E \otimes A)$, and the element $\tau(c)$ is induced by a class

$$\sigma \in H^2(X, S^2\Omega^1 \otimes \mathrm{Hom}(E, E))$$

which is nothing but the left-hand side of 1.2.2 for E. Theorem is proved.

The case $A = \mathcal{O}_X$ does not lead to anything interesting. Indeed, we have

PROPOSITION 2.3.2. *The Lie algebra structure on $H^{\bullet-1}(X, T)$ given by α_{TX}, is trivial (all brackets are zero). Similarly, the module structure on $H^{\bullet-1}(X, E)$ is trivial.*

Proof. Let $a \in H^i(X, T)$, $b \in H^j(X, T)$. Using Proposition 2.2.1, the bracket $[a, b] \in H^{i+j+1}(X, T)$ is obtained by applying to $a \cup b \in H^{i+j}(X, S^2T)$ the boundary homomorphism δ: $H^{i+j}(X, S^2T) \to H^{i+j+1}(X, T)$ of the second symbol sequence. But we have a pairing of sheaves

$$T \otimes_{\mathbf{C}} T \to \mathcal{D}^{\leqslant 1} \otimes_{\mathbf{C}} \mathcal{D}^{\leqslant 1} \to \mathcal{D}^{\leqslant 2} \to \mathcal{D}^{\leqslant 2}/\mathcal{D}^{\leqslant 0},$$

induced by the composition of differential operators. Therefore we get an element $a \sqcup b \in H^{i+j}(X, \mathcal{D}^{\leqslant 2}/\mathcal{D}^{\leqslant 0})$ mapping into $a \cup b$. But this implies that $\delta(a \cup b) = 0$.

For the bundle case the argument is similar. We note that α_E is represented by the symbol sequence (1.1.4) and that we have a pairing of sheaves

$$T \otimes_{\mathbf{C}} E \hookrightarrow \mathcal{D}^{\leqslant 1} \otimes_{\mathbf{C}} E \to \mathcal{D}^{\leqslant 1} \otimes_{\mathcal{O}} E.$$

2.4. WEAK LIE ALGEBRAS AND THEIR MODULES

Theorem 2.3 is a global cohomological statement about the Atiyah class. We now want to give a local, cochain level strenghtening of this result. Each time when the cohomology of some sheaf forms a graded Lie algebra, it is natural to seek an underlying structure on the space of cochains. The standard way for doing this is by using the concept of weak Lie algebras (or shLA's), see [S]. Let us recall this concept.

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DEFINITION 2.4.1. A weak Lie algebra is a **Z**-graded **C**-vector space $\mathbf{g}^\bullet$ equipped with a differential d of degree $+1$ and (graded) antisymmetric n-linear operations

$$b_n\colon (\mathbf{g}^\bullet)^{\otimes n} \to \mathbf{g}^\bullet, \quad x_1 \otimes \cdots \otimes x_n \mapsto [x_1, \ldots, x_n]_n, n \geqslant 2,$$
$$\deg(b_n) = 2 - n,$$

satisfying the conditions (generalized Jacobi identities)

$$d(b_n) = \sum_{p+q=n} \sum_{\sigma \in \mathrm{Sh}(p,q)} \mathrm{sgn}(\sigma) b_{p+1}(b_q \otimes 1)\sigma, \tag{2.4.2}$$

where $\mathrm{Sh}(p,q)$ is the set of (p,q)-shuffles and $d(b_n)$ is the value at b_n of the natural differential in $\mathrm{Hom}((\mathbf{g}^\bullet)^{\otimes n}, \mathbf{g}^\bullet)$.

In particular, b_2 is a morphism of complexes ($d(b_2) = 0$), and it makes $H^\bullet_d(\mathbf{g}^\bullet)$ into a graded Lie algebra. The higher b_n are the compensating terms for the violation of the Jacobi identity on the level of cochains rather than cohomology.

An equivalent formulation of (2.4.2) is as follows [S]. Consider $\hat{S}(\mathbf{g}^*[-1])$, the completed symmetric algebra of the shifted dual space to $\mathbf{g}$. Each b_n gives a map $b_n^*\colon \mathbf{g}^*[-1] \to S^n(\mathbf{g}^*[-1])$ of degree 1. Let d_n be the unique odd derivation of the algebra $\hat{S}(\mathbf{g}^*[-1])$ extending b_n^*. Then the identities (2.4.2) all together can be expressed as one condition

$$D^2 = 0, \qquad D = d + \sum_{n \geqslant 2} d_n. \tag{2.4.3}$$

Let $(\mathbf{g}^\bullet, (b_n))$ be a weak Lie algebra. A weak $\mathbf{g}^\bullet$-module is a graded vector space $M^\bullet$ equipped with a differential d of degree $+1$ and maps

$$c_n\colon (\mathbf{g}^\bullet)^{\otimes(n-1)} \otimes M^\bullet \to M^\bullet, n \geqslant 2, \quad \deg(c_n) = 2 - n, \tag{2.4.4}$$

antisymmetric in the first $n-1$ arguments and satisfying the identities which it is convenient to express right away in the form similar to (2.4.3). Namely, let us extend $c_n^*\colon M^\bullet \to M^\bullet \otimes S^{n-1}(\mathbf{g}^*[-1])$ to a derivation d_n^M of the $\hat{S}(\mathbf{g}^*[-1])$-module $M^\bullet \otimes \hat{S}(\mathbf{g}^*[-1])$. Then the condition on the c_n is that

$$(1 \otimes D + D_M)^2 = 0, \qquad D_M = \sum_{n \geqslant 2} d_n^M. \tag{2.4.5}$$

2.5. Weak Lie algebra in Kähler geometry

Let X be a complex manifold. We now unravel the Jacobi identity for the Atiyah class on the level of Dolbeault forms. Since we will work with holomorphic as well as with antiholomorphic objects, let us agree that in the remainder of this section

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$T = TX$ will mean the holomorphic tangent bundle of X, while $\Omega_X^{p,q}$ will signify the space of global C^∞ forms of type (p, q). Similarly, for a holomorphic vector bundle E on X we will denote by $\Omega^{p,q}(E)$ the space of all C^∞ forms of type (p, q) with values in E.

Suppose that X is equipped with a Kähler metric h.

Let ∇ be the canonical $(1, 0)$-connection in T associated with h, so that (1.3)

$$[\nabla, \nabla] = 0 \quad \text{in } \Omega^{2,0}(\mathrm{End}(T)). \tag{2.5.1}$$

Set $\tilde{\nabla} = \nabla + \bar{\partial}$, where $\bar{\partial}$ is the $(0, 1)$-connection defining the complex structure. The curvature of $\tilde{\nabla}$ is just

$$R = [\bar{\partial}, \nabla] \in \Omega^{1,1}(\mathrm{End}(T)) = \Omega^{0,1}(\mathrm{Hom}(T \otimes T, T)). \tag{2.5.2}$$

This is a Dolbeault representative of the Atiyah class α_{TX}, in particular

$$\bar{\partial} R = 0 \quad \text{in} \quad \Omega^{0,2}(\mathrm{Hom}(T \otimes T, T)) \tag{2.5.3}$$

(Bianchi identity). Further, the condition for h to be Kähler is equivalent, as it is well known, to torsion-freeness of ∇, so actually

$$R \in \Omega^{0,1}(\mathrm{Hom}(S^2 T, T)). \tag{2.5.4}$$

Let us now define tensor fields R_n, $n \geqslant 2$, as higher covariant derivatives of the curvature

$$R_n \in \Omega^{0,1}(\mathrm{Hom}(S^2 T \otimes T^{\otimes(n-2)}, T)), \qquad R_2 := R,\ R_{i+1} = \nabla R_i. \tag{2.5.5}$$

PROPOSITION 2.5.6. *Each R_n is totally symmetric, i.e., $R_n \in \Omega^{0,1}(\mathrm{Hom}(S^n T, T))$.*

Proof. Follows immediately from (2.5.1).

Except for $R_2 = R$ the forms R_n are not, in general, $\bar{\partial}$-closed. Let $\Omega^{0,\bullet}(T)$ be the Dolbeault complex of global smooth $(0, i)$-forms with values in T, and $\Omega^{0,\bullet-1}(T)$ be the shifted complex.

THEOREM 2.6. *The maps*

$$b_n\colon \Omega^{0,j_1}(T) \otimes \cdots \otimes \Omega^{0,j_n}(T) \to \Omega^{0,j_1+\cdots+j_n+1}(T), \quad n \geqslant 2,$$

given by composing the wedge product (with values in $\Omega^{0,\bullet}(T^{\otimes n})$) with $R_n \in \Omega^{0,1}(\mathrm{Hom}(T^{\otimes n}, T))$, make the shifted Dolbeault complex $\Omega^{0,\bullet-1}(T)$ into a weak Lie algebra.

COROLLARY 2.6.1. *If X is a Hermitian symmetric space, then R makes $\Omega^{0,\bullet-1}(T)$ into a genuine Lie dg-algebra.*

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Proof. We need to establish the generalized Jacobi identities (2.4.1) for the R_n. For this, write

$$\bar{\partial} R_n = \bar{\partial} \nabla \cdots \nabla R, \tag{2.6.2}$$

(with $(n-2)$ instances of ∇) and use the commutation relation (2.5.2) together with (2.5.3). This gives

$$\bar{\partial} R_n = \sum_{a+b=n-2} \nabla^a \circ R_* \circ \nabla^b R, \tag{2.6.3}$$

where

$$R_* \in \Omega^{1,1}(\mathrm{End}(\mathrm{Hom}(S^{b+2}T, T)))$$

is the operator-valued (1, 1)-form induced by R. By evaluating R_*, we find

$$\bar{\partial} R = \sum_{p+q=n} \sum_{\sigma \in \mathrm{Sh}(p,q)} R_{p+1}(R_q \otimes 1)\sigma,$$

which differs from the right-hand side of the generalized Jacobi identity only by the absense of the signs $\mathrm{sgn}(\sigma)$. These signs, however, constitute exactly the effect of shift from $H^\bullet$ to $H^{\bullet-1}$. Theorem is proved.

Remark. The first instance of Theorem 2.6 (that R_3 cobounds the Jacobi identity for the curvature) was communicated to me by L. Rozansky.

2.7. Companion theorem for vector bundles

Let now (E, h_E) be a Hermitian holomorphic vector bundle on a Kähler manifold X, and let ∇_E be its canonical (0, 1)-connection, so that

$$[\nabla_E, \nabla_E] = 0 \quad \text{in} \quad \Omega^{2,0}(\mathrm{End}(E)). \tag{2.7.1}$$

Let

$$F = [\bar{\partial}, \nabla_E] \in \Omega^{1,1}(\mathrm{End}(E)) = \Omega^{0,1}(\mathrm{Hom}(T \otimes E, E)) \tag{2.7.2}$$

be the total curvature of ∇_E. Then

$$\bar{\partial} F = 0 \quad \text{in} \quad \Omega^{2,0}(\mathrm{Hom}(T \otimes E, E)) \tag{2.7.3}$$

and F is the Dolbeault representative of the Atiyah class α_E. Define the tensor fields

$$F_n \in \Omega^{0,1}(\mathrm{Hom}(S^{n-1}T \otimes E, E)) \tag{2.7.4}$$

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by setting

$$F_2 = F, \quad F_n = \nabla F_{n-1}, n \geqslant 3. \tag{2.7.5}$$

As before, the required symmetry of F follows from (2.7.1).

THEOREM 2.7.6. *The maps*

$$c_n\colon (\Omega^{0,\bullet-1}(T))^{\otimes(n-1)} \otimes \Omega^{0,\bullet-1}(E) \to \Omega^{0,\bullet-1}(E)$$

given by composing the wedge product with F_n*, make the Dolbeault complex* $\Omega^{0,\bullet-1}(E)$ *into a weak module over the weak Lie algebra* $\Omega^{0,\bullet-1}(T)$.

The proof, using (2.7.1–3), is almost identical to that of Theorem 2.6 and is left to the reader.

COROLLARY 2.7.7. *If* (E, h_E) *is a homogeneous Hermitian bundle over a Hermitian symmetric space* X*, then* F *makes* $\Omega^{0,\bullet-1}(E)$ *into a dg-module over the dg-Lie algebra* $\Omega^{0,\bullet-1}(T)$.

2.8. INTERPRETATION VIA $D^2 = 0$

In the notation of Section 2.5, let

$$R_n^* \in \Omega^{0,1}(\mathrm{Hom}(T^*, S^n T^*))$$

be the partial transpose of R_n. Consider the completed symmetric algebra $\hat{S}(T^*)$ (this is a sheaf of ungraded $\mathcal{O}_X$-algebras) and introduce in the algebra $\Omega^{0,\bullet}(\hat{S}(T^*))$ the grading induced from that on $\Omega^{0,\bullet}$. Let $\tilde{R}_n^*$ be the odd derivation of this algebra induced by R_n^*. Theorem 2.6 can be reformulated as follows (I am grateful to V. Ginzburg for suggesting that I do this).

REFORMULATION 2.8.1. *The derivation* $D = \bar{\partial} + \Sigma_{n\geqslant 2}\tilde{R}_n^*$ *of* $\Omega^{0,\bullet}(\hat{S}(T^*))$ *satisfies* $D^2 = 0$.

This is not exactly the result of applying (2.4.3) to $\mathbf{g}^\bullet = \Omega^{0,\bullet-1}(T)$ because we take symmetric powers over $\mathcal{O}_X$ rather than **C** and also do not seem to dualize the spaces $\Omega^{0,\bullet}$. But because the maps R_n are $\mathcal{O}_X$-linear and because the formal adjoint of $\bar{\partial}\colon \Omega^{0,i} \to \Omega^{0,i+1}$ is $\bar{\partial}\colon \Omega^{0,r-i-1} \to \Omega^{0,r-i}$ ($r = \dim(X)$), this change of context is justified.

Let us now view D geometrically. The sheaf $\hat{S}(T^*)$ is the sheaf of functions on $X_{TX}^{(\infty)}$, the formal neighborhood of X (regarded as the zero section) in (the total space of) TX. More formally, denoting by $\pi\colon X_{TX}^{(\infty)} \to X$ the natural projection, we can write

$$\hat{S}(T^*) = \pi_*(\mathcal{O}_{X_{TX}^{(\infty)}}).$$

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The derivation D in $\Omega^{0,\bullet}(\hat{S}(T^*))$ can thus be regarded as a non-linear $(0,1)$-connection $\mathbf{D}$ in the fiber bundle $\pi\colon X^{(\infty)}_{TX} \to X$. The condition $D^2 = 0$ means that $\mathbf{D}$ is integrable, i.e., defines a new holomorphic structure in $X^{(\infty)}_{TX}$. We are going to describe this new structure and at the same time give a very natural explanation of the previous constructions. Namely, consider $X^{(\infty)}_{X\times X}$, the formal neighborhood of the diagonal $X \subset X \times X$. This is a fiber bundle over X (with respect to the projection to, say, the second factor) whose fiber over $x \in X$ is $x^{(\infty)}_X = \mathrm{Spf}(\hat{\mathcal{O}}_{X,x})$, the formal neighborhood of x in X. Clearly, this fiber bundle has a holomorphic structure induced from that on X.

THEOREM 2.8.2. *The bundle $X^{(\infty)}_{TX}$ with the new complex structure $\mathbf{D}$ is naturally isomorphic to $X^{(\infty)}_{X\times X}$.*

The proof is given in the next subsection.

2.9. The holomorphic exponential map

We want now to recall a classical but not very well known construction in Kähler geometry [C]. We preserve the notations from the previous subsections.

Let $x \in X$ be a point. Recall that by T_xX we denote $T^{1,0}_xX$, the 'holomorphic' tangent space which we want to distinguish from $T^{\mathbf{R}}_xX$, the tangent space to X considered as a real manifold. More precisely, let $I\colon T^{\mathbf{R}}_xX \to T^{\mathbf{R}}_xX$ be the complex structure, $I^2 = -1$, and $T^{\mathbf{C}}_xX = \mathbf{C} \otimes_{\mathbf{R}} T^{\mathbf{R}}_xX$. Then T_xX is the $(+i)$-eigenspace of $1 \otimes I$ on $T^{\mathbf{C}}_xX$. The correspondence $\xi \mapsto \xi - iI\xi$ defines an isomorphism of complex vector spaces $(T^{\mathbf{R}}_xX, I) \to (T_xX, i)$.

Now, the geodesic exponential map at x (for X considered as a real manifold)

$$\exp^{\mathbf{R}}_x\colon T^{\mathbf{R}}_xX \to X$$

is not, in general, holomorphic. Suppose first that our Kähler metric is real analytic. Then so is $\exp^{\mathbf{R}}_x$, and we can take its analytic continuation 'to the complex domain'. In other words, let $X' = X$ and X'' be X with the opposite complex structure. Then the image of the diagonal embedding $X \hookrightarrow X' \times X''$ is totally real, so $X' \times X''$ can be seen as the complexification of X. Therefore $\exp^{\mathbf{R}}_x$ continues to a holomorphic map

$$\exp^{\mathbf{C}}_x\colon T^{\mathbf{C}}_xX = T_xX \oplus T^{0,1}_xX \to X' \times X'',$$

defined in some neighborhood of 0.

LEMMA 2.9.1. *Suppose the Kähler metric on X is real analytic. Then, the restriction of $\exp^{\mathbf{C}}_x$ to T_xX takes values in $X' \times \{x\}$ and thus gives (via the holomorphic identification $X' \to X$) a holomorphic map $\exp_x\colon T_xX \to X$ defined in some neighborhood of 0, and whose differential at 0 is the identity.*

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Proof. The complexified Riemannian connection on $T^{\mathbf{C}}X$ is $\tilde{\nabla} = \nabla + \bar{\partial}$. Its analytic continuation is a holomorphic connection $\tilde{\nabla}^{\mathbf{C}} = \nabla^{\mathbf{C}} + \bar{\partial}^{\mathbf{C}}$ in the holomorphic tangent bundle of $X' \times X''$, defined in some neighborhood of X. The summands $\nabla^{\mathbf{C}}$ and $\bar{\partial}^{\mathbf{C}}$ have types (1, 0) and (0, 1) with respect to the decomposition

$$T_{(x',x'')}(X' \times X'') = T_{x'}X' \oplus T_{x''}X''.$$

This decomposition being flat for $\nabla^{\mathbf{C}}$ and holomorphic, the exponential map for $\nabla^{\mathbf{C}}$ at a diagonal point $(x, x), x \in X$ takes T_xX' into $X' \times \{x\} \simeq X$. But $T_xX' \subset T_{(x,x)}(X' \times X'')$ is precisely $T_x^{1,0}X \subset T_x^{\mathbf{C}}X$, and the exponential map for $\nabla^{\mathbf{C}}$ is just the restriction of $\exp_x^{\mathbf{C}}$ to $T_x^{1,0}$. Lemma is proved.

The map $\exp_x$ can be called the *holomorphic exponential map*. It was rediscovered in 1994, in the physical paper [BCOV] and called ‘canonical coordinates’. Note that even when the metric is not analytic but only smooth, consideration of the Taylor expansion of $\exp_x^{\mathbf{R}}$ in coordinates $z_i, \bar{z}_i$ (where z_i form a local holomorphic coordinate system), furnishes an isomorphism of formal neighborhoods

$$\exp_x\colon 0_{T_xX}^{(\infty)} \to x_X^{(\infty)}, \tag{2.9.2}$$

which will be sufficient for the purposes we have in mind.

EXAMPLE 2.9.3. Let $X = \mathbf{C}P^1$ with the Fubini–Study metric. As a Riemannian manifold, X is the unit sphere $S^2 \subset \mathbf{R}^3$. Choose a point $x \in X$ and introduce in T_xX a linear coordinate system (u, v) by means of an orthogonal frame. Then identify a neighborhood of x with T_xX by means of the stereographic projection from the opposite point, thereby introducing a coordinate system in X. Elementary trigonometry gives

$$\exp_x^{\mathbf{R}}(u, v) = \frac{2\sin\sqrt{u^2+v^2}}{\sqrt{u^2+v^2}(1+\cos\sqrt{u^2+v^2})}(u, v),$$

(this is real analytic since $\sin(z)/z$ and $\cos(z)$ are even functions). Now, the complex structure in $T_x^{\mathbf{R}}X$ is $I(a, b) = (b, -a)$. Thinking now of u, v as complex variables and substituting $u = a - ib, v = b + ia$ with $a, b \in \mathbf{R}$ (which means that we restrict to $T_x^{1,0} \subset T_x^{\mathbf{C}}$) we find that the radicals vanish and we get $\exp_x(z) = z$, $z \in T_xX$. So the holomorphic exponential map is, in this case, exactly the stereographic projection, i.e., the affine coordinate on $\mathbf{C}P^1$ for which the point opposite to x serves as the infinity. In a similar way, for X a Grassmannian the map $\exp_x$ provides an affine identification of T_xX with an open Schubert cell.

Let us now prove Theorem 2.8.2. Consider, for any $x \in X$, the formal isomorphism (2.9.2). These isomorphisms unite into a fiberwise holomorphic isomorphism of fiber bundles

$$\exp\colon X_{TX}^{(\infty)} \to X_{X\times X}^{(\infty)}.$$

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The variation with respect to x of the $\exp_x$ is not, in general, holomorphic in the usual sense. However, we have the following statement which implies our theorem.

PROPOSITION 2.9.4. *The map* exp *is holomorphic with respect to the complex structure* **D** *on* $X_T^{(\infty)}$.

Proof. We will consider the real analytic case. The general case presents only notational complications in that we replace X' and X'' below by working in the variables z_i and $\bar{z}_i$.

By considering the connection $\nabla^{\mathbf{C}}$ on $X' \times X''$, we reduce ourselves to the following purely holomorphic problem.

Suppose given a complex manifold X and a family $\nabla = (\nabla_s)_{s\in S}$ of flat torsion-free connections in TX parametrized by some complex manifold S. Let p, q be the projections of $X \times S$ to X and S respectively. Then the variation (derivative) of the ∇_s with respect to s is a section

$$R \in \Gamma(X \times S,\, q^*\Omega^1_S \otimes p^*\mathrm{Hom}(S^2TX, TX)).$$

We can apply to each restriction $R|_{X\times\{s\}}$ the covariant derivative ∇_s several times, getting tensor fields

$$R_n = \nabla^{n-2}R \in \Gamma(X \times S,\, q^*\Omega^1_S \otimes p^*\mathrm{Hom}(S^nTX, TX)), \quad n \geqslant 2.$$

On the other hand, for every x, s the connection ∇_s gives rise to the exponential map

$$\exp_{x,s}\colon T_xX \to X, \qquad 0 \mapsto x, \qquad d_0 \exp_{x,s} = \mathrm{Id},$$

whose variation with respect to s is, for each fixed x, a 1-form on X with values in analytic vector fields on (some neighborhood of 0 in) T_xX with vanishing constant and linear terms. Recall that for any vector space V the space of formal vector fields on V at 0 is the product $\prod_{n\geqslant 0}\mathrm{Hom}(S^nV, V)$. Thus we can write the Taylor expansion of the variation as

$$\exp_{x,s}^{-1} d_s\, \exp_{x,s} \in \Gamma\left(X \times S,\, q^*\Omega^1_S \otimes \prod_{n\geqslant 2} p^*\mathrm{Hom}(S^nTX, TX)\right).$$

In order to establish Proposition 2.9.4, it is enough to prove the following.

PROPOSITION 2.9.5. *In the described situation* R_n *is the* nth *homogeneous component of* $\exp_{x,s}^{-1} d_s\, \exp_{x,s}$.

Proof. Fix some $x_0 \in X, s_0 \in S$ and identify $T_{x_0}X$ with $\mathbf{C}^r$, $r = \dim(X)$ by means of some linear isomorphism. Then use $\exp_{x_0,s_0}$ as a coordinate system on

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X near x_0. For any s the connection ∇_s is then defined in our coordinates by its connection matrix $\Gamma(s) \in \Gamma(\mathbf{C}^r, \mathrm{Hom}(S^2T, T))$, so that $R = d_s\Gamma(s)$ is just its derivative with respect to s. For $s = s_0$ we have $\Gamma(s_0) = 0$, because the exponential map for a flat torsion free connection takes it into the standard Euclidean connection on the tangent space. This implies that the higher covariant derivatives $\nabla^i_{s_0} R|_{X\times\{s_0\}}$ are the same as the usual derivatives, with respect to our chosen coordinates, of $R_{s_0} = d_s|_{s=s_0}\Gamma(s)$. By the same token as before, for arbitrary s the flatness of ∇_s allows us to describe it as the connection induced from the standard Euclidean connection on $\mathbf{C}^r$ by the change of coordinates given by $\exp_{x_0,s}$. So our statement reduces to the following lemma.

LEMMA 2.9.6. *Let* $v = \Sigma_{i=1}^r v_i(z)\partial/\partial z_i$ *be a holomorphic vector field on* (*some domain of*) $\mathbf{C}^r$. *Regarding* v *as an infinitesimal diffeomorphism* (*i.e., the tangent to a family of diffeomorphisms* $g(s)\colon \mathbf{C}^r \to \mathbf{C}^r$, $s \in \mathbf{C}$, $g(0) = \mathrm{Id}$), *let* $\Gamma \in \Gamma(\mathbf{C}^r, \mathrm{Hom}(S^2T, T))$ *be the corresponding infinitesimal variation of the connections* (*induced by the* $g(s)$ *from the Euclidean one*). *Then the components of* Γ *are*

$$\Gamma^i_{jk}(z) = \frac{\partial^2 v_i}{\partial z_j \partial z_k}.$$

The proof of this lemma is straightforward from the standard formulas of differential geometry.

3. Operadic interpretation

As we saw, for any sheaf A of commutative algebras on X, the Atiyah class $\alpha_{TX} \in H^1(X, \mathrm{Hom}(S^2T, T))$ makes each $H^{\bullet-1}(X, T\otimes A)$, into a graded Lie algebra. Each composite m-ary operation in this algebra (such as, e.g., $[[x_1, x_2], [x_3, x_4]]$ for $m = 4$) is represented by a certain class in $H^{m-1}(X, \mathrm{Hom}(T^{\otimes m}, T))$ composed out of α_{TX}. In this section we study these classes by themselves rather than by using the operations on $H^{\bullet-1}(X, T\otimes A)$ represented by them. For this, we use the language of operads and PROPs, see [Ad] [GiK], [GeK1-2] [KM].

3.1. Reminder on operads, PROPs and modules

Recall that an operad $\mathcal{P}$ is a collection of vector spaces $\mathcal{P}(n)$, $n \geqslant 0$, together with the action of S_n, the symmetric group, on $\mathcal{P}(n)$ for each n and composition maps

$$\circ_i\colon \mathcal{P}(m)\otimes\mathcal{P}(n) \to \mathcal{P}(m+n-1), \quad i = 1, \ldots, m$$

satisfying appropriate equivariance and associativity axioms. Informally, elements of $\mathcal{P}(m)$ can be thought of as m-ary operations, the S_m-action as permutation of arguments in the operations, and $p \circ_i q$ as the operation

$$p(x_1, \ldots, x_{i-1}, q(x_i, \ldots, x_{i+n-1}), x_{i+n}, \ldots, x_{m+n-1}). \tag{3.1.1}$$

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An algebra over an operad $\mathcal{P}$ is a vector space A together with S_n-invariant maps $\mu_n\colon \mathcal{P}(n) \otimes A^{\otimes n} \to A$ satisfying the associativity properties which mean that the compositions $\circ_i$ in $\mathcal{P}$ indeed go, under the μ_n, into the substitution of one operation inside another, as described in (3.1.1).

The concept of a PROP (see [Ad]) is slightly more general. While operads describe algebras A with operations of the form $A^{\otimes n} \to A$, PROPs allow for more general operations $A^{\otimes n} \to A^{\otimes m}$ (which may or may not be deducible from the former ones).

Thus a PROP Π is a family of vector spaces $\Pi(n, m)$, $n, m \geqslant 0$, equipped with a left S_n-action and a right S_m-action, commuting with each other, as well as the following structures:

(3.1.2) Composition maps $\Pi(n, p) \otimes \Pi(m, n) \to \Pi(m, p)$, making Π into a category with the set of objects $[m], m \in \mathbf{Z}_+$ and $\mathrm{Hom}([n], [m]) = \Pi(n, m)$.

(3.1.3) Juxtaposition maps $\Pi(n, m) \otimes \Pi(n', m') \to \Pi(n+n', m+m')$, making Π into a symmetric monoidal category with monoidal operation on objects defined by $[n] \odot [n'] = [n + n']$.

A (right) module over an operad $\mathcal{P}$ is (see [M]) a collection $\mathcal{M}$ of S_n-modules $\mathcal{M}(n)$, $n \geqslant 0$ and compositions

$$\circ_i\colon \mathcal{M}(m) \otimes \mathcal{P}(n) \to \mathcal{M}(m + n - 1), \quad i = 1, \ldots, m$$

satisfying the equivariance and associativity axioms obtained by polarizing those of an operad.

EXAMPLE 3.1.4. (a) For any vector space V we have its *endomorphism operad* $\mathcal{E}_V$ with components $\mathcal{E}_V(n) = \mathrm{Hom}(V^{\otimes n}, V) = (V^*)^{\otimes n} \otimes V$. The space V is canonically an algebra over $\mathcal{E}_V$. For any operad $\mathcal{P}$ a structure of $\mathcal{P}$-algebra on a vector space A is the same as a morphism of operads $\mathcal{P} \to \mathcal{E}_A$.

Similarly, we have a PROP END_V with $\mathrm{END}_V(n, m) = \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})$. An algebra over a PROP Π is a vector space A together with a morphism of PROPs $\Pi \to \mathrm{END}_A$. For example, the class of Hopf algebras can be described by a PROP but not an operad.

(b) Any operad is a module over itself. If Π is a PROP, then the spaces $\mathcal{P}(n) = \Pi(n, 1)$ form an operad. For every k the spaces $\Pi_a(n) = \Pi(n, a)$ form a module over this operad.

3.2. DG-OPERADS AND PROPs

All the above constructions can be carried out in any symmetric monoidal category. By a differential graded (dg-) operad we mean an operad in the symmetric monoidal category of differential graded vector spaces, i.e., cochain complexes

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(in that category the symmetry isomorphisms are given by the Koszul sign rule). For a dg-vector space $V^\bullet$ we define its shifts $V^\bullet[m]$ by $(V^\bullet[m])^i = V^{m+i}$. For a dg-operad $\mathcal{P}$ its *suspension* $\Sigma(\mathcal{P})$ is a new dg-operad formed by the shifted spaces $\Sigma(\mathcal{P})(n) = \mathcal{P}(n)[1-n]$ with the symmetric group action differing from that on $\mathcal{P}(n)$ be tensoring with the sign representation, see [GeK1] for the explicit formulas for the compositions. If $A^\bullet$ is a differential graded $\mathcal{P}$-algebra, then $\mathcal{A}[1]$ is a $\Sigma(\mathcal{P})$-algebra. For $p \in \mathcal{P}(n)$ let $\Sigma(p)$ be the corresponding element of $\Sigma(\mathcal{P})(n)$. The conventions for PROPs are similar. Thus, the suspension $\Sigma\Pi$ of a dg-PROP Π has $\Sigma\Pi(n,m) = \Pi(n,m)[m-n]$. We will view graded vector spaces as dg-vector spaces with zero differential.

3.3. A PROP FROM AN OPERAD

Let $\mathcal{P}$ be an operad. We define a $\mathcal{P}$-module $\mathcal{P}(-,0) = \{\mathcal{P}(n,0)\}$ called the module of natural forms (on $\mathcal{P}$-algebras). It is defined as the $\mathcal{P}$-module generated by symbols

$$\mathrm{tr}(p) \in \mathcal{P}(n,0), \qquad p \in \mathcal{P}(n+1), \tag{3.3.1}$$

subject to the following relations

$$\mathrm{tr}(p\sigma) = \mathrm{tr}(p)\sigma, \qquad \sigma \in S_n \subset S_{n+1}, \tag{3.3.2}$$

$$\mathrm{tr}(p \circ_i q) = \mathrm{tr}(p) \circ_i q, \;\; p \in \mathcal{P}(a+1), \;\; q \in \mathcal{P}(b+1), \;\; i \neq a+1, \tag{3.3.3}$$

$$\begin{aligned} &\mathrm{tr}(p \circ_{a+1} q) = \mathrm{tr}(q \circ_{b+1} p)\tau, \\ &\tau = \begin{pmatrix} 1 & 2 & \dots & a & \dots a+b \\ a+1 & \dots & a+b & 1 & \dots \quad a \end{pmatrix}, \\ &p \in \mathcal{P}(a+1), \qquad q \in \mathcal{P}(b+1). \end{aligned} \tag{3.3.4}$$

Motivation: if A is a finite-dimensional $\mathcal{P}$-algebra, then any $p \in \mathcal{P}(n+1)$ gives a morphism $\mu_p\colon A^{\otimes(n+1)} \to A$, and we can take its trace $\mathrm{tr}_{n+1}(\mu_p)\colon A^{\otimes n} \to \mathbf{C}$ with respect to the last contravariant argument and the only covariant argument. The requirements on the $\mathrm{tr}(p)$ are the axiomatizations of the properties of these traces.

We now define a PROP, denoted $\Pi_\mathcal{P}$ to be generated by formal juxtapositions and permutations from $\mathcal{P}(n,0) = \Pi_\mathcal{P}(n,0)$ and $\mathcal{P}(n) \subset \Pi_\mathcal{P}(n,1)$. In other words

$$\Pi_\mathcal{P}(n,m) = \bigoplus_{\substack{\{1,\dots,n\}= \\ A_1\cup\dots\cup A_m \cup B_1 \cup\dots\cup B_r}} \bigotimes_i \mathcal{P}(A_i) \otimes \bigotimes_j \mathcal{P}(B_j, 0), \tag{3.3.5}$$

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where $\mathcal{P}(A), \#(A) = a$, is the notation for the functor on the category of a-element sets and their bijections associated to the S_a-module $\mathcal{P}(a)$.

PROPOSITION 3.3.6. *If A is a finite-dimensional $\mathcal{P}$-algebra, then it is also a $\Pi_{\mathcal{P}}$-algebra.*

3.4. THE LIE OPERAD AND PROP

We denote by $\mathcal{L}ie$ the Lie operad, whose algebras are Lie algebras in the usual sense, see [GeK1-2] [GiK]. Explicitly, $\mathcal{L}ie(n)$ is a subspace in the free Lie algebra on generators $x_1, \ldots, x_n$ spanned by Lie monomials containing each x_i exactly once. Thus $\mathcal{L}ie(2)$ is one-dimensional and spanned by $[x_1, x_2]$ (which is anti-invariant under S_2), while $\mathcal{L}ie(3)$ is two-dimensional and spanned by three elements

$$[x_1, [x_2, x_3]], \qquad [x_2, [x_1, x_3]], \qquad [x_3, [x_1, x_2]],$$

whose sum is zero (Jacobi identity). Given an arbitrary operad $\mathcal{P}$ and an element $p \in \mathcal{P}(2)$, we will say that p is a *Lie element*, if p is antisymmetric and satisfies the Jacobi identity. In other words, p is a Lie element if there is a morphism of operads $\mathcal{L}ie \to \mathcal{P}$ which takes the generator $[x, y] \in \mathcal{L}ie(2)$ into p. Such a morphism is unique, if it exists.

We denote the PROP $\Pi_{\mathcal{L}ie}$ by LIE. The new generators in LIE (apart from the bracket $[x_1, x_2] \in \mathrm{LIE}(2, 1)$) form the space $\mathrm{LIE}(n, 0) = \mathcal{L}ie(n, 0)$. An example of an element of the latter space is given by

$$\kappa_n = \mathrm{tr}([x_1[x_2 \ldots [x_n, x_{n+1}] \ldots]). \tag{3.4.1}$$

Here $[x_1[x_2 \ldots [x_n, x_{n+1}] \ldots]$ is regarded as an element of $\mathcal{L}ie(n+1)$. It follows from (3.3.4) that κ_n is cyclically symmetric, i.e.,

$$\kappa_n t = \kappa_n, \quad t = (12 \ldots n) \in \mathbf{Z}_n \subset S_n. \tag{3.4.2}$$

If $\mathbf{g}$ is a finite-dimensional Lie algebra, then κ_n gives the nth Killing form on $\mathbf{g}$

$$x_1 \otimes \cdots \otimes x_n \mapsto \mathrm{tr}(\mathrm{ad}(x_1) \ldots \mathrm{ad}(x_n)). \tag{3.4.3}$$

PROPOSITION 3.4.4. *The space $\mathcal{L}ie(n, 0)$ has dimension $(n-1)!$ and a basis there is formed by the elements $\kappa_n \sigma$, $\sigma \in S_n/\mathbf{Z}_n$.*

Proof. This follows from the fact that a basis in $\mathcal{L}ie(n+1)$ is formed by the Lie monomials

$$[x_{\sigma_1}[\ldots [x_{\sigma(n)}, x_{n+1}] \ldots], \quad \sigma \in S_n.$$

3.5. The Atiyah class as a Lie element

Let now X be a complex manifold, $T = TX$ its tangent bundle. We have a sheaf of operads $\mathcal{E}_T$ and a sheaf of PROPs END_T on X defined by

$$\mathcal{E}_T(n) = \mathrm{Hom}(T^{\otimes n}, T), \qquad \mathrm{END}_T(n,m) = \mathrm{Hom}(T^{\otimes n}, T^{\otimes m}).$$

By applying the functor $H^\bullet(X, -)$ from sheaves to graded vector spaces, we get a graded operad $H^\bullet(X, \mathcal{E}_T)$ an a graded PROP $H^\bullet(X, \mathrm{END}_T)$. Recall also (1.4) that we have the 'big Chern classes' $\hat{c}_m(T) \in H^m(X, \mathrm{Cyc}^m(\Omega^1))$ of the tangent bundle. Now, a more inclusive formulation of the properties of the Atiyah class is by using the suspension of the above PROP and goes as follows.

THEOREM 3.5.1. *The element* $\Sigma^{-1}\alpha_{TX} \in \Sigma^{-1}H^\bullet(X, \mathcal{E}_T)(2)$ *is a Lie element. Furthermore, the correspondence*

$$[x_1, x_2] \in \mathrm{LIE}(2,1) \mapsto \Sigma^{-1}\alpha_{TX} \in \Sigma^{-1}H^\bullet(X, \mathrm{END}_T)(2,1),$$

$$\kappa_n \in \mathrm{LIE}(n,0) \mapsto \Sigma^{-1}\hat{c}_m(T) \in \Sigma^{-1}H^\bullet(X, \mathrm{END}_T)(n,0)$$

defines a morphism of PROPs

$$\mathrm{LIE} \to \Sigma^{-1}H^\bullet(X, \mathrm{END}_T) = H^\bullet(X, \mathrm{END}_{T[-1]}).$$

The proof follows readily from the cohomological Bianchi identity (1.2).

3.6. Weak Lie operad and PROP

We denote by $\mathcal{WL}ie$ the dg-operad governing weak Lie algebras (2.4). It is generated by elements $\beta_n \in \mathcal{WL}ie(n)$, $\deg(\beta_n) = 2-n$, $n \geqslant 2$, which are antisymmetric with respect to S_n and satisfy the conditions obtained from Definition 2.4.1. Thus, the cohomology operad $H_d^\bullet(\mathcal{WL}ie)$ is just $\mathcal{L}ie$.

The operad $\mathcal{WL}ie$ can be also described as the cobar-construction of the commutative operad [GiK]. Explicitly, this means that a basis in $\mathcal{WL}ie(n)$ is formed by certain trees. More precisely, by an n-tree we mean a connected oriented graph Γ with no loops, equipped with structures satisfying the conditions listed below.

(1) Each vertex of Γ has valency at least 3. In addition, Γ has $n+1$ legs, i.e., edges bounded by a vertex from one side only.
(2) For every vertex v all edges incident to v, except exactly one, are oriented towards v. The set of such edges is denoted by $\mathrm{In}(v)$.
(3) It follows that all the legs of Γ except exactly one, are oriented towards Γ. The set of such legs is denoted by $\mathrm{In}(\Gamma)$.
(4) The set $\mathrm{In}(\Gamma)$ is identified with $\{1, 2, \ldots, n\}$.

Let $\mathcal{T}(n)$ be the set of isomorphism classes of n-trees. For $\Gamma \in \mathcal{T}(n)$ set

$$\det(\Gamma) = \bigotimes_{v \in \mathrm{Vert}(\Gamma)} \bigwedge^{\max} (\mathbf{C}^{\mathrm{In}(v)}). \tag{3.6.1}$$

PROPOSITION 3.6.2. *We have an identification of graded vector spaces*

$$\mathcal{WL}ie(n) = \bigoplus_{\Gamma \in \mathcal{T}(n)} \det(\Gamma)^*, \qquad \deg(\det(\Gamma)^*) = \sum_{v \in \mathrm{Vert}(\Gamma)} (2 - |\mathrm{In}(v)|),$$

with the differential being dual to the map given by contraction of edges, and the operad structure given by the grafting of trees, see [*GiK*].

Proof. The identification is obtained by associating to β_n the unique n-tree with one vertex ('corolla') and to any composition of the β_n the tree describing the composition. The terms in the generalized Jacobi identity correspond, in geometric language, to all possible n-trees with exactly two vertices and one edge (so that the corolla is obtained from such a tree by contracting this unique edge).

Let WLIE be the (dg-) PROP corresponding to the dg-operad $\mathcal{WL}ie$ as described in (3.3). It also has a natural graphical description. Namely, call an (n, m)-graph a (not necessarily connected) oriented graph Γ with $n + m$ legs, of which n are inputs and are labelled with numbers $1, \ldots, n$, and m are outputs and are labelled by $1, \ldots, m$, and which satisfy the conditions (1)–(2) above. Each component of an (n, m) graph is either a tree satisfying (1)–(3), or a graph with no output. Let $\mathcal{G}(n, m)$ be the set of isomorphism classes of (n, m)-graphs. Retaining the same notations Vert, In, det, as for trees, we easily conclude the following.

PROPOSITION 3.6.3. *We have identifications*

$$\mathrm{WLIE}(n, m) = \bigoplus_{\Gamma \in \mathcal{G}(n,m)} \det(\Gamma)^*, \qquad \deg(\det(\Gamma)^*) = \sum_{v \in \mathrm{Vert}(\Gamma)} (2 - |\mathrm{In}(v)|),$$

with the differential being dual to the map given by contraction of edges, composition maps given by grafting of graphs, and juxtaposition maps given by disjoint union.

If $\mathcal{P}$ is any dg-operad, a family of elements $p_n \in \mathcal{P}(n)$, $n \geqslant 2$, is called *a weak Lie family*, if the correspondence $\beta_n \mapsto p_n$ gives a morphism of dg-operads $\mathcal{WL}ie \to \mathcal{P}$. If (p_n) is a weak Lie family, then the class of p_2 in $H^0(\mathcal{P}(2))$ is a Lie element in $H^\bullet(\mathcal{P})$. In this case the p_n give also a morphism of PROPs $\mathrm{WLIE} \to \Pi_{\mathcal{P}}$.

3.7. Differential covariants and the weak Lie PROP

We now want to restate Theorem 2.6 (which describes the unraveling of the Jacobi identity for the Atiyah class in the framework of Kähler geometry) in a more universal form.

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Notice, first of all, that the structure we really used, was not the Kähler metric itself but only its canonical $(1,0)$-connection ∇. So let us call a *semiflat manifold* a pair (X,∇) where X is a complex manifold and ∇ is a $(1,0)$-connection in TX such that $[\nabla,\nabla]=0$. For such a connection we define $R=[\bar\partial,\nabla]$ and all the considerations of (2.6), 2.8 hold true.

Fix d,i,m,n and let $\mathcal{C}_r$ be the sheaf of semiflat (0,1)-connections on $\mathbf{C}^r$. Following Gilkey [Gil] and Epstein [E], introduce the space $V_r^i(n,m)$ of (not necessarily linear) differential operators of finite order $\mathcal{C}_\Gamma\to\Omega^{0,i}\otimes\mathrm{Hom}(T^{\otimes n},T^{\otimes m})$ defined in some neighborhood of 0, and let $U_r^i(n,m)\subset V_r^i(n,m)$ be the subspace of operators equivariant under the group of holomorphic diffeomorphisms. Elements of the latter space will be called differential covariants of type (i,n,m) of r-dimensional semiflat manifolds, since for each such manifold (X,∇) they produce natural tensors in $\Omega^{0,i}(\mathrm{Hom}(T^{\otimes n},T^{\otimes m}))$. In particular, they do so for each Kähler manifold. In fact, we can say that elements of $U_r^i(n,m)$ are differential covariants of Kähler manifolds which depend only on the canonical connection. The differential $\bar\partial$ makes $U_r^\bullet(n,m)$ into a complex; taken for all n,m, these complexes form a dg-PROP $U_r^\bullet$.

For example, $R_n=\nabla^{n-2}R$ (the covariant derivative of the curvature) is an element of $U_r^1(n,1)$. Furthermore, let Γ be a (n,m)-graph with N vertices. For every vector space W we have the contraction map

$$p_\Gamma\colon \bigotimes_{v\in\mathrm{Vert}(\Gamma)}\mathrm{Hom}(S^{|\mathrm{In}(v)|}W,W)\to\mathrm{Hom}(W^{\otimes n},W^{\otimes m}). \tag{3.7.1}$$

Applying this to the tensor product of the $R_{|\mathrm{In}(v)|}\in\Omega^{0,1}(\mathrm{Hom}(S^{|\mathrm{In}(v)|}T,T))$, we get a covariant

$$R_\Gamma=p_\Gamma\Big(\bigotimes R_{|\mathrm{In}(v)|}\Big)\in U_r^N(n,m). \tag{3.7.2}$$

Because of the symmetry of the R_i, the desuspended element $\Sigma^{-1}R_\Gamma$ can be viewed as a morphism

$$\Sigma^{-1}R_\Gamma\colon\ \det(\Gamma)^*\to\Sigma^{-1}U_r^\bullet(n,m). \tag{3.7.3}$$

THEOREM 3.7.4. (a) *The maps $\Sigma^{-1}R_\Gamma$ define a morphism of dg-PROPs ρ*: WLIE $\to\Sigma^{-1}U_r^\bullet$.

(b) *For any i,n,m the morphism of vector spaces $\rho^i_{n,m}$*: $\mathrm{WLIE}^i(n,m)\to(\Sigma^{-1}U_r)^i(n,m)$ *is surjective.*

(c) *If $r\gg i,m,n$, then the morphism $\rho^i_{n,m}$ is in fact bijective.*

Part (c) means that the ‘stabilized’ PROP of differential covariants is just the suspension of the weak Lie PROP.

Proof. (a) Follows from Theorem 2.6.

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(b) Covariants of ∇ can be viewed as covariants of the total connection $\tilde{\nabla} = \nabla + \bar{\partial}$. It is known classically that all covariants of an affine connection are obtained from the covariant derivatives of the curvature (and torsion) by performing 'tensorial contractions'. For example, the argument sketched in [E] exhibits the Taylor expansion of the Christoffel symbols in the normal coordinates in such a form, and this clearly suffices. In our case, the covariant derivatives of the curvature of $\tilde{\nabla}$ all reduce to the R_n, while a way to perform the contractions produces an (n, m)-graph.

(c) This follows from the main theorem of invariant theory which implies that for $\dim(W) \gg n_1, \ldots, n_N, N$, the space of all $\mathrm{GL}(W)$-equivariant maps

$$\bigotimes_{i=1}^{N} \mathrm{Hom}(S^{n_i}W, W) \to \mathrm{Hom}(W^{\otimes n}, W^{\otimes m})$$

has as its basis, the maps p_Γ for various (n, m)-graphs Γ with N vertices of valencies n_i.

4. The weak lie operad in formal geometry and Gelfand–Fuks cohomology

4.1. THE COCHAINS

In this section we describe another way of unraveling the Jacobi identity for the Atiyah class which uses 'formal geometry' (analysis in the space of infinite jets, see [B] [GGL] [GKF]) instead of Kähler geometry. This approach has the advantage of being purely holomorphic. Instead of Dolbeault cochains, we will use the following lemma to represent necessary cohomology classes.

LEMMA 4.1.1. *Let X be a complex manifold and p: $A \to X$ be a locally trivial fibration with fibers isomorphic to $\mathbf{C}^N$ for some N. Then for any coherent sheaf $\mathcal{F}$ on X we have a natural morphism*

$$\tau\colon \Gamma(A, \Omega^\bullet_{A/X} \otimes p^*\mathcal{F}) \to R\Gamma(X, \mathcal{F}).$$

If A is a Stein manifold, then τ is a quasi-isomorphism.

The first statement means that any closed relative i-form on A with values in $p^*\mathcal{F}$ gives rise to a class in $H^i(X, \mathcal{F})$. The second statement means that if A is Stein, then this correspondence is 1-to-1.

Proof. Follows from the quasiisomorphism $\mathcal{O}_X \to p_*\Omega^\bullet_{A/X}$ (i.e., from the acyclicity of the global holomorphic de Rham complex of $\mathbf{C}^N$).

It was proved by Jouanolou [J] that if X is a quasi-projective algebraic manifold, then there always exists an A as above which is an affine variety (therefore Stein). We will be interested in some natural $\mathbf{C}^N$-fibrations which, though not Stein in general, still give the holomorphic cohomology classes we need.

4.2. Formal exponential maps

Let X be a complex manifold. Consider the space $\Phi^{(n)}(X) \xrightarrow{p_n} X$ of 'nth order exponential maps', cf. [B]. By definition, for $x \in X$ the fiber $\Phi^{(n)}_x(X)$ is the space of nth order jets of holomorphic maps ϕ: $T_xX \to X$ such that $\phi(0) = x$, $d_0\phi = \mathrm{Id}$. Thus $\Phi^{(2)}(X) = \Phi(X)$ is the affine fibration (2.2) defining torsion-free connections. Thus we have a chain of projections

$$X \leftarrow \Phi^{(2)}(X) \leftarrow \Phi^{(3)}(X) \leftarrow \cdots. \tag{4.2.1}$$

Each $\Phi^{(n+1)}$ is an affine bundle over $\Phi^{(n)}$ whose associated vector bundle is $p_n^*\mathrm{Hom}(S^{n+1}TX, TX)$. Thus each fiber of $\Phi^{(n)}(X)$ is isomorphic to $\mathbf{C}^N$ for some N and Lemma 4.1.1 is applicable: every closed relative form on $\Phi^{(n)}(X)$ gives a holomorphic cohomology class on X.

EXAMPLE 4.2.2. Since the space $\Phi(X) = \Phi^{(2)}(X)$ is an affine bundle over $\mathrm{Hom}(S^2TX, TX)$, it carries a tautological 1-form $\alpha_2 \in \Omega^1_{\Phi(X)/X} \otimes p^*\mathrm{Hom}(S^2TX, TX)$. This form is relatively closed and represents, via Lemma 4.1.1, the Atiyah class α_{TX}.

Let $J^{(n)}(TX) \to X$ be the group bundle whose fiber over $x \in X$ is the group of nth jets of biholomorphisms ψ: $T_xX \to T_xX$ with $\psi(0) = 0$, $d_0\psi = \mathrm{Id}$. Then $\Phi^{(n)}(X)$ is a bundle of $J^{(n)}(TX)$-torsors. Let $\mathbf{j}^{(n)}(TX)$ be the bundle of Lie algebras associated to $J^{(n)}(TX)$. Note that we have a natural splitting

$$\mathbf{j}^{(n)}(TX) = \bigoplus_{i=2}^{n} \mathrm{Hom}(S^iTX, TX), \tag{4.2.3}$$

induced by the action of $\mathrm{GL}(TX)$ on $\mathbf{j}^{(n)}(TX)$.

Let now $\Phi := \Phi^{(\infty)}(X) \xrightarrow{p} X$ be the inverse limit of the diagram (4.2.1), i.e., the space of *formal exponential maps*. It is a bundle of $J^{(\infty)}(X)$-torsors, where $J^{(\infty)}(TX) = \lim J^{(n)}(TX)$. The Lie algebra bundle of the bundle of proalgebraic groups $J^{(\infty)}(TX)$ is just

$$\mathbf{j}^{(\infty)}(TX) = \prod_{n\geqslant 2} \mathrm{Hom}(S^nT, T), \quad T = TX. \tag{4.2.4}$$

Denote by $p^{(n)}$: $\Phi \to \Phi^{(n)}(X)$ the projection and set

$$\Omega^\bullet_{\Phi/X} = \bigcup_n (p^{(n)})^*\Omega^\bullet_{\Phi^{(n)}(X)/X}. \tag{4.2.5}$$

As with any bundle of torsors, we have the tautological relative 1-form

$$\omega \in \Omega^1_{\Phi/X} \otimes p^*\mathbf{j}^{(\infty)}(TX). \tag{4.2.6}$$

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Projecting ω to the nth graded component in (4.2.4), we get the *tautological form*

$$\alpha_n \in \Omega^1_{\Phi/X} \otimes p^*\mathrm{Hom}(S^n T, T). \tag{4.2.7}$$

These forms are formal geometry analogs of the covariant derivatives of the curvature in (2.5) and satisfy very similar identities, as we shall explain later.

For every coherent sheaf $\mathcal{F}$ on X set

$$A^\bullet_\infty(\mathcal{F}) = \Gamma(\Phi, \Omega^\bullet_{\Phi/X} \otimes p^*\mathcal{F}).$$

This is a complex naturally mapping into $R\Gamma(X, \mathcal{F})$. Accordingly, for any coherent sheaf of operads $\mathcal{P}$ on X (i.e., an operad in the category of coherent sheaves) we have a dg-operad $A^\bullet_\infty(\mathcal{P})$. Similarly for PROPs.

Let us consider, in particular, the sheaf of operads $\mathcal{E}_T = \{\mathrm{Hom}(T^{\otimes n}, T)\}$ and the sheaf of PROPs $\mathrm{END}_T = \{\mathrm{Hom}(T^{\otimes n}, T^{\otimes m})\}$. The tautological form α_n, $n \geqslant 2$, gives an element of $A^\bullet_\infty \mathcal{E}_T(n) \subset A^\bullet_\infty \mathrm{END}_T(n,1)$, which is antisymmetric and has degree 1. Consider the desuspended dg-PROP $\Sigma^{-1} A^\bullet_\infty \mathrm{END}_T$. The shifted tautological forms $\Sigma^{-1}\alpha_n$ becomes antisymmetric of degree $2-n$. Further, let Γ be an (n,m)-graph (3.6) with l vertices. We denote by $\alpha_\Gamma \in A^l_\infty \mathrm{END}_T(n,m)$ the composition of the tautological forms $\alpha_{|\mathrm{In}(v)|}$ for all vertices v of Γ, by using the contractions along the edges of Γ. Then, because of the antisymmetry of the $\Sigma^{-1}\alpha_n$, we have that

$$\Sigma^{-1}\alpha_\Gamma \in \mathrm{Hom}(\det(\Gamma)^*, \Sigma^{-1} A^\bullet_\infty \mathrm{END}_T(n,m)).$$

Now, a formal geometry version of Theorem 2.6 is as follows.

THEOREM 4.3. *The elements* $\Sigma^{-1}\alpha_n \in \Sigma^{-1} A^\bullet_\infty \mathcal{E}_T(n)$ *form a weak Lie family. Moreover, the maps*

$$\Sigma^{-1}\alpha_\Gamma \colon \det(\Gamma)^* \to \Sigma^{-1} A^\bullet_\infty \mathrm{END}_T(n,m), \quad \Gamma \in \mathcal{G}(n,m),$$

define a morphism of PROPs WLIE $\to \Sigma^{-1} A^\bullet_\infty \mathrm{END}_T$. *In particular, the complex of sheaves* $p_*\Omega^{\bullet-1}_{\Phi/X} \otimes T$ *on* X *(quasi-isomorphic to* $T[-1]$*) has a natural structure of a weak Lie algebra.*

To formulate the companion theorem for a vector bundle E, we can proceed in a similar way, by working on the product $\Phi \times_X C$, where C is the fiber bundle over X whose fiber at x consists of infinite jets of fiberwise linear isomorphisms $E_x \times X \to E$ identical over x. We leave this to the reader.

We know three proofs of Theorem 4.3. The first two will be sketched, and the third one given in more detail.

First proof (sketch) (4.3.1). We can mimic all features of Kähler geometry but on the space Φ. First of all, the bundle $\Phi \to X$ (like other infinite jet bundles, see [GKF] [GGL]), carries a natural (non-linear) formally integrable connection D. Its covariantly constant sections over a simply connected $U \subset X$ correspond to affine structures on U, i.e., embeddings of U into an affine space of the same dimension, modulo affine equivalence. This decomposes the tangent space $T_\phi \Phi$ at every point into a direct sum $T_\phi^{1,0}\Phi + T_\phi^{0,1}\Phi$, where $T_\phi^{0,1}$ is the tangent space to the fiber of p passing through ϕ and $T_\phi^{1,0}$ is the horizontal subspace of the connection. Accordingly, we have decompositions

$$\Omega_\Phi^m \simeq \bigoplus_{a+b=m} \Omega_\Phi^{a,b}, \qquad \Omega_\Phi^{a,b} = p^*\Omega_X^a \otimes \Omega_{\Phi/X}^b,$$

and the de Rham differential is decomposed as

$$d = d' + d'', \qquad d'' = d_{\Phi/X}, \qquad (d')^2 = (d'')^2 = [d', d''] = 0.$$

We can speak therefore about $(0,1)$ and $(1,0)$-connections in fiber bundles on Φ. Every bundle of the form p^*E, lifted from X, has a canonical integrable $(0,1)$-connection. The bundle p^*T has, in addition, a natural integrable $(1,0)$-connection ∇ satisfying the identities

$$[d'', \nabla] = \alpha_2, \qquad \nabla\alpha_n = \alpha_{n+1},$$

which imply our theorem in the same way as in the Kähler case.

Second proof (sketch) (4.3.2). In line with 2.8, we consider the odd derivation D of the algebra

$$\Omega_{\Phi/X}^\bullet \otimes p^*\hat{S}(T^*)$$

obtained by extending $d_{\Phi/X} + \Sigma_{n\geqslant 2}\alpha_n^*$. Then we have only to prove that $D^2 = 0$. To do this, we consider, as in 2.8, the fiber bundles

$$\pi\colon X_T^{(\infty)} \to X, \qquad \rho\colon X_{X\times X}^{(\infty)} \to X,$$

where $X_T^{(\infty)}$ is the formal neighborhood of the zero section of TX and $X_{X\times X}^{(\infty)}$ is the formal neighborhood of the diagonal in $X \times X$. The algebra $\hat{S}(T^*)$ is just $\mathcal{O}_{X_T^{(\infty)}}$. The pullback to Φ of the nonlinear bundle ρ possesses an integrable connection along the fibers, which gives rise to an algebra differential Δ in $\Omega_{\Phi/X}^\bullet \otimes p^*\mathcal{O}_{X_{X\times X}^{(\infty)}}$

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satisfying $\Delta^2 = 0$. On the other hand, on Φ we have the tautological exponential map which is a nonlinear isomorphism of fiber bundles

$$\mathrm{Exp}\colon p^* X_T^{(\infty)} \to p^* X_{X\times X}^{(\infty)},$$

and one can verify that Exp is taking D into Δ, thereby proving the theorem.

4.4. Tautological Forms and Gelfand–Fuks Cohomology

Another way of proof of Theorem 4.3 is to reduce it to known results about the cohomology of the Lie algebra of formal vector fields, by making use of the general relationship between this cohomology and tautological forms. Let us first recall this relationship [B] [GKF].

Let $r \geqslant 1$ be fixed. Denote by $G^{(n)}$ the group of nth jets of biholomorphisms $\phi\colon \mathbf{C}^r \to \mathbf{C}^r$ with $\phi(0) = 0$, and by $J^{(n)} \subset G^{(n)}$ the normal subgroup formed by ϕ with $d_0\phi = \mathrm{Id}$. So we have an exact sequence

$$1 \to J^{(n)} \to G^{(n)} \to \mathrm{GL}_r \to 1, \tag{4.4.1}$$

which, moreover, canonically splits (by considering jets of linear transformations), making $G^{(n)}$ a semidirect product.

If X is an r-dimensional complex manifold and $x \in X$, let $F_x^{(n)}(X)$ be the space of nth jets of biholomorphisms $\phi\colon \mathbf{C}^r \to X$ with $\phi(0) = x$. This is a $G^{(n)}$-torsor. These torsors unite into a principal $G^{(n)}$-bundle $F^{(n)}(X) \xrightarrow{q_n} X$ called the bundle of nth order frames. The quotient $F^{(n)}(X)/\mathrm{GL}_r$ is $\Phi^{(n)}(X)$, the space of nth jets of exponential maps from (4.2).

Let $\mathbf{g} = \mathbf{h} \oplus \mathbf{k}$ be a Lie algebra split into a semidirect product of two subalgebras of which $\mathbf{k}$ is an ideal. Let M be an $\mathbf{h}$-module. Because of the identification $\mathbf{g}/\mathbf{k} = \mathbf{h}$, we can regard M as a $\mathbf{g}$-module, and form the relative cochain complex

$$C^\bullet(\mathbf{g}, \mathbf{h}, M) = \mathrm{Hom}(\Lambda^\bullet(\mathbf{g}/\mathbf{h}), M)^{\mathbf{h}}.$$

Recall the following standard fact about this complex.

PROPOSITION 4.4.3. *If* $\mathbf{g}$, $\mathbf{h}$, $\mathbf{k}$ *are the Lie algebras of connected Lie groups* G, H, K *so that* G *is a semidirect product* HK, *then for every* G-*torsor* P *we have a natural identification*

$$C^\bullet(\mathbf{g}, \mathbf{h}, M) = \Gamma(P/H, \Omega^\bullet_{P/H} \otimes M)^K.$$

Let us apply this to $G = G^{(n)}, H = \mathrm{GL}_r, K = J^{(n)}$. Let $\mathbf{g}^{(n)}$ be the Lie algebra of $G^{(n)}$. A representation M of GL_d gives rise, in a standard way, to the functor from the category of r-dimensional vector spaces and their isomorphisms to the category of vector spaces called the Schur functor and denoted by $W \mapsto S^M(W)$.

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In particular, the vector bundle $S^M(TX)$ over X is defined. We denote by V the standard r-dimensional representation of GL_r. Take also P to be (fibers of) the principal $G^{(n)}$-bundle $F^{(n)}(X) \to X$. We obtain the following.

PROPOSITION 4.4.4. *We have a natural identification of complexes of sheaves on* X

$$\mathcal{O}_X \otimes C^\bullet(\mathbf{g}^{(n)}, \mathbf{gl}_r, M) \simeq p_{n*}\left(\Omega^\bullet_{\Phi^{(n)}(X)/X} \otimes q_n^*(S^M(TX))\right)^{J^{(n)}(TX)}.$$

Note that the tautological forms from (4.2.7) give global sections of the complex in (4.4.4). They correspond to $M = \mathrm{Hom}(S^n(V), V)$.

By passing to the limit $n \to \infty$, we consider

$$\mathrm{Vect}_r^0 = \varprojlim \mathbf{g}^{(n)} = \prod_{n \geqslant 1} \mathrm{Hom}(S^n V, V), \tag{4.4.5}$$

the Lie algebra of formal vector fields on $\mathbf{C}^r$ vanishing at 0. This is a topological Lie algebra and we will consider its continuous cohomology.

4.5. The Lie operad in Gelfand–Fuks cohomology. Results of Fuks

Taking, for every $n, m \geqslant 0$, the relative cochain complex

$$C^\bullet(\mathrm{Vect}_r^0, \mathbf{gl}_r, \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})) = \Pi_r^\bullet(n, m), \tag{4.5.1}$$

we get a dg-PROP $\Pi_r^\bullet$. Let $\mathcal{H}_r^\bullet$ be the graded PROP formed by the cohomology of $\Pi_r^\bullet$. By the above, an element of $\mathcal{H}_r^i(n)$ gives, for each r-dimensional complex manifold X, a class in $H^i(X, \mathrm{Hom}(T^{\otimes n}, T^{\otimes m}))$. Let

$$a_n \in C^1(\mathrm{Vect}_r^0, \mathbf{gl}_r, \mathrm{Hom}(V^{\otimes n}, V)), \quad n \geqslant 2,$$

be the tautological cochain which associates to a formal vector field its degree n homogeneous component (lying in $\mathrm{Hom}(S^n V, V)$). For any (n, m)-graph Γ with N vertices let

$$a_\Gamma \in C^N(\mathrm{Vect}_r^0, \mathbf{gl}_r, \mathrm{Hom}(V^{\otimes n}, V^{\otimes m}))$$

be the cochain obtained by contracting the cup product of the $a_{|\mathrm{In}(v)|}$, $v \in \mathrm{Vert}(\Gamma)$ along the edges of Γ, cf. 4.2. Now, Theorem 4.3 can be reformulated as follows.

THEOREM 4.5.2. *Let the maps*

$$\Sigma^{-1} a_\Gamma\colon\ \det(\Gamma)^* \to \Pi_r^N(n, m)$$

define a morphism of PROPs WLIE $\to \Sigma^{-1}\Pi_r^\bullet$.

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In this formulation the theorem follows at once from results of D. B. Fuks [Fuk] who studied stable cohomology of $\Pi_r^\bullet(m,n)$ (when r is big compared to m,n and the number of the cohomology) and identified it with the cohomology of a certain graph complex. Translated into our language, his results immediately imply the relation with LIE and WLIE. More precisely, we deduce the following fact.

THEOREM 4.5.3. *As $r \to \infty$, each term of the complex $\Pi_r^\bullet(m,n)$ stabilizes, so that we have a limit complex $\Pi^\bullet(m,n)$. Taken for all m,n, these complexes form a dg-PROP, which is isomorphic to $\Sigma^{-1}(\mathrm{WLIE})$.*

Proof. The existence of the stabilization and its identification with a graph complex is completely explicit in [Fuk]. Namely, the space $\Pi_r^N(n,m)$ consists of GL_r-invariant antisymmetric continuous maps

$$\bigwedge^N\left(\prod_{i\geqslant 2}\mathrm{Hom}(S^iV,V)\right)\to\mathrm{Hom}(V^{\otimes n},V^{\otimes m}). \tag{4.5.4}$$

Thus

$$\Pi_r^N(n,m) = \bigoplus_{\substack{(N_i\in\mathbf{Z}_+)_{i\geqslant 2}\\ \sum N_i=N}}\mathrm{Hom}\left(\bigotimes_{i\geqslant 2}\bigwedge^{N_i}\mathrm{Hom}(S^iV,V),\ \mathrm{Hom}(V^{\otimes n},V^{\otimes m})\right)^{\mathrm{GL}_r}. \tag{4.5.5}$$

Let Γ be an (n,m)-graph (3.1) with N vertices. Then we have a natural contraction map

$$p_\Gamma\colon \bigotimes_{v\in\mathrm{Vert}(\Gamma)}\mathrm{Hom}(S^{|\mathrm{In}(v)|}(V),V)\to\mathrm{Hom}(V^{\otimes n},V^{\otimes m}), \tag{4.5.6}$$

which is obviously invariant. Moreover, when $r\gg 0$, then by the main theorem of invariant theory such contraction maps for various Γ provide a basis in the space of all invariant maps. This implies the stabilization of the $\Pi_r^\bullet(n,m)$. Let $N_i(\Gamma)$ be the number of $v\in\mathrm{Vert}(\Gamma)$ with $|\mathrm{In}(v)|=i$, and let

$$t(\Gamma)\colon \bigotimes_{i\geqslant 2}\bigwedge^{N_i(\Gamma)}\mathrm{Hom}(S^iV,V)\to\mathrm{Hom}(V^{\otimes n},V^{\otimes m}) \tag{4.5.7}$$

be the antisymmetrization of p_Γ. Then

$$t\colon\ \det(\Gamma)^*\mapsto t(\Gamma)$$

is the desired isomorphism of complexes $\mathrm{WLIE}(n,m) \to \Pi^\bullet(n,m)$ of degree $m-n$. To finish the proof, it remains to identify the composition structure in $\Pi^\bullet$ with that in $\Sigma^{-1}(\mathrm{WLIE})$, which is straightforward.

4.6. Generalization to other operads

Theorem 4.5.3 can be straightforwardly generalized to any quadratic Koszul operad $\mathcal{Q}$ in the sense of [GiK]. Namely, let $\mathrm{Vect}_r^0(\mathcal{Q})$ be the Lie algebra of derivations of $F_{\mathcal{Q}}(r)$, the free $\mathcal{Q}$-algebra on r generators.

THEOREM 4.6.1. *Set*

$$A^\bullet_{r,\mathcal{Q}}(m,n) = C^\bullet(\mathrm{Vect}_r^0(\mathcal{Q}), \mathbf{gl}_r, \mathrm{Hom}(V^{\otimes n}, V^{\otimes m}).$$

Then, as $r\to\infty$, each term of $A^\bullet_{r,\mathcal{Q}}(m,n)$ stabilizes, and the stable complexes $A^\bullet_{\mathcal{Q}}(m,n)$ form a dg-PROP $A^\bullet_{\mathcal{Q}}$. This PROP is isomorphic to $\Pi_{\mathbf{D}(\mathcal{Q})}$, the PROP associated (2.10) *to the dg-operad $\mathbf{D}(\mathcal{Q})$, the cobar-construction of $\mathcal{Q}$. In particular, the graded PROP formed by the cohomology of $A^\bullet_{\mathcal{Q}}$ is isomorphic to $\Sigma^{-1}\Pi_{\mathcal{Q}^!}$ where $\mathcal{Q}^!$ is the Koszul dual quadratic operad.*

This statement provides a non-symplectic analog of the result of M. Kontsevich [K1] describing the stable cohomology of the algebra of hamiltonial vector fields. Theorem 4.5.3 corresponds to the case when $\mathcal{Q} = \mathcal{C}om$, the operad describing commutative algebras.

4.7. Example: noncommutativization

As we could see before, all the properties of the Atiyah class, including the detailed unraveling of the Jacobi identity, can be deduced from the careful study of the non-linear fiber bundle on X whose fiber over x is the formal neighborhood of x, i.e., the spectrum of the completed local algebra $\hat{\mathcal{O}}_{X,x}$. This algebra is free, i.e., isomorphic to $\mathbf{C}[[t_1,\dots,t_d]]$, $d=\dim(X)$, but there is no canonical identification, the Atiyah class being an obstruction to choosing such identifications for all x in a holomorphic way. Taken for all $x\in X$, the algebras $\hat{\mathcal{O}}_{X,x}$ arrange themselves into a sheaf of complete commutative $\mathcal{O}_X$-algebras $\mathcal{O}_{X^{(\infty)}_{X\times X}}$ (functions on the formal neighborhood of the diagonal), which is locally on X isomorphic to $\mathcal{O}_X[[t_1,\dots,t_d]]$.

For any commutative ring R let $R\langle\langle t_1,\dots,t_d\rangle\rangle$ be the algebra of non-commutative formal power series in $t_1,\dots,t_d$, with coefficients in R, i.e., the completion of the free associative algebra on the x_i. Now let us make the following definition.

DEFINITION 4.7.1. Let X be a d-dimensional complex manifold. A noncommutative structure on X is a sheaf of complete associative $\mathcal{O}_X$-algebras $\mathbf{O}$ on X which

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locally on X is isomorphic to $\mathcal{O}_X\langle\langle t_1, \ldots, t_d\rangle\rangle$, together with an isomorphism $\mathbf{O}/[\mathbf{O}, \mathbf{O}] \to \mathcal{O}_{X^{(\infty)}_{X\times X}}$.

In other words, such a structure gives, for every $x \in X$ a 'non-commutative formal neighborhood' whose ring of functions is $\mathbf{O}_x$, the fiber of $\mathbf{O}$ at x. These rings are noncanonically isomorphic to $\mathbf{C}\langle\langle t_1, \ldots, t_d\rangle\rangle$.

EXAMPLE 4.7.2. A natural class of examples of manifolds and, more generally, stacks with noncommutative structure is provided by the moduli spaces of vector bundles (as opposed to more general principal G-bundles). Namely, if E is a vector bundle on an algebraic variety Z. Suppose that $H^0(X, \mathrm{Hom}(E, E)) = \mathbf{C}$. Let $\mathcal{M}$ be Kuranishi deformation space of E, so that we have a distinguished point $[E] \in \mathcal{M}$. Then, by the general principles of deformation theory [GM], the formal neighborhood of $[E]$ in $\mathcal{M}$ is the spectrum of $\mathbf{H}^0_{\mathrm{Lie}}(R\Gamma(Z, \mathrm{End}(E)))$, the zeroth Lie algebra hypercohomology of the dg-Lie algebra $R\Gamma(Z, \mathrm{End}(E))$. Here we regard $\mathrm{End}(E)$ as a sheaf of Lie algebras with respect to the bracket $[a, b] = ab - ba$, thereby ignoring a richer structure of an associative algebra. If we do not ignore this structure, we get an associative dg-algebra structure on $R\Gamma(Z, \mathrm{End}(E))$. Therefore, the associative algebra hypercohomology

$$\mathbf{H}^0_{\mathrm{Ass}}(R\Gamma(Z, \mathrm{End}(E))),$$

(to be precise, here we mean the Hochshild cohomology with $\mathbf{C}$ coefficients and the algebra should be modified so as to get rid of the unity), will give us an associative algebra whose quotient by the commutant maps naturally into the Lie algebra cohomology, i.e., into the completed local ring of $\mathcal{M}$ at $[E]$, and under suitable conditions (when the bundle is simple and unobstructed) this is an isomorphism.

A more down-to earth explanation of this phenomenon can be obtained as follows. The moduli space of G-bundles for any G gives rise to a representable functor (stack) which is a contravariant functor on the category of affine schemes, or, what is the same, a covariant functor on the category of commutative algebras. In the case when $G = \mathrm{GL}_r$, i.e., we are dealing with rank r vector bundles, this functor can be naturally extended to all associative algebras, i.e., we can meaningfully speak about families of rank r vector bundles parametrized by 'Spec(A)' where A is any associative algebra. Such a family is just given, in a Čech covering $\{U_i\}$, by transition functions ϕ_{ij} which are sections of $\mathrm{GL}_r(\mathcal{O}_X \otimes A)$ over $U_i \cap U_j$. For example, if $A = \mathrm{Mat}_n(\mathbf{C})$, then a 'family' of rank r bundles parametrized by A is just a rank rn bundle on X.

Remark. As M. Kontsevich communicated to the author upon reading the manuscript, he also has had the idea equivalent to Definition 4.7.1 and was aware of Example 4.7.2.

The considerations of this and the earlier sections revolve, as it is clear from contemplating Theorem 4.5.3, around the Koszul dual pair of operads $(\mathcal{C}om, \mathcal{L}ie)$:

manifolds are described by commutative algebras of functions, while the curvature data lead to Lie algebras. So they can be generalized to manifolds with a noncommutative structure, if we consider instead the dual pair $(\mathcal{A}ss, \mathcal{A}ss)$, where $\mathcal{A}ss$ is the (self-dual) operad governing associative algebras, see [GiK]. Let us summarize briefly this generalization.

THEOREM 4.7.3. *Let X be a complex manifold with a noncommutative structure, $T = TX$ is its usual tangent bundle. Then:*

(a) *The second-order obstruction to global trivialization of the noncommutative formal neighborhoods is a certain class $\alpha_X \in H^1(X, T \otimes T)$ (the noncommutative Atiyah class), whose symmetrization is the usual Atiyah class α_{TX}.*
(b) *The desuspension $\Sigma^{-1}\alpha_X$, regarded as an element of the operad $\Sigma^{-1}H^\bullet(X, \mathcal{E}_T)$, is an associative element, i.e., it defines a morphism of operads $\mathcal{A}ss \to \Sigma^{-1}H^\bullet(X, \mathcal{E}_T)$. In particular, for any sheaf A of commutative $\mathcal{O}_X$-algebras the shifted cohomology $H^{\bullet-1}(X, T \otimes A)$ has a natural structure of an associative algebra, given by α_X.*
(c) *The graded Lie algebra structure on $H^{\bullet-1}(X, T \otimes A)$ defined by the usual Atiyah class, is obtained from the associative structure in* (b) *by the standard formula $[a, b] = ab \pm ba$. In particular, if $A = \mathcal{O}_X$, then the structure of an associative algebra on $H^{\bullet-1}(X, T)$ is in fact commutative.*

5. The symplectic Atiyah class

5.1. Symmetry of the Atiyah class

Let now X be a complex manifold equipped with a holomorphic symplectic structure. Let $\omega \in \Gamma(X, \Omega^2)$ be the symplectic form. We will identify the tangent bundle $T = TX$ with its dual T^* by means of ω. After this identification, we can view the Atiyah class α_{TX} as an element of $H^1(X, S^2(T) \otimes T)$.

PROPOSITION 5.1.1. *The element α_{TX} is totally symmetric, i.e., it lies in the summand $H^1(X, S^3(T))$.*

Proof. Let Symp(X) be the sheaf of connections in TX preserving the symplectic form ω. Since for a symplectic vector space V the Lie algebra sp(V) of infinitesimal symplectic transformations is identified with $S^2(V)$, the sheaf Symp(X) is, by 1.5, a torsor over $\Omega^1 \otimes \mathrm{sp}(T) \simeq \Omega^1 \otimes S^2(T) \simeq T \otimes S^2(T)$. This shows that α_{TX} is symmetric with respect to the permutation of the second and third argument. Since it is already symmetric in the first two arguments, the assertion follows.

Remarks 5.1.2. One can right away exhibit an $S^3(T)$-torsor from which α_{TX} is obtained by change of scalars. This is the torsor $\mathrm{Symp}_{tf}(X)$ of torsion-free

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symplectic connections. As in (2.2), it can be materialized as the sheaf of sections of the fiber bundle $\Psi(X) \to X$ whose fiber $\Psi_x(X)$ for $x \in X$ is the space of second jets of holomorphic symplectomorphisms ϕ: $T_x X \to X$ such $\phi(0) = x, d_o\phi = \mathrm{Id}$. Clearly, $\Psi_x(X)$ is an affine space over $S^3(T_x X)$, and sections of Ψ are the same as torsion-free symplectic connections.

5.2. The IHX relation for the Atiyah class

Let V be a finite-dimensional symplectic vector space whose symplectic form is denoted by ω. Then V^* is also a symplectic vector space, with respect to the inverse form ω^{-1}. Let Γ be a finite 3-valent graph with possibly several legs (non-compact edges bound by a vertex from one side only, cf. [GeK]). Denote by $\mathrm{Vert}(\Gamma), \mathrm{Ed}(\Gamma), \mathrm{Leg}(\Gamma)$ the sets of vertices, (compact) edges and legs of Γ. Let also $\mathrm{Flag}(\Gamma)$ be the set of all flags consisting of a vertex and an incident half-edge (including a leg) of Γ. For a vertex v let $\mathrm{Flag}(v)$ be the 3-element set of flags having v as a vertex. We will distinguish between arbitrary automorphisms of Γ and strict automorphisms (i.e., those fixing each leg).

For a finite-dimensional vector space W we will denote by $\det(W)$ the top exterior power of W. If I is a finite set, then $\det(\mathbf{C}^I)$ will be abbreviated to $\det(I)$. Note that $\det(I)^{\otimes 2}$ is canonically (i.e., $\mathrm{Aut}(I)$-equivariantly) isomorphic to $\mathbf{C}$. For an edge e of Γ we denote by $\mathrm{OR}(e)$ the orientation line of e, i.e., $\mathrm{OR}(e) = \det(\partial e)$ where $\partial e \subset \mathrm{Flag}(\Gamma)$ is the set formed by the two flags with edge e.

With these notations, note that we have a natural $\mathrm{Sp}(V)$-equivariant projection

$$p_\Gamma\colon (S^3(V))^{\otimes \mathrm{Vert}(\Gamma)} \to (V^{\otimes \mathrm{Leg}(\Gamma)}) \otimes \bigotimes_{e \in \mathrm{Ed}(\Gamma)} \mathrm{OR}(e), \tag{5.2.1}$$

obtained by applying the form $\omega\colon V \otimes V \to \mathbf{C}$ to any edge of Γ. The factors $\mathrm{OR}(e)$ appear because of the antisymmetry of ω.

For example, there is a unique, up to scalar, $\mathrm{Sp}(V)$-equivariant antisymmetric map

$$p_{IHX}\colon S^3(V) \otimes S^3(V) \to S^4(V), \qquad p_{IHX}(a \otimes b) = \{a, b\}, \tag{5.2.2}$$

the Poisson bracket of a and b considered as cubic polynomial functions on V^*. By working out the definition of the Poisson bracket, we find that the composition of p_{IHX} with the embedding $S^4(V) \hookrightarrow V^{\otimes 4}$ can be represented as the sum of three projections

$$p_{IHX} = p_{\mathbf{I}} + p_{\mathbf{H}} + p_{\mathbf{X}} : S^3(V) \otimes S^3(V) \to V^{\otimes 4}, \tag{5.2.3}$$

where $\mathbf{I}, \mathbf{H}, \mathbf{X}$ are the three possible (up to strict isomorphism) trivalent graphs with two vertices and the set of legs identified with $\{1, 2, 3, 4\}$.

Now the Bianchi identity (1.2.2) gives, after the symmetrization, that the Atiyah class $\alpha_{TX} \in H^1(X, S^3(T))$ satisfies the so-called IHX relation

$$p_{IHX}(\alpha_{TX} \cup \alpha_{TX}) = \{\alpha_{TX}, \alpha_{TX}\} = 0 \quad \text{in } H^2(X, S^4(T)). \tag{5.2.4}$$

Of course, this can be understood from the point of view of the Lie operad, as in (3.5–6), the graphs $\mathbf{I}, \mathbf{H}, \mathbf{X}$ corresponding to the three terms in the Jacobi identity.

5.3. Rozansky–Witten classes

Let now Γ be a trivalent graph without legs having l vertices. Then the projection p_Γ from (5.2.1) takes values in $\mathbf{C}$. By applying it to $(\alpha_{TX})^l \in H^l(X, (S^3(T))^{\otimes \mathrm{Vert}(\Gamma)})$ we get elements

$$c_\Gamma(X) \in H^l(X, \mathcal{O}) \otimes \det(\mathrm{Vert}(\Gamma)) \otimes \bigotimes_e \mathrm{OR}(e). \tag{5.3.1}$$

The factor $\det(\mathrm{Vert}(\Gamma)$ appears because of the anticommutativity of the multiplication in the cohomology, while the origin of the $\mathrm{OR}(e)$ was explained in 5.2. The following lemma shows that the sign factor in (5.3.1) is the same as the one considered by Rozansky–Witten [RW] and Kontsevich [K1].

LEMMA 5.3.2. *For a trivalent graph* Γ *without edge-loops there is a natural* (*i.e.,* $\mathrm{Aut}(\Gamma)$-*equivariant*) *identification of* 1-*dimensional vector spaces*

$$\begin{aligned} \det(\mathrm{Vert}(\Gamma)) \otimes \bigotimes_{e \in \mathrm{Ed}(\Gamma)} \mathrm{OR}(e) &\simeq \det(\mathrm{Ed}(\Gamma)) \otimes \det(H_1(\Gamma, \mathbf{C})) \\ &\simeq \bigotimes_{v \in \mathrm{Vert}(\Gamma)} \det(\mathrm{Flag}(v)). \end{aligned}$$

Proof. We start with the first isomorphism. Note that

$$\det(\mathrm{Ed}(\Gamma)) \otimes \bigotimes_{e \in \mathrm{Ed}(\Gamma)} \mathrm{OR}(e) \simeq \det\left(\bigoplus_{e \in \mathrm{Ed}(\Gamma)} \mathrm{OR}(e) \right),$$

while the consideration of the chain complex of Γ

$$\bigoplus_{e \in \mathrm{Ed}(\Gamma)} \mathrm{OR}(e) \to \mathbf{C}^{\mathrm{Vert}(\Gamma)},$$

gives

$$\det\left(\bigoplus \mathrm{OR}(e)\right) \simeq \det(\mathbf{C}^{\mathrm{Vert}(\Gamma)}) \otimes \det(H_1(\Gamma, \mathbf{C})).$$

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This implies that the tensor product of the left and the right-hand sides of the first proposed isomorphism in (3.3.2), is canonically trivial. Because $\det(I)^{\otimes 2} \simeq \mathbf{C}$ for any finite set I, we get the first isomorphism.

To establish the second isomorphism, consider the projections

$$\mathrm{Vert}(\Gamma) \xleftarrow{\phi} \mathrm{Flag}(\Gamma) \xrightarrow{\psi} \mathrm{Ed}(\Gamma).$$

The consideration of fibers of ψ gives

$$\begin{aligned}\det(\mathrm{Flag}(\Gamma)) &= \det\left(\mathrm{Ed}(\Gamma) \oplus \bigoplus_e \mathrm{OR}(e)\right) \\ &= \det(\mathrm{Ed}(\Gamma)) \otimes \det(\mathrm{Vert}(\Gamma)) \otimes \det(H_1(\Gamma, \mathbf{C})),\end{aligned}$$

and the consideration of fibers of ϕ gives that

$$\det(\mathrm{Flag}(\Gamma)) = \det(\mathrm{Vert}(\Gamma)) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma)} \det(\mathrm{Flag}(v)),$$

whence the statement.

For a 3-element set I a choice of direction of the real line $\det(\mathbf{R}^I)$ is the same as a cyclic order on I. Thus the classes $c_\Gamma(X)$ can be seen as being elements of $H^l(X, \mathcal{O})$ but defined on graphs with cyclic orders on each $\mathrm{Flag}(v)$ and changing the sign under the changing of the cyclic order. Further, it follows from (5.2.3) that the c_Γ thus understood satisfy the IHX relation in the sense of [RW]. So we get the first part of the following statement.

THEOREM 5.4. *For any holomorphic symplectic manifold X the classes $c_\Gamma(X)$ defined before, give rise to invariants of* 3*-manifolds with values in $H^l(X, \mathcal{O})$. If X is compact and hyper-Kähler, then the c_Γ coincide with the coefficients defined by Rozansky and Witten*.

The second part just follows from the fact that the curvature represents the Atiyah class (Proposition 1.3.1).

It is convenient to consider as in [RW], numerical invariants constructed from the $c_\Gamma(X)$. Namely, let us put

$$\bar{c}_\Gamma(X) = \omega^{l/2} c_\Gamma(X) \in H^l(X, \Omega^l). \tag{5.4.1}$$

Here $\omega \in H^0(X, \Omega^2)$ is the symplectic form and l is the (necessarily even) number of vertices of the 3-valent graph Γ. Further, if X is compact and L is a line bundle on X, we define the number

$$b_\Gamma(X, L) = \int_X \bar{c}_\Gamma(X) \cdot c_1(L)^{\dim(X) - l} \in \mathbf{C}. \tag{5.4.2}$$

PROPOSITION 5.4.3. *The classes* $\bar{c}_\Gamma(X)$ *and the numbers* $b_\Gamma(X, L)$ *remain unchanged if the symplectic form* ω *on* X *is replaced by* $\lambda\omega$, $\lambda \in \mathbf{C}^*$.

Proof. The class $\alpha_{TM} \in H^1(X, S^2T^* \otimes T)$ does not depend on ω at all. When we write it in the totally symmetric form, we in fact apply the isomorphism

$$\phi_\omega\colon S^2T^* \otimes T \to S^2T \otimes T,$$

which is homogeneous in ω of degree (-2). Since every pairing corresponding to an edge of Γ is homogeneous in ω of degree $+1$, we find that $c_\Gamma(X)$ is homogeneous of degree

$$-|\mathrm{Vert}(\Gamma)| + |\mathrm{Ed}(\Gamma)| = -(1/2)|\mathrm{Vert}(\Gamma)| = -l/2,$$

where we used the fact that Γ is 3-valent. Therefore $\bar{c}_\Gamma(X)$ is homogeneous of degree 0.

5.5. Calculation of the c_Γ via non-symplectic connections

Note that the Atiyah class α_{TX} used to construct the c_Γ, is defined in terms of the tangent bundle TX alone, without any symplectic structure. Provided such structure ω is given, α_{TX} just *happens to be* totally symmetric in all three arguments (if we identify $T \simeq T^*$ by means of ω). This means that a Dolbeault representative of α_{TX} in $\Omega^{0,1} \otimes S^3(TX)$ can be found by forcibly symmetrizing the $(1, 1)$-part of the curvature of any $(1, 0)$-connection in TX. In particular, we can take any Kähler (not necessarily hyper-Kähler) metric, write its curvature as a section of $\Omega^{0,1} \otimes (\Omega^{1,0} \otimes \Omega^{1,0} \otimes T)$, identify T with $\Omega^{1,0}$ via the symplectic form and then just symmetrize with respect to the last 3 arguments. Denoting by R the corresponding $(0, 1)$-form with values in $S^3(T)$, we find:

THEOREM 5.5.1. *The class* c_Γ *is represented by the* $(0, l)$*-form* $p_\Gamma(R^{\mathrm{Vert}(\Gamma)})$.

5.6. Reminder on modular operads and graph complexes

We are now going to upgrade the operadic analysis of the properties of the Atiyah class and the curvature for general complex manifolds to the symplectic case. For this, we need the concept of a modular operad [GeK2]. Let us briefly recall this concept.

A stable $((g, n))$-graph is a connected graph Γ with the following structures and properties:

(5.6.1) The set of legs of Γ is identified with $\{1, \ldots, n\}$.

(5.6.2) A function g: $\mathrm{Vert}(\Gamma) \to \mathbf{Z}_+$ is given such that $2(g(v) - 1) + |v| > 0$ for each vertex v.

(5.6.3.) $\dim(H^1(\Gamma, \mathbf{C})) + \Sigma_v g(v) = g$.

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Let $I((g,n))$ be the set of isomorphism classes of stable $((g,n))$-graphs, and $\tilde{I}((g,n))$ be the similar set in which we allow disconnected graphs as well.

In [GeK2], several versions of modular operads were introduced, differing by the sign conventions ('cocycles') entering the definition. In the present paper we are going to use only one of them. Namely, for a stable $((g,n))$-graph Γ we set

$$\mathfrak{R}(\Gamma) = \bigotimes_{e \in \mathrm{Ed}(\Gamma)} \mathrm{OR}(e). \tag{5.6.4}$$

The spaces $\mathfrak{R}(\Gamma)$ define a cocycle $\mathfrak{R}$ on the category of graphs in the sense of [GeK2], Section 4. By a modular operad we will in the sequel always mean a $\mathfrak{R}$-modular operad. Explicitly, this is an ordinary operad $\mathcal{P}$ with the following additional structures:

(5.6.5) Symmetry between the inputs and the output, i.e., a S_{n+1}-action on $\mathcal{P}(n)$. To emphasize this symmetry, we write $\mathcal{P}((n+1))$ for $\mathcal{P}(n)$.

(5.6.6) A decomposition $\mathcal{P}((n)) = \bigoplus_{2g-2+n>0} \mathcal{P}((g,n))$ into S_n-invariant subspaces. The S_n-module $\mathcal{P}((g,n))$ defines, in a standard way, a functor on the category of n-element sets and bijections, whose value on a set J will be denote of $\mathcal{P}((g,J))$.

(5.6.7) Graphical composition maps

$$\mathfrak{R}(\Gamma) \otimes \bigotimes_{v \in \mathrm{Vert}(\Gamma)} \mathcal{P}((g(v), \mathrm{Leg}(v))) \to \mathcal{P}((g,n)).$$

These structures are required to satisfy the compatibility properties given in [GeK2], n. 4.2.

EXAMPLES 5.7. (a) If V is a vector space with a skew-symmetric inner product B, then we have the *endomorphism modular operad* $\mathcal{E}[V]$ with $\mathcal{E}[V]((g,n)) = V^{\otimes n}$ for all g, the S_n-action being the standard one and the compositions defined by contracting with help of B. Accordingly, for a holomorphic symplectic manifold X we have a graded modular operad $H^\bullet(X, \mathcal{E}[T]) = \{H^\bullet(X, T^{\otimes n})\}$.

(b) The suspended Lie operad $\Sigma\mathcal{L}ie$ is a modular operad with $\Sigma\mathcal{L}ie((g,n)) = 0$ for $g > 0$ and $\Sigma\mathcal{L}ie(n-1)$ (i.e., the space $\mathcal{L}ie(n-1)$ placed in degree $n-2$) for $g = 0$, and with the S_n-action described in [K1], [GeK1-2]. This action arises naturally from the consideration of Lie algebras with an invariant inner product. In the same way, $\Sigma\mathcal{WL}ie$ is a modular dg-operad.

(c) The graph complexes of Kontsevich [K1], once generalized to allow graphs with legs, form a modular operad. More precisely, let $\mathcal{G}((g,n)) \subset I((g,n))$ be the set of Γ for which all the numbers $g(v)$ are 0. Then the condition (5.6.2) just means $|v| \geqslant 3$ for all v, and (5.6.3) means that the number of loops in Γ is g. Set

$$\mathcal{F}((g,n)) = \bigoplus_{\Gamma \in \mathcal{G}((g,n))} \delta(\Gamma)), \tag{5.7.2}$$

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$$\delta(\Gamma) = \det(\mathrm{Ed}(\Gamma))^* \otimes \det(H^1(\Gamma, \mathbf{C}))^*, \ \deg(\delta(\Gamma)) = |\mathrm{Vert}(\Gamma)|. \tag{5.7.2}$$

Then the $\mathcal{F}((g,n))$ form a modular dg-operad $\mathcal{F}$ with compositions given by grafting of graphs and the differential dual to the one contracting edges. In fact, $\mathcal{F}$ is a certain twist of $F\mathcal{C}om$, the Feynman transform of the commutative operad defined in [GeK2]. We chose the present version to avoid dealing in this paper with twists and suspensions of modular operads and the resulting sign issues.

Note that the tree part of $\mathcal{F}$ is

$$\mathcal{F}((0,n)) = \Sigma\mathcal{WL}ie((n)) = \Sigma\mathcal{WL}ie(n-1). \tag{5.7.3}$$

Further, $\mathcal{F}((1,n))$ (the part formed by 1-loop graphs) can be expressed in terms of the weak Lie PROP, namely it is the subcomplex in $\Sigma\mathrm{WLIE}(n,1)$ formed by connected graphs. The legless part $\mathcal{F}((g,0))$ is the graph complex defined by Kontsevich in [K1].

(d) Let $\tilde{\mathcal{G}}((g,n))$ be the subset of $\tilde{I}((g,n))$, see 5.6 formed by graphs with all $g(v) = 0$. Let $\tilde{\mathcal{F}}((g,n))$ be the space defined similarly to (5.7.1) but by summing over $\tilde{\mathcal{G}}((g,n))$. They form a modular dg-operad $\tilde{\mathcal{F}}$.

5.8. Operadic interpretation of the Jacobi identity: the level of cohomology

Let (X,ω) be a holomorphic symplectic manifold. Then we have a graded modular operad $H^\bullet(X,\mathcal{E}[T])$, see the example 5.7(a). The Atiyah class α_{TX} can be regarded as an element of a modular operad

$$\alpha_{TX} \in H^\bullet(X,\mathcal{E}[T])((0,3)).$$

THEOREM 5.8.1. *The correspondence*

$$\Sigma([x_1,x_2]) \in \Sigma\mathcal{L}ie(2) = \Sigma\mathcal{L}ie((0,3)) \mapsto \alpha_{TX},$$

(*with* Σ *meaning the suspension*) *defines a morphism of modular operads* $\Sigma\mathcal{L}ie \to H^\bullet(X,\mathcal{E}[T])$.

Proof. The only new property here, as compared to Theorem 3.5.1, is that we have a morphism of modular operads, i.e., that it is invariant with respect to the action of larger symmetric groups. But this follows from the total symmetry of α_{TX}.

5.9. Operadic interpretation: Dolbeault forms

We now look at the modular dg-operad $\Omega^{0,\bullet}(\mathcal{E}[T]) = \{\Omega^{0,\bullet}(T^{\otimes n})\}$. Assume that X is equipped with a hyper-Kähler metric h . Then the canonical (0,1)-connection ∇ of h, see 2.5, preserves the symplectic structure. The covariant derivatives of the curvature, which we denoted $R_n, n \geqslant 2$, are also totally symmetric

$$R_n \in \Omega^{0,1}(S^{n+1}T). \tag{5.9.1}$$

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For a graph $\Gamma \in \tilde{\mathcal{G}}((g,n))$ and a symplectic vector space W let

$$P_\Gamma\colon \bigotimes_{v\in \mathrm{Vert}(\Gamma)} S^{|v|}W \to W^{\otimes n} \tag{5.9.2}$$

be the natural contraction map. Let

$$R_\Gamma = p_\Gamma\left(\bigotimes_{v\in \mathrm{Vert}(\Gamma)} R_{|v|-1}\right) \in \Omega^{0,n}(T^{\otimes n}), \qquad N = |\mathrm{Vert}(\Gamma)|. \tag{5.9.3}$$

Then R_Γ can be regarded as a morphism

$$R_\Gamma\colon \delta(\Gamma) \to \Omega^{0,\bullet}(T^{\otimes n}). \tag{5.9.4}$$

THEOREM 5.9.5. *The morphisms R_Γ extend to a morphism of modular dg-operads*

$$\tilde{\mathcal{F}} \to \Omega^{0,\bullet}(\mathcal{E}[T]).$$

In particular, for connected graphs with no legs the tensors R_i define a morphism from $\mathcal{F}((g,0))$ (Kontsevich's graph complex) into the Dolbeault complex $\Omega^{0,\bullet}$.

Proof. This follows from Theorem 2.6, once we take into account the additional symmetries of the R_i.

5.10. Operadic interpretation: formal geometry

Similarly to 4.2, let $p\colon \Psi \to X$ be the fiber bundle whose fiber at x consists of all formal *symplectic* exponential maps $T_xX \to X$. This bundle carries the tautological forms

$$\bar{\alpha}_n \in \Omega^1_{\Psi/X} \otimes p^*S^{n+1}T, \tag{5.10.1}$$

from which we construct the forms

$$\bar{\alpha}_\Gamma \in \Omega^N_{\Psi/X} \otimes p^*T^{\otimes n}, \quad \Gamma \in \widetilde{\mathcal{G}}((g,n)), |\mathrm{Vert}(\Gamma)| = N, \tag{5.10.2}$$

similarly to (5.9.3).

THEOREM 5.10.3. *The forms $\bar{\alpha}_\Gamma$ give rise to a morphism of modular dg-operads*

$$\tilde{\mathcal{F}} \to H^0(X, (p_*\Omega^\bullet_{\Psi/X}) \otimes \mathcal{E}[T]).$$

In particular, for connected graphs with no legs these form define a map of $\mathcal{F}((g,0))$ into $p_*\Omega^\bullet_{\Psi/X}$.

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The proof can be obtained by embedding $e\colon \Psi \hookrightarrow \Phi$ where Φ is the space of all formal exponential maps from 4.2 and noticing that $\bar{\alpha}_n = e^*\alpha_n$, where α_n is the tautological form from (4.2.7). Our statement, which amounts to calculating $d_{\Psi/X}\bar{\alpha}_n$, follows from Theorem 4.3.

5.11. Operadic interpretation: Lie algebra cohomology

The construction of 5.10 comes close to the original approach of [K2]: even though we do not use the $\bar{\partial}$-foliation on X and its universal characteristic classes, the formal geometry framework can be regarded as a holomorphic replacement of the $\bar{\partial}$-theory. In particular, the cohomology of the Lie algebra of formal Hamiltonian vector fields has direct interpretation in both frameworks. The role of Fuks' theorem (4.5.3) from the non-symplectic case is played here by the result of Kontsevich [K1]. Let us formulate it in a more general form, allowing graphs with legs so that the operadic formalism is applicable.

Let r be an even integer and $V = \mathbf{C}^r$ be the standard symplectic vector space of dimension r. Let

$$\mathrm{Ham}_r^0 = \prod_{n\geqslant 2} S^n V \tag{5.11.1}$$

be the Lie algebra of formal Hamiltonian vector fields on V with trivial constant term. Its degree 2 part is $S^2V = \mathbf{sp}_r$, the Lie algebra of linear symplectomorphisms. As in 4.4, any relative cochain of $(\mathrm{Ham}_r^0, \mathbf{sp}_r)$ with coefficients in some tensor power of V gives rise to a natural relative form on Ψ/X with values in the corresponding tensor power of p^*TX. In particular, the tautological cochain

$$\bar{a}_n \in C^1(\mathrm{Ham}_r^0, \mathbf{sp}_r, S^{n+1}V), \tag{5.11.2}$$

associating to a vector field its degree $n+1$ part, corresponds to the form $\bar{\alpha}_n$. For any graph $\Gamma \in \mathcal{G}((g,n))$ define

$$\bar{a}_\Gamma \in C^N(\mathrm{Ham}_r^0, \mathbf{sp}_r, S^{n+1}V), \tag{5.11.3}$$

as in 5.8.1. Furthermore, the complexes

$$P_r^\bullet((g,n)) := C^\bullet(\mathrm{Ham}_r^0, \mathbf{sp}_r, S^nV) \tag{5.11.4}$$

define a modular dg-operad $P_r^\bullet$. The symplectic analog of Theorem 4.5.2 is as follows.

THEOREM 5.11.5. (a) *The maps*

$$\bar{a}_\gamma\colon \delta(\Gamma) \to P_r^\bullet((g,n)), \quad \Gamma \in \tilde{\mathcal{G}}((g,n))$$

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define a morphism of modular operads $\tau\colon \tilde{\mathcal{F}} \to P_r^\bullet$.

(b) *If* $r \gg g, n, i$, *then the map of vector spaces*

$$\tau\colon \tilde{\mathcal{F}}^i((g,n)) \to P_r^i((g,n))$$

is an isomorphism.

This is proved in the same way as the result in [K1] (Theorem 1.1, which concerns graphs without legs, i.e., cohomology with trivial coefficients) or [Fuk].

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