ON THE EXISTENCE OF VARIOUS BOUNDED HARMONIC FUNCTIONS WITH GIVEN PERIODS

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1. Consider a pair \((R, \Gamma)\) of a Riemann surface \(R\) and a period \(\Gamma\). By a period \(\Gamma\) we mean a real-valued function \(\Gamma(\gamma)\) on one-dimensional cycles \(\{\gamma\}\) of the Riemann surface \(R\). Let \(O_\delta^B\) be the class of pairs \((R, \Gamma)\) such that there is no harmonic function on the Riemann surface \(R\) which satisfies a boundedness property \(\delta\) and

\[\int_{\gamma}^* du = \Gamma(\gamma)\]

for every cycle \(\gamma\). As for \(\delta\) we let \(B\) stand for boundedness, \(D\) for the finiteness of the Dirichlet integral, \(BD\) for \(B\) and \(D\). The relations to standard notations \(O_{\delta\delta}\) in the classification theory of Riemann surfaces (cf. [1]) should be clear. For example, \(R \in O_{\delta\delta}\) means that \((R, \Gamma_0) \in O_\delta^B\), where \(\Gamma_0(\gamma) = 0\) for every cycle \(\gamma\), and \(R \in O_{\delta\delta}\) means that \((R, \Gamma_0) \in O_{\delta\delta}^B\). From our standpoint H. Widom’s articles [3] and [4] may be considered as the study of the class \(O_\delta^B\). Our study may be also be considered as being in the frame work of that of Riemann matrices.

The well known Virtanen identity \(O_{\delta\delta} = O_{\delta\delta}^BD\) is one of the beautiful results in the classification theory; what’s more, the space \(HBD(R)\) is dense in \(HD(R)\) in the \(CD\)-topology (cf. [1, p. 178]). Therefore there exists a sequence \(\{u_n\}\) in \(HBD(R)\) convergent to a given \(u \in HD(R)\) so that \(\int_{\gamma}^* du_n\) converges to \(\int_{\gamma}^* du\) for every cycle \(\gamma\). In this connection one naturally asks whether \(O_\delta^B = O_\delta^BD\). The question also relates to the unsettled strictness question \(O_{\delta\delta} \subset O_{\delta\delta}\). The main result of this paper is the following strict inclusion:

**Theorem.** \(O_\delta^B < O_{\delta\delta}^B\).

We will show that there exists a planar region \(O^*\) such that there

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exist $HD$-functions on $Ω^*$ which have the same period as the given $HB$-
functions on $Ω^*$ but there exists no $HB$-function on $Ω^*$ which has the
same period as some $HD$-function on $Ω^*$.

2. Let $Ω$ denote the right half plane of the complex plane and
$Ω[a, b]$ the right half plane less the interval $[a, b]$ on the real axis. The
function

$$g(z, z_0) = \log \left| \frac{z - z_0}{z + z_0} \right|$$

is the Green’s function for the region $Ω$ with pole at $z_0$. The function

$$u[a, b](z) = \int_a^b \log \left| \frac{z + t}{z - t} \right| \, dt \quad (0 < a < b)$$

is the potential whose support is the interval $[a, b]$. Therefore $u[a, b](z)$
is positive and harmonic on the region $Ω[a, b]$ and vanishes on the
imaginary axis, and furthermore has the following properties:

**Lemma 1.** Let $β$ be a simple curve oriented clockwise enclosing the
interval $[a, b]$. Then $u[a, b]$ is continuous on the region $Ω$ and

1. \[ \int_β du[a, b] = 2π(b - a); \]
2. \[ D(u[a, b]) = π((2b)^2 \log 2b - 2(a + b)^2 \log (a + b) + (2a)^2 \log 2a) \]
   \[ + 2π(b - a)^2 \log \frac{1}{b - a}. \]

**Proof.** Put $u = u[a, b]$. For $a ≤ x ≤ b$,

$$u(x) = \int_a^x \log \left| \frac{x + t}{x - t} \right| \, dt$$

$$= \int_a^x \log (x + t)dt - \int_a^x \log (x - t)dt - \int_x^b \log (t - x)dt$$

$$= (x + b) \log (x + b) - (x + a) \log (x + a)$$

$$- (x - a) \log (x - a) - (b - x) \log (b - x).$$

Thus $u(x)$ is continuous on the interval $[a, b]$ which is the support of
potential $u$, and therefore it follows from the continuity principle (cf.
[2, p. 54]) that $u$ is continuous on the region $Ω$.

Fix $x, a < x < b$, and consider
\[ f(z) = \int_a^b \log \frac{z + t}{z - t} \, dt \]
on the upper plane. Observe that
\[ f'(z) = \int_a^b \left( \frac{1}{z + t} - \frac{1}{z - t} \right) \, dt. \]

Since
\[ \lim_{z \to -z} \Im \left( \int_a^b \frac{1}{z + t} \, dt \right) = 0 \]
and
\[ \int_a^b \frac{1}{t - z} \, dt = \log (b - z) - \log (a - z), \]
whose imaginary part is the angle formed by the lines \( \overline{za} \) and \( \overline{zb} \), we conclude that
\[ \lim_{z \to -z} \Im (f'(z)) = \pi. \]

From this it follows that \( *du = \pi \) on the interval \((a \ b)\) considered as the degenerate closed curve traced in the negative direction.

Therefore (1) is trivially true. By
\[ D(u) = 2\pi \int_a^b u(t) \, dt \]
and direct calculations, we obtain (2).

**Corollary.** For \( a \geq e \),
\[ (3) \quad \int_a^b *du[a \ a + 1] = 2\pi; \]
\[ (4) \quad D(u[a \ a + 1]) \leq 10\pi \log a. \]

**Proof.** The relation (3) is trivial and (4) is seen by direct calculations.

3. We denote by \( D_e \) the interior of the ellipse, whose horizontal axis is of length \( \frac{1}{2}((1/r) + r) = c \) and vertical axis \( \frac{1}{2}((1/r) - r)(0 < r < 1) \), less the interval with length 1 in the center on the horizontal axis. Let
\( v_c \) denote the harmonic measure of the interval with respect to the region \( D_c \).

**Lemma 2.** Let \( \beta \) be a simple curve oriented clockwise enclosing the interval. Then

\[
\int_{\beta}^*dv_c \leq 2\pi (\log c)^{-1}.
\]

**Proof.** Suppose that the center of the ellipse is the origin. The function \( z = \frac{1}{2}(1/w + w) \) maps the annulus \( \{ r < |w| < 1 \} \) conformally onto \( D_c \), the circle \( |w| = r \) onto the ellipse and the circle \( |w| = 1 \) onto the interval. The harmonic measure of the circle \( \{ |w| = 1 \} \) with respect to the annulus \( \{ r < |w| < 1 \} \) is the function

\[
\log \frac{|w|}{r} \log \frac{1}{r}
\]

whose flux is \( 2\pi (\log 1/r)^{-1} \). Therefore

\[
\int_{\beta}^*dv_c = 2\pi \left( \log \frac{1}{r} \right)^{-1} = 2\pi (\log (c + (c^2 - 1)^{1/2}))^{-1} \leq 2\pi (\log 2c)^{-1}.
\]

4. Put

\[
a_n = \exp \left( \sum_{k=0}^{n} 2^k \right)
\]

and

\[
\Omega^* = \bigcap_{n=1}^{\infty} \Omega[a_n a_n + 1]
\]

and \( u_n = u[a_n a_n + 1] \) and \( u = \sum_{n=1}^{\infty} n2^{-n}u_n \). Let \( \gamma_n \) be a simple curve oriented clockwise enclosing \( [a_n a_n + 1] \) so that \( \gamma_m \) and \( \gamma_n \) are disjoint if \( m \neq n \). Then \( \{ \gamma_n \}_{n=1}^{\infty} \) is a homology basis of \( \Omega^* \).

In order to prove our theorem it is sufficient to show the following lemma:

**Lemma 3.** The region \( \Omega^* \) has the following properties:

(i) The function \( u \) belongs to \( HD(\Omega^*) \);

(ii) No function belong to \( HB(\Omega^*) \) has the same period as the function \( u \);

(iii) Give any function \( v \) belonging to \( HB(\Omega^*) \),

\[\text{...}\]


\[ v^* = \frac{1}{2\pi} \sum_{n=1}^{\infty} \left( \int_{\gamma_n} *dv \right) u_n \]

belongs to \( HD(\Omega^*) \) and has the same period as the function \( v \).

**Proof.** Since

\[
D(u_n) \leq 10\pi \log a_n = 10\pi \sum_{k=0}^{n} 2^k \leq 20\pi 2^n ,
\]

\[ \sum_{n=0}^{\infty} n2^{-n}(D(u_n))^4 \leq (20\pi)^4 n(2^{-1})^n < \infty . \]

Noticing this and using properties of \( CD \)-topology [1, p. 149], the function \( u \) belongs to the class \( HD(\Omega^*) \), i.e. \( (i) \) is true.

To prove \( (ii) \) it suffices to show that

\[
\lim_{n \to \infty} \frac{\int_{\gamma_n} *dv}{\int_{\gamma_n} *du} = 0
\]

for every \( v \in HB(\Omega^*) \). We may, without loss of generality, assume that \( M - 1 > v > 1 \). Let \( D_n \) denote the region \( D, c = a_n - a_{n-1} - \frac{1}{2} \), whose outer boundary is an ellipse having the center at \( a_n + \frac{1}{2} \) and passing \( a_{n-1} + 1 \), and let \( v_n \) denote \( 2Mv_c \). For \( \frac{1}{2} < t < 1 \), the set \( \{ z \in D_n : tv_n > v \} \) contains a neighbourhood of the interval \([a_n a_{n+1}]\) and does not contain a neighbourhood of the ellipse. By the maximum principle, this set is a region and we can choose some \( t \) so that the set \( \{ z \in D_n : tv_n = v \} \) is a simple regular closed curve, which is denoted by \( \delta_n \), homologous to \( \gamma_n \).

Since

\[
\int_{\delta_n} *dv_n > \int_{\delta_n} *dv ,
\]

and

\[
\int_{\gamma_n} *dv_n = \int_{\delta_n} *dv_n > t \int_{\delta_n} *dv_n = \int_{\delta_n} *dtv_n > \int_{\delta_n} *dv = \int_{\gamma_n} *dv .
\]

By Lemma 2,

\[
0 < \int *dv_n \leq 2\pi M (\log ((a_n - a_{n-1}) - 1/2))^{-1}
\]

\[ \leq 2\pi M \left( \log \frac{a_n}{a_{n-1}} \right)^{-1} = 2\pi M 2^{-n} . \]
From
\[ \int_{\mathcal{R}_n} *du = n2^{-n} \int_{\mathcal{R}_n} *du_n = 2\pi n2^{-n}, \]

it follows that
\[ \int_{\mathcal{R}_n} *du > \frac{1}{M} n \int_{\mathcal{R}_n} *dv_n > \frac{1}{M} n \int_{\mathcal{R}_n} *dv. \]

Since \( M - 1 > M - v > 1 \), by the same arguments,
\[ \int_{\mathcal{R}_n} *du \geq -\frac{1}{M} n \int_{\mathcal{R}_n} *dv. \]

The proof of (ii) is herewith complete.

Since \( \int_{\mathcal{R}_n} *dv = o(n2^{-n}) \), by the same argument as for the function \( u \), we can show that the function \( v^* \) belongs to \( HD(\Omega^*) \). It is trivial that the function \( v^* \) has the same period as the function \( v \), and (iii) is obtained.

REFERENCES


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