CONTINUOUS AND NUMERICAL ANALYSIS
OF A BOUNDARY SHOCK PROBLEM

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A quasilinear singularly perturbed boundary value problem whose solution has a shock layer is investigated. Estimates of the derivatives of the solution are derived. Based on these estimates, a new independent variable is introduced. Then the transformed problem is solved numerically using finite - difference schemes. The transformation corresponds to solving the original problem on a mesh which is dense in the layer. The linear convergence uniform in the perturbation parameter is proved in the discrete L^1 norm. Numerical results show uniform pointwise convergence too.

1. Introduction

The aim of the paper is to construct a uniform numerical method for solving the following quasilinear boundary value problem with a small positive perturbation parameter ε:

(1) $$-\varepsilon u'' - ub(u)u' + uc(x, u) = 0, \ u(0) = 0, \ u(1) = B > 0.$$

Under appropriate conditions, which we shall state in the next Section, it follows that there exists a unique solution, u_ε, to the problem (1). We are interested in the boundary shock layer behaviour of u_ε, in the sense of [2]. In Section 2 we derive estimates of the derivatives of u_ε. They are necessary to prove that our numerical method is uniform in ε.

The numerical method is given in Section 3. Essentially, it uses finite-difference (upwind) schemes on a special non-equidistant mesh which is dense in the layer. However, the mesh is introduced indirectly: the problem (1) is transformed by changing the independent variable and then the resulting problem is solved numerically on an equidistant mesh. This approach can be found in [3], as well as in some earlier papers of these authors, and in [10]. We use here the same transformation as in [10], where the non-turning point case of a quasilinear singularly perturbed boundary value problem was considered. Our main result is the uniform (that is, uniform in ε) first order

convergence of the numerical solution towards the restriction of u_ε on the mesh, in the discrete L^1 norm. The result follows from the principle: uniform stability + uniform consistency $\Rightarrow$ uniform convergence. Uniform stability can be proved in the discrete L^1 norm (see [1, 4, 5, 6]) and this is the reason why this norm is used. But, on the other hand, we are not able to prove uniform pointwise convergence, which is present, as numerical examples in Section 4 show.

   This paper might be regarded as a step towards uniform numerical methods for more general problems whose solutions have interior shock layers, such as the Lagerstrom–Cole model problem, see [2]. In these cases the problem of locating shocks has to be resolved.

   Singularly perturbed quasilinear problems have been solved numerically in [1, 7] and [4, 5, 6] (the case $\varepsilon = 0$), just to mention some of the papers. Note that none of these papers deals with the proof of uniform convergence in the above sense.

2. ESTIMATES OF THE DERIVATIVES

   Thus we shall consider the problem:

(2a) $$Tu = -\varepsilon u'' - ub(u)u' + uc(x, u) = 0, x \in I = [0, 1],$$

(2b) $$Ru = (u(0), u(1)) = (0, B), B > 0,$$

where $' = d/dx$, $0 < \varepsilon \leqslant \varepsilon^* < 1$ (usually $\varepsilon^* << 1$).

Let $U = [0, B]$.

We assume

(3) $$b \in C^2(U), \quad c \in C^2(I \times U)$$

(4) $$c^* \geqslant c(x, u) \geqslant c_* \geqslant 0, \quad x \in I, \quad u \in U,$$

(5) $$b^* \geqslant b(u) \geqslant b_* > 0, \quad u \in U.$$

Because of (4) 0 and B are lower and upper solutions to (2), thus the problem (2) has a solution, $u_\varepsilon \in C^4(I)$, and

$$u_\varepsilon(x) \in U, x \in I.$$

Furthermore, from (4) we have $u'_\varepsilon(x) \geqslant 0$ for $x \in I$.

   The following inequality will be of interest:

(6) $$B > Q = \{b^* c^* + [b^* c^*(b^* c^* - b_* c_*)]^{1/2}\}/(b^* b_*).$$

We suppose that the conditions (3-6) hold throughout the section.

   In this and the next Section we shall denote by M any (in the sense of $O(1)$) positive constant which is independent of ε. In particular, some of these constants will be denoted by m, m', m_0 et cetera.

LEMMA 1. $u_\varepsilon'(0) \geqslant m_1/\varepsilon$.

PROOF: First we show that

(7) $$u_\varepsilon(x) \leqslant u^*(x) = (c_*/b^*)(x-1) + B, \; x \in I.$$

Let z_ε be the solution to the following special case of the problem (2):

$$\tilde{T}u = -\varepsilon u'' - b^* u u' + c_* u = 0, \quad Ru = (0, B).$$

Because of (4) and (5) we have

$$\tilde{T}u_\varepsilon = (b(u_\varepsilon) - b^*)u_\varepsilon' + (c_* - c(x, u_\varepsilon))u_\varepsilon \leqslant 0, \quad x \in I,$$

and since the operator $\left(\tilde{T}, R \right)$ is inverse monotone, we get

(8) $$u_\varepsilon(x) \leqslant z_\varepsilon(x), \quad x \in I.$$

Now introduce the linear operator:

$$Lu = -\varepsilon u'' - b^* z_\varepsilon u'.$$

We have $$L(u^* - z_\varepsilon) = 0, \quad (u^* - z_\varepsilon)(1) = 0,$$
and (6) implies $$u^*(0) > 0.$$

Thus by the inverse monotonicity of (L, R) we get

$$z_\varepsilon(x) \leqslant u^*(x), \quad x \in I,$$

which together with (8) completes the proof of (7).

Now rewrite (2a) in the form:

(9a) $$-\varepsilon u'' - f(u)' + uc(x, u) = 0,$$

(9b) $$f(u) = \int_0^u sb(s)ds,$$

and integrate (9a) from 0 to 1. Using $u_\varepsilon'(1) \geqslant 0$ we get

$$\varepsilon u_\varepsilon'(0) \geqslant f(B) - \int_0^1 u_\varepsilon(x)c(x, u_\varepsilon(x))dx \geqslant$$

$$f(B) - c^* \int_0^1 u^*(x)dx = (b_*/2)B^2 - c^* B + c^* c_*/(2b^*) = m_1.$$

We have $m_1 > 0$ because of (6).   □

REMARK. Condition (6) implies $B > c^*/b_*$ which by Peano's theorem guarantees the existence of a solution u_R to the reduced problem

$$-ub(u)u' + uc(x, u) = 0, \, x \in I, \, u(1) = B.$$

Moreover, $u'_R(x) \geqslant 0$ for $x \in I$, and $u_R(0) > 0$, hence u_ε has a layer at $x = 0$.

The condition (6) is artificial to some extent. Note, however, that in the constant coefficient case $(b = b^* = b_*, \, c = c^* = c_*)$ it reduces to

$$B > c/b,$$

which is the necessary condition for existence of a boundary layer at $x = 0$. If $B \leqslant c/b$, the interior crossing phenomenon occurs, see [2].

Let
$$F(x) = \int_0^x u_\varepsilon(s)b(u_\varepsilon(s))ds.$$

We have:

LEMMA 2. $u'_\varepsilon(x) \leqslant M(1 + \varepsilon^{-1}\exp(-F(x)/\varepsilon)), \, x \in I.$

PROOF: From (2a) it follows that

$$\varepsilon(\exp(F(x)/\varepsilon)u'_\varepsilon(x))' = u_\varepsilon(x)c(x, u_\varepsilon(x))\exp(F(x)/\varepsilon).$$

Expressing $u'_\varepsilon(x)$ by integration we get

$$u'_\varepsilon(x) = \left[\varepsilon^{-1}\int_0^x g(t)u_\varepsilon(t)b(u_\varepsilon(t))\exp(F(t)/\varepsilon)dt + u'_\varepsilon(0)\right]\exp(-F(x)/\varepsilon)$$
$$= [g(\theta)(\exp(F(x)/\varepsilon) - 1) + u'_\varepsilon(0)]\exp(-F(x)/\varepsilon),$$

where $g(x) = c(x, u_\varepsilon(x))/b(u_\varepsilon(x)), \, \theta \in (0, x)$. Thus:

(10) $u'_\varepsilon(x) = g(\theta)(1 - \exp(-F(x)/\varepsilon)) + u'_\varepsilon(0)\exp(-F(x)/\varepsilon).$

Now integrate (10) from 0 to 1 and express $u'_\varepsilon(0)$. It follows:

$$u'_\varepsilon(0) \leqslant M/J, \quad J = \int_0^1 \exp(-F(x)/\varepsilon)dx.$$

Since $J \geqslant \int_0^1 \exp(-b^*Bx/\varepsilon)dx = \varepsilon(1 - \exp(-b^*B/\varepsilon))/(b^*B),$

we get $u'_\varepsilon(0) \leqslant M/\varepsilon,$

which together with (10) gives the assertion.   □

LEMMA 3.

(11a) $$F(x) \geqslant m_2 x^2/\varepsilon \text{ for } x \in [0, \varepsilon],$$

(11b) $$F(x) \geqslant m_3(x - \varepsilon) \text{ for } x \in [\varepsilon, 1].$$

PROOF: Let $x \in [0, \varepsilon]$. Then we use

(12) $$u_\varepsilon(x) = x u'_\varepsilon(\sigma), \quad \sigma \in (0, \varepsilon).$$

From (10) it follows that

$$u'_\varepsilon(\sigma) \geqslant u'_\varepsilon(0) \exp\left(-F(\sigma)/\varepsilon\right)$$
$$\geqslant m_1 \exp\left(-b^* B\right)/\varepsilon = m_4/\varepsilon,$$

where we have used Lemma 1. Thus from (12) we get

$$u_\varepsilon(x) \geqslant m_4 x/\varepsilon, \quad x \in [0, \varepsilon],$$

and (11a) follows with $m_2 = b_* m_3/2$.

Now let $x \in [\varepsilon, 1]$. We have

$$u_\varepsilon(x) \geqslant u_\varepsilon(\varepsilon) = \varepsilon u'_\varepsilon(\gamma), \quad \gamma \in (0, \varepsilon),$$

thus, using the above technique we get

(13) $$u_\varepsilon(x) \geqslant m_4.$$

Now $$F(x) \geqslant \int_\varepsilon^x u_\varepsilon(s) b(u_\varepsilon(s)) ds \geqslant b_* m_4 (x - \varepsilon),$$

and (11b) is proved with $m_3 = b_* m_4$.   ☐

LEMMA 4. $|u''_\varepsilon(x)| \leqslant M\left(1 + \varepsilon^{-3} x \exp\left(-F(x)/\varepsilon\right)\right), x \in I$.

PROOF: Differentiate (2a) and use the same technique as in the proof of Lemma 2 to obtain:

$$u''_\varepsilon(x) = \varepsilon^{-1} \exp\left(-F(x)/\varepsilon\right) \int_0^x G(t) \exp\left(F(t)/\varepsilon\right) dt,$$

(note that $u''_\varepsilon(0) = 0$) where from Lemma 2 it follows that

$$|G(t)| \leqslant M\left(1 + \varepsilon^{-1} \exp\left(-F(t)/\varepsilon\right) + \varepsilon^{-2} \exp\left(-2F(t)/\varepsilon\right)\right).$$

Thus:
$$|u_\varepsilon''(x)| \leqslant M\varepsilon^{-1} \exp\left(-F(x)/\varepsilon\right) \int_0^x [\varepsilon^{-2} + \exp\left(F(t)/\varepsilon\right)]dt,$$

and in order to prove the lemma we have to show that

$$(14) \qquad S = \varepsilon^{-1} \int_0^x \exp\left((F(t) - F(x))/\varepsilon\right)dt \leqslant M.$$

Let $x \in [0, \varepsilon]$. Then obviously $S \leqslant 1$. Now let $x \in [\varepsilon, 1]$. We have

$$(15) \qquad S \leqslant 1 + \int_\varepsilon^x \exp\left((F(t) - F(x))/\varepsilon\right)dt.$$

Since for $\varepsilon \leqslant t \leqslant x \leqslant 1$ we have

$$F(t) - F(x) \leqslant b_* m_4(t - x),$$

where m_4 is the same as in (13), from (15) we get (14).   ☐

THEOREM 1.

$$\left|u_\varepsilon^{(k)}(x)\right| \leqslant M\left(1 + \varepsilon^{-k} \exp\left(-mx/\varepsilon\right)\right),$$
$$x \in I, \quad k = 1, 2, 3.$$

PROOF: The proof of the cases $k = 1, 2$ follows immediately from Lemmas 2, 3 and 4, with apppropriate M and m. It remains to prove the case $k = 3$. First differentiate (2a) and use Lemma 2 to get $\left|u_\varepsilon^{(3)}(0)\right| \leqslant M\varepsilon^{-3}$. Then diffenentiate (2a) once more and obtain:

$$\left|u_\varepsilon^{(3)}(x)\right| \leqslant M \exp\left(-F(x)/\varepsilon\right) \left\{\varepsilon^{-3} + \varepsilon^{-1} \int_0^x \left[1 + \varepsilon^{-1} \exp\left(-F(t)/\varepsilon\right)\right.\right.$$
$$+ \varepsilon^{-2} \exp\left(-2F(t)/\varepsilon\right) + \varepsilon^{-3} \exp\left(-3F(t)/\varepsilon\right) + \varepsilon^{-3}t \exp\left(-F(t)/\varepsilon\right)$$
$$\left.+ \varepsilon^{-4}t \exp\left(-2F(t)/\varepsilon\right)\right] \exp\left(F(t)/\varepsilon\right)dt\Big\}$$
$$\leqslant M \exp\left(-F(x)/\varepsilon\right)\{\varepsilon^{-3} + \varepsilon^{-1} \int_0^x [\varepsilon^{-3} + \varepsilon^{-4}t + \exp\left(F(t)/\varepsilon\right)]dt\}$$
$$\leqslant M\left(\varepsilon^{-3} + \varepsilon^{-4}x + \varepsilon^{-5}x^2\right) \exp\left(-F(x)/\varepsilon\right) + S,$$

where S is the same as in (14), and the assertion follows because of Lemma 3.   ☐

3. The numerical method

We shall give here the numerical method for the problem (2). Additionally, we have to assume

(16) $(uc(x, u))_u \geqslant c_1 > 0, \, x \in I, \, u \in U.$

Let $\lambda(t) = \begin{cases} \omega(t) = \beta\varepsilon t/(\gamma - t), \, t \in [0, \alpha] \\ \pi(t) = \delta(t - \alpha)^3 + \omega''(\alpha)(t - \alpha)^2/2 \\ \qquad +\omega'(\alpha)(t - \alpha) + \omega(\alpha), \, t \in [\alpha, 1] \end{cases}$

where $\alpha \in (0, 1)$ is given and $\gamma = \alpha + \varepsilon^{1/4}$. Obviously, $\lambda \in C^2(I)$ and $\lambda \in C^\infty[0, \alpha]$, $\lambda \in C^\infty[\alpha, 1]$. Coefficient δ is determined from the condition $\pi(1) = 1$, while coefficient β is chosen to satisfy:

$$0 < \beta\varepsilon^{1/4}\left(\gamma(1 - \alpha)^2 + \varepsilon^{1/4}\gamma(1 - \alpha) + \varepsilon^{1/2}\alpha\right) \leqslant 1,$$

so that $\delta \geqslant 0$ and $\pi''(t)$ is non-decreasing. From this we have

$$\pi''(t) \geqslant \pi''(\alpha) = \omega''(\alpha) > 0, \quad t \in [\alpha, 1],$$
hence $\pi'(t) \geqslant \pi'(\alpha) = \omega'(\alpha) > 0, \quad t \in [\alpha, 1],$
and obviously $\omega'(t) > 0$ for $t \in [0, \alpha].$

Thus, the function λ can be used to introduce a new independent variable t via $x = \lambda(t)$. Let $y(t) = u(\lambda(t))$. Then the transformed problem (9), (2b) reads:

(17a) $Py = -\varepsilon(\mu(t)y')' - f(y)' + q(t, y) = 0,$
(17b) $Ry = (0, B),$

where $' = d/dt$,

$$\mu(t) = 1/\lambda'/(t),$$
$$q(t, y) = yc(\lambda(t), y)\lambda'(t).$$

The same function λ was used in [10]. Its part ω is a certain modification of the inverse of the boundary layer function $\exp(-mx/\varepsilon)$, see [8, 9, 3].

Let I_h be an equidistant mesh with points

$$t_i = ih, \, i = 0, 1, \ldots, n, \quad h = 1/n, \quad n \in \mathbb{N} \quad (n \geqslant 4),$$
and let $t_{i+1/2} = t_i \pm h/2, \quad i = 1, 2, \ldots, n - 1.$

We form the discrete problem corresponding to (17):

(18) $P_h w_h = 0,$

where $w_h = [w_1, w_2, \ldots, w_{n-1}]^T \in \mathbf{R}^{n-1}$ is a mesh function on $I_h \setminus \{0, 1\}$ and $P_h : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ is

$$
\begin{aligned}
P_h w_1 &= \varepsilon h^{-2}[(\mu(t_{1/2}) + \mu(t_{3/2}))w_1 - \mu(t_{3/2})w_2] \\
&\quad - h^{-1}[f(w_2) - f(w_1)] + q(t_1, w_1), \\
P_h w_i &= \varepsilon h^{-2}\left[-\mu(t_{i-1/2})w_{i-1} + (\mu(t_{i-1/2}) + \mu(t_{i+1/2}))w_i\right. \\
&\quad \left. - \mu(t_{i+1/2})w_{i+1}\right] - h^{-1}[f(w_{i+1}) - f(w_i)] \\
&\quad + q(t_i, w_i), \, i = 2, 3, \ldots, n - 2, \\
P_h w_{n-1} &= \varepsilon h^{-2}\left[-\mu(t_{n-3/2})w_{n-2} + (\mu(t_{n-3/2}) + \mu(t_{n-1/2}))w_{n-1}\right. \\
&\quad \left. - \mu(t_{n-1/2})B\right] - h^{-1}[f(B) - f(w_{n-1})] + q(t_{n-1}, w_{n-1}).
\end{aligned}
$$

Let $\|\cdot\|_1$ denote the usual vector (matrix) norm in $\mathbf{R}^{n-1}$ $(\mathbf{R}^{n-1,n-1})$. Let $e_h = [1, 1, \ldots, 1]^T \in \mathbf{R}^{n-1}$ and let

$$
U_h = \{w_h \in \mathbf{R}^{n-1} : Be_h \geqslant w_h \geqslant 0\},
$$

(the inequality sign in $\mathbf{R}^{n-1}$ should be understood componentwise). We have:

THEOREM 2. *Let (3-5) and (16) hold. Then in U_h there exists a unique solution $w_{e,h}$ to the discrete problem (18). Moreover, for any $w_h, v_h \in U_h$ the following stability inequality holds*

(19) $$\sum_{i=1}^{n-1} \lambda'(t_i) |w_i - v_i| \leqslant c_1^{-1} \|P_h w_h - P_h v_h\|_1.$$

PROOF: Let us prove (19). For $w_h, v_h \in U_h$ we set $z_i = w_i \lambda'(t_i)$, $g_i = v_i \lambda'(t_i)$, $i = 1, 2, \ldots, n-1$, and introduce a new operator:

$$
\tilde{P}_h z_i = P_h(z_i/\lambda'(t_i)), \quad i = 1, 2, \ldots, n-1.
$$

Then the Frechet derivative of $\tilde{P}_h$ satisfies:

$$
\left\|\left(\tilde{P}_h'(z_h)\right)^{-1}\right\|_1 \leqslant 1/c_1,
$$

see [5, 6]. Indeed, $\tilde{P}_h'(z_h)$ is an L-matrix and

$$
\left(\tilde{P}_h'(z_h)\right)^T e_h \geqslant c_1 e_h.
$$

It follows that $\|z_h - g_h\|_1 \leqslant c_1^{-1} \left\|\tilde{P}_h z_h - \tilde{P}_h g_h\right\|_1,$

and (19) is immediate. The inequalities

$$P_h(Be_h) \geqslant 0 \geqslant P_h 0,$$

complete the proof.   ☐

Let us now introduce the norm:

$$\|w_h\|_h = \sum_{i=1}^{n-1} \overline{h}_i \, |w_i|,$$

where
$$\overline{h}_i = (x_{i+1} - x_{i-1})/2, \, i = 1, 2, \ldots, n-1,$$
$$x_i = \lambda(t_i), \, i = 0, 1, \ldots, n.$$

This is the standard L^1 discrete norm, see [1]. Let $w_{\varepsilon,h}$ be as in Theorem 2 and let

$$u_{\varepsilon,h} = [u_\varepsilon(x_1), \, u_\varepsilon(x_2), \, \ldots, \, u_\varepsilon(x_{n-1})]^T.$$

Moreover, in this section we let the constants M be independent of h as well. Then we have:

THEOREM 4. *Let (3-6) and (16) hold and let the function λ be given with $\alpha = t_j$ for some $j \in (1, 2, \ldots, n-1)$. Then:*

(20) $$\|u_{\varepsilon,h} - w_{\varepsilon,h}\|_h \leqslant Mh.$$

PROOF: From (19) we have

(21) $$\sum_{i=1}^{n-1} \lambda'(t_i) \, |u_\varepsilon(x_i) - w_{\varepsilon,i}| \leqslant c_1^{-1} \|r_h\|_1,$$

where
$$r_h = P_h u_{\varepsilon,h}.$$

The components of the consistency-error-vector r_h are:

$$r_i = P_h y_\varepsilon(t_i) - (Py_\varepsilon)(t_i), \quad i = 1, 2, \ldots, n-1,$$

where $y_\varepsilon(t) = u_\varepsilon(\lambda(t))$. The following estimates are proved in [10]:

$$\varepsilon \left| \mu^{(k)}(t) \right| \leqslant M, \, k = 0, 1, 2,$$
$$\left| y_\varepsilon^{(k)}(t) \right| \leqslant M, \, k = 1, 2, 3,$$

where $t \in (t_{i-1}, t_i)$, $i = 1, 2, \ldots, n$, (note that these derivatives exist in (t_{i-1}, t_i)). It follows that

(22a)
$$|r_i| \leqslant Mh, \quad i = 1, 2, \ldots, n-1.$$

Furthermore, it is easy to prove that

(22b)
$$\left| h\lambda'(t_i) - \overline{h}_i \right| \leqslant Mh^3, \quad i = 1, 2, \ldots, n-1,$$

and, obviously:

(22c)
$$|u_\varepsilon(x_i) - w_{\varepsilon,i}| \leqslant M, \quad i = 1, 2, \ldots, n-1.$$

Now multiply (21) by h and use (22) to obtain (20).   □

4. NUMERICAL RESULTS

Let us consider the problem

$$-\varepsilon u'' - uu' + u = p(x), \quad u(0) = 0, \quad u(1) \approx 2,$$

where $p(x)$ is given in such a way that the solution is

$$u_\varepsilon(x) = x + 1 - \exp\left(-x/\varepsilon\right).$$

Although this problem is more general than the problem (2), we shall use it to compare the numerical solution obtained by the method from this Section with the exact one. Note that u_ε behaves in the way described by Theorem 1.

Let
$$E_\infty = \max_{1 \leqslant i \leqslant n-1} |u_\varepsilon(x_i) - w_{\varepsilon,i}|$$
and
$$E_h = \|u_{\varepsilon,h} - w_{\varepsilon,h}\|_h.$$

In Table 1 we present the results of our method. The function λ is taken with $\alpha = 0.5$ and $\beta = 1$. For $\varepsilon = 10^{-2}$ this gives about 40% of the points x_i in the interval $[0, \varepsilon]$ representing the layer. The percentage changes as ε does: for $\varepsilon = 10^{-3}$ it is about 33%, for $\varepsilon = 10^{-6}$ - about 26%, and for $\varepsilon = 10^{-9}$ - about 25%. By changing α and β one can change the density of the points x_i in the layer and a prescribed percentage can be achieved, see [9, 10].

Table 1 confirms our theoretical results, but it shows the uniform pointwise convergence as well. This is the advantage of our method in comparison with the numerical solution obtained on equidistant meshes without transforming the original problem.

These results are given in Table 2 and they also show the uniform convergence in the norm $\|\cdot\|_h = h \|\cdot\|_1$ (and even more than that). However, the uniform pointwise convergence is absent.

Table 1

n	ε	$1.-2$	$1.-3$	$1.-6$	$1.-9$
50	E_∞	$6.41-2$	$7.44-2$	$7.86-2$	$7.88-2$
	E_h	$3.30-2$	$3.48-2$	$3.53-2$	$3.54-2$
100	E_∞	$3,30-2$	$3.82-2$	$4.03-2$	$4.03-2$
	E_h	$1.69-2$	$1.78-2$	$1.81-2$	$1.81-2$
200	E_∞	$1.68-2$	$1.94-2$	$2.04-2$	$2.04-2$
	E_h	$8.55-3$	$9.02-3$	$9.17-3$	$9.19-3$

Table 2

n	ε	$1.-2$	$1.-3$
50	E_∞	0.200	$4.02-2$
	E_h	$8.78-3$	$3.60-3$
100	E_∞	0.106	$8.97-2$
	E_h	$4.66-3$	$2.38-3$
200	E_∞	$5.22-2$	0.170
	E_h	$2.20-3$	$1.74-3$

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