COMPOSITIO MATHEMATICA

Homomorphisms between standard modules over finite-type KLR algebras

Alexander S. Kleshchev and David J. Steinberg

Compositio Math. **153** (2017), 621–646.

doi:10.1112/S0010437X16008204

Compositio Math. 153 (2017) 621–646
doi:10.1112/S0010437X16008204

Homomorphisms between standard modules over finite-type KLR algebras

Alexander S. Kleshchev and David J. Steinberg

Abstract

Khovanov–Lauda–Rouquier (KLR) algebras of finite Lie type come with families of standard modules, which under the Khovanov–Lauda–Rouquier categorification correspond to PBW bases of the positive part of the corresponding quantized enveloping algebra. We show that there are no non-zero homomorphisms between distinct standard modules and that all non-zero endomorphisms of a standard module are injective. We present applications to the extensions between standard modules and modular representation theory of KLR algebras.

1. Introduction

Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra and $U_q^+(\mathfrak{g})$ the positive part of the quantized enveloping algebra of $\mathfrak{g}$. The algebra $U_q^+(\mathfrak{g})$ is Q^+-graded, where Q^+ is the non-negative part of the corresponding root lattice. On the other hand, to every $\alpha \in Q^+$ and a commutative unital ground ring $\Bbbk$, Khovanov, Lauda and Rouquier associate an infinite-dimensional associative unital graded $\Bbbk$-algebra H_α. Then an informal statement of the Khovanov–Lauda–Rouquier categorification theorem [KL09, KL11, Rou08, Rou12] is that representation theory of the KLR algebras H_α categorifies $U_q^+(\mathfrak{g})$ as a twisted bialgebra. This is important because, for example, it sheds new light on various interesting bases of $U_q^+(\mathfrak{g})$, such as canonical and dual canonical bases as well as PBW bases. The categorification is our first motivation for studying KLR algebras.

Our second motivation comes from the results of [BK09a, BK09b, BKW11, Rou08], which reveal deep connections between KLR algebras and classical representation theory of various Hecke algebras. These connections are especially important for modular representation theory and, more specifically, questions related to reduction modulo p (although in modular representation theory we are most interested in KLR algebras of affine Lie type A). This paper touches upon both aspects of interest of KLR algebras described above.

KLR algebras of finite Lie type possess *affine quasi-hereditary structures* [Kat14, BKM14, KLM13, KL15, KX12, Kle15b]. In particular, they come with important families of modules which are called *standard*. Under the Khovanov–Lauda–Rouquier categorification, standard modules correspond to PBW monomials in $U_q^+(\mathfrak{g})$; see [BKM14, Kat14].

Affine quasihereditary structures are parametrized by *convex orders* on the sets of positive roots of the corresponding root systems. In this paper we work with an arbitrary convex order and an arbitrary finite Lie type. When working with the KLR algebra H_α for any $\alpha \in Q^+$, the

Received 29 May 2015, accepted in final form 17 November 2016, published online 1 March 2017.
2010 Mathematics Subject Classification 16G99, 16E05, 17B37 (primary).
Keywords: representations of Khovanov–Lauda–Rouquier algebras, categorification, affine quasihereditary algebras.

This research was supported by the NSF grant DMS-1161094.

standard modules $\Delta(\lambda)$ are labeled by $\lambda \in \mathrm{KP}(\alpha)$, where $\mathrm{KP}(\alpha)$ is the set of Kostant partitions of α. With these conventions, our main result is the following.

Theorem A. *Let $\alpha \in Q^+$ and $\lambda, \mu \in \mathrm{KP}(\alpha)$. If $\lambda \neq \mu$, then*

$$\mathrm{Hom}_{H_\alpha}(\Delta(\lambda), \Delta(\mu)) = 0.$$

When $\lambda \nleq \mu$, it is clear that $\mathrm{Hom}_{H_\alpha}(\Delta(\lambda), \Delta(\mu)) = 0$, but for $\lambda < \mu$ we found this fact surprising. Theorem A is proved in § 3.

The case of $\lambda = \mu$ is also well understood. In fact, the endomorphism algebras of the standard modules are naturally isomorphic to certain algebras of symmetric functions; see Theorem 2.17. Now, Theorem A can be complemented by the following (folklore) observation and compared with the main result of [BCGM11].

Theorem B. *Let $\alpha \in Q^+$ and $\lambda \in \mathrm{KP}(\alpha)$. Then every non-zero H_α-endomorphism of $\Delta(\lambda)$ is injective.*

For the reader's convenience, we prove Theorem B in § 2.3.

Theorem A turns out to have some applications to modular representation theory of KLR algebras, which are pursued in § 4. Note that KLR algebras are defined over an arbitrary commutative unital ground ring $\Bbbk$, and when we wish to emphasize this fact, we use the notation $H_{\alpha,\Bbbk}$. Using the p-modular system (F, R, K) with $F = \mathbb{Z}/p\mathbb{Z}$, $R = \mathbb{Z}_p$ and $K = \mathbb{Q}_p$, we can reduce modulo p any irreducible $H_{\alpha,K}$-module. An important problem is to determine when these reductions remain irreducible; see [KR11, Wil14]. This problem can be reduced to homological questions involving standard modules.

In § 4, we show that standard modules have universal R-forms $\Delta(\lambda)_R$ such that $\Delta(\lambda)_R \otimes_R \Bbbk \cong \Delta(\lambda)_\Bbbk$ for any field $\Bbbk$. Then (denoting graded dimension by $\dim_q$) an application of the universal coefficient theorem and Theorem A yields the following result.

Theorem C. *Let $\alpha \in Q^+$ and $\lambda, \mu \in \mathrm{KP}(\alpha)$. Then the R-module*

$$\mathrm{Ext}^1_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R)$$

is torsion-free. Moreover,

$$\dim_q \mathrm{Ext}^1_{H_{\alpha,F}}(\Delta(\lambda)_F, \Delta(\mu)_F) = \dim_q \mathrm{Ext}^1_{H_{\alpha,K}}(\Delta(\lambda)_K, \Delta(\mu)_K)$$

if and only if the R-module $\mathrm{Ext}^2_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R)$ is torsion-free.

As a final application, using a universal extension procedure, we construct R-forms $Q(\lambda)_R$ of the projective indecomposable modules $P(\lambda)_K$ and prove the following theorem.

Theorem D. *Let $\alpha \in Q^+$. Then reductions modulo p of all irreducible $H_{\alpha,K}$-modules are irreducible if and only if one of the following equivalent conditions holds:*

(i) *$Q(\lambda)_R \otimes_R F$ is a projective $H_{\alpha,F}$-module for all $\lambda \in \mathrm{KP}(\alpha)$;*

(ii) *$\mathrm{Ext}^1_{H_{\alpha,F}}(Q(\lambda)_R \otimes_R F, \Delta(\mu)_F) = 0$ for all $\lambda, \mu \in \mathrm{KP}(\alpha)$;*

(iii) *the R-module $\mathrm{Ext}^2_{H_{\alpha,R}}(Q(\lambda)_R, \Delta(\mu)_R)$ is torsion-free for all $\lambda, \mu \in \mathrm{KP}(\alpha)$.*

2. Preliminaries

2.1 KLR algebras

We follow closely the set-up of [BKM14]. In particular, Φ is an irreducible root system with simple roots $\{\alpha_i \mid i \in I\}$ and Φ^+ is the corresponding set of positive roots. Denote by Q the root lattice and by $Q^+ \subset Q$ the set of $\mathbb{Z}_{\geqslant 0}$-linear combinations of simple roots, and write $\mathrm{ht}(\alpha) = \sum_{i\in I} c_i$ for $\alpha = \sum_{i\in I} c_i\alpha_i \in Q^+$. The standard symmetric bilinear form $Q \times Q \to \mathbb{Z}$, $(\alpha, \beta) \mapsto \alpha \cdot \beta$ is normalized so that $d_i := (\alpha_i \cdot \alpha_i)/2$ is equal to 1 for the short simple roots α_i. We also set $d_\beta := (\beta \cdot \beta)/2$ for all $\beta \in \Phi^+$. The Cartan matrix is $C = (c_{i,j})_{i,j\in I}$ with $c_{i,j} := (\alpha_i \cdot \alpha_j)/d_i$.

Fix a commutative unital ring $\Bbbk$ and an element $\alpha \in Q^+$ of height n. The symmetric group S_n with simple transpositions $s_r := (r\ r+1)$ acts on the set

$$I^\alpha := \left\{ \boldsymbol{i} = i_1 \cdots i_n \in I^n \,\middle|\, \sum_{j=1}^n \alpha_{i_j} = \alpha \right\}$$

on the left by place permutations. Choose signs $\epsilon_{i,j}$ for all $i, j \in I$ with $c_{ij} < 0$ so that $\epsilon_{i,j}\epsilon_{j,i} = -1$. With this data, Khovanov and Lauda [KL09, KL11] and Rouquier [Rou08] define the $\Bbbk$-algebra H_α with unit 1_α, called the *KLR algebra*, given by generators

$$\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^\alpha\} \cup \{x_1, \ldots, x_n\} \cup \{\tau_1, \ldots, \tau_{n-1}\}$$

subject only to the following relations:

- $x_r x_s = x_s x_r$;
- $1_{\boldsymbol{i}} 1_{\boldsymbol{j}} = \delta_{\boldsymbol{i},\boldsymbol{j}} 1_{\boldsymbol{i}}$ and $\sum_{\boldsymbol{i}\in I^\alpha} 1_{\boldsymbol{i}} = 1_\alpha$;
- $x_r 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}} x_r$ and $\tau_r 1_{\boldsymbol{i}} = 1_{s_r\cdot\boldsymbol{i}}\tau_r$;
- $(x_t\tau_r - \tau_r x_{s_r(t)})1_{\boldsymbol{i}} = \delta_{i_r,i_{r+1}}(\delta_{t,r+1} - \delta_{t,r})1_{\boldsymbol{i}}$;
- $\tau_r^2 1_{\boldsymbol{i}} = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ \varepsilon_{i_r,i_{r+1}}(x_r{}^{-c_{i_r,i_{r+1}}} - x_{r+1}{}^{-c_{i_{r+1},i_r}})1_{\boldsymbol{i}} & \text{if } c_{i_r,i_{r+1}} < 0, \\ 1_{\boldsymbol{i}} & \text{otherwise;} \end{cases}$
- $\tau_r\tau_s = \tau_s\tau_r$ if $|r - s| > 1$;
- $(\tau_{r+1}\tau_r\tau_{r+1} - \tau_r\tau_{r+1}\tau_r)1_{\boldsymbol{i}} = \begin{cases} \displaystyle\sum_{a+b=-1-c_{i_r,i_{r+1}}} \varepsilon_{i_r,i_{r+1}} x_r^a x_{r+2}^b 1_{\boldsymbol{i}} & \text{if } c_{i_r,i_{r+1}} < 0 \text{ and } i_r = i_{r+2}, \\ 0 & \text{otherwise.} \end{cases}$

The KLR algebra is graded with $\deg 1_{\boldsymbol{i}} = 0$, $\deg(x_r 1_{\boldsymbol{i}}) = 2d_{i_r}$ and $\deg(\tau_r 1_{\boldsymbol{i}}) = -\alpha_{i_r} \cdot \alpha_{i_{r+1}}$.

For each element $w \in S_n$, fix a reduced decomposition $w = s_{r_1} \cdots s_{r_l}$ and set $\tau_w = \tau_{r_1} \cdots \tau_{r_l} \in H_\alpha$ (this element depends in general on the choice of reduced decomposition).

Theorem 2.1 (Basis theorem [KL09, Theorem 2.5]). *The sets*

$$\{\tau_w x_1^{a_1} \cdots x_n^{a_n} 1_{\boldsymbol{i}}\} \quad \textit{and} \quad \{x_1^{a_1} \cdots x_n^{a_n} \tau_w 1_{\boldsymbol{i}}\}, \tag{2.2}$$

with w running over S_n, a_r running over $\mathbb{Z}_{\geqslant 0}$ and $\boldsymbol{i}$ running over I^α, are $\Bbbk$-bases for H_α.

It follows that H_α is Noetherian if $\Bbbk$ is, which we shall always assume from now on. It also follows that for any $1 \leqslant r \leqslant n$, the subalgebra $\Bbbk[x_r] \subseteq H_\alpha$ generated by x_r is isomorphic to the polynomial algebra $\Bbbk[x]$; this fact will be used often without further comment. Moreover, for each $\boldsymbol{i} \in I^\alpha$, the subalgebra $\mathcal{P}(\boldsymbol{i}) \subseteq 1_{\boldsymbol{i}} H_\alpha 1_{\boldsymbol{i}}$ generated by $\{x_r 1_{\boldsymbol{i}} \mid 1 \leqslant r \leqslant n\}$ is isomorphic to a polynomial algebra in n variables. By defining $\mathcal{P}_\alpha := \bigoplus_{\boldsymbol{i}\in I^\alpha} \mathcal{P}(\boldsymbol{i})$, we obtain a linear action of S_n on $\mathcal{P}_\alpha$ given by

$$w \cdot x_1^{a_1} \cdots x_n^{a_n} 1_{\boldsymbol{i}} = x_{w(1)}^{a_1} \cdots x_{w(n)}^{a_n} 1_{w\cdot\boldsymbol{i}}$$

for any $w \in S_n$, $\boldsymbol{i} \in I^\alpha$ and $a_1, \ldots, a_n \in \mathbb{Z}_{\geqslant 0}$. Setting $\Lambda(\alpha) := \mathcal{P}_\alpha^{S_n}$, we have the following result.

THEOREM 2.3 [KL09, Theorem 2.9]. *$\Lambda(\alpha)$ is the center of H_α.*

Let $\Bbbk$ be Noetherian. If H is a Noetherian graded $\Bbbk$-algebra, we denote by H-mod the category of finitely generated graded left H-modules. The morphisms in this category are all homogeneous degree-zero H-homomorphisms, which we denote by $\hom_H(-,-)$. For $V \in H$-mod, let $q^d V$ denote its grading shift by d, so if V_m is the degree-m component of V, then $(q^d V)_m = V_{m-d}$. More generally, for a Laurent polynomial $a = a(q) = \sum_d a_d q^d \in \mathbb{Z}[q, q^{-1}]$ with non-negative coefficients, we set $aV := \bigoplus_d (q^d V)^{\oplus a_d}$.

For $U, V \in H$-mod, we set $\operatorname{Hom}_H(U, V) := \bigoplus_{d \in \mathbb{Z}} \operatorname{Hom}_H(U, V)_d$, where

$$\operatorname{Hom}_H(U, V)_d := \hom_H(q^d U, V) = \hom_H(U, q^{-d} V).$$

We define $\operatorname{ext}^m_H(U, V)$ and $\operatorname{Ext}^m_H(U, V)$ similarly. Since U is finitely generated, $\operatorname{Hom}_H(U, V)$ can be identified in the obvious way with the set of all H-module homomorphisms ignoring the gradings. A similar result holds for $\operatorname{Ext}^m_H(U, V)$, since U has a resolution by finitely generated projective modules. We use $\cong$ to denote an isomorphism in H-mod and $\simeq$ an isomorphism up to a degree shift, i.e. $V \simeq W$ if and only if $V \cong q^n W$ for some $n \in \mathbb{Z}$.

Let q be a variable, and let $\mathbb{Z}((q))$ be the ring of Laurent series. The quantum integers $[n] = (q^n - q^{-n})/(q - q^{-1})$ and expressions like $1/(1 - q^2)$ are always interpreted as elements of $\mathbb{Z}((q))$.

From now on until the end of § 3, we assume that $\Bbbk$ is a field. A graded $\Bbbk$-vector space $V = \bigoplus_{m \in \mathbb{Z}} V_m$ is said to be *Laurentian* if the graded components V_m are finite dimensional for all $m \in \mathbb{Z}$ and $V_m = 0$ for $m \ll 0$. The *graded dimension* of a Laurentian vector space V is

$$\dim_q V := \sum_{m \in \mathbb{Z}} (\dim V_m) q^m \in \mathbb{Z}((q)).$$

We always work in the category H_α-mod. Note that H_α is Laurentian as a vector space; therefore so is any $V \in H_\alpha$-mod, and then so are all $1_{\boldsymbol{i}} V$ for $\boldsymbol{i} \in I^\alpha$. The *formal character* of $V \in H_\alpha$-mod is an element of $\bigoplus_{\boldsymbol{i} \in I^\alpha} \mathbb{Z}((q)) \cdot \boldsymbol{i}$ defined as

$$\operatorname{ch}_q V := \sum_{\boldsymbol{i} \in I^\alpha} (\dim_q 1_{\boldsymbol{i}} V) \cdot \boldsymbol{i}.$$

Note that $\operatorname{ch}_q (q^d V) = q^d \operatorname{ch}_q (V)$, where the first q^d means the degree shift. We refer to $1_{\boldsymbol{i}} V$ as the *$\boldsymbol{i}$-weight space* of V and to its vectors as *vectors of weight $\boldsymbol{i}$*.

There is an anti-automorphism $\iota : H_\alpha \to H_\alpha$ which fixes all the generators. Given $V \in H_\alpha$-mod, we let

$$V^{\circledast} := \operatorname{Hom}_{\Bbbk}(V, \Bbbk),$$

viewed as a left H_α-module via ι. Note that in general $V^{\circledast}$ is not finitely generated as an H_α-module, but we will apply $\circledast$ only to finite-dimensional modules. In that case, we have $\operatorname{ch}_q V^{\circledast} = \overline{\operatorname{ch}_q V}$, where the bar means the *bar-involution*, i.e. the automorphism of $\mathbb{Z}[q, q^{-1}]$ that swaps q and q^{-1} extended to $\bigoplus_{\boldsymbol{i} \in I^\alpha} \mathbb{Z}[q, q^{-1}] \cdot \boldsymbol{i}$.

Let $\beta_1, \ldots, \beta_m \in Q^+$ and $\alpha = \beta_1 + \cdots + \beta_m$. Consider the set of concatenations

$$I^{\beta_1, \ldots, \beta_m} := \{\boldsymbol{i}^1 \cdots \boldsymbol{i}^m \mid \boldsymbol{i}^1 \in I^{\beta_1}, \ldots, \boldsymbol{i}^m \in I^{\beta_m}\} \subseteq I^\alpha.$$

There is a natural (non-unital) algebra embedding $H_{\beta_1} \otimes \cdots \otimes H_{\beta_m} \to H_\alpha$, which sends the unit $1_{\beta_1} \otimes \cdots \otimes 1_{\beta_m}$ to the idempotent

$$1_{\beta_1, \ldots, \beta_m} := \sum_{\boldsymbol{i} \in I^{\beta_1, \ldots, \beta_m}} 1_{\boldsymbol{i}} \in H_\alpha. \tag{2.4}$$

We have an exact induction functor

$$\mathrm{Ind}^{\alpha}_{\beta_1,\dots,\beta_m} = H_\alpha 1_{\beta_1,\dots,\beta_m} \otimes_{H_{\beta_1}\otimes\cdots\otimes H_{\beta_m}} - \quad : \ (H_{\beta_1}\otimes\cdots\otimes H_{\beta_m})\text{-mod} \to H_\alpha\text{-mod}.$$

For $V_1 \in H_{\beta_1}$-mod$, \dots, V_m \in H_{\beta_m}$-mod, we denote by $V_1 \boxtimes\cdots\boxtimes V_m$ the vector space $V_1\otimes\cdots\otimes V_m$, considered naturally as an $(H_{\beta_1}\otimes\cdots\otimes H_{\beta_m})$-module, and set

$$V_1 \circ\cdots\circ V_m := \mathrm{Ind}^{\alpha}_{\beta_1,\dots,\beta_m} V_1 \boxtimes\cdots\boxtimes V_m.$$

2.2 Standard modules

The KLR algebras H_α are known to be *affine quasihereditary* in the sense of [Kle15b]; see [Kat14, BKM14, KL15]. Central to this theory is the notion of *standard modules*, whose definition depends on the choice of a certain partial order. We first fix a *convex order* on Φ^+, i.e. a total order such that whenever γ, β and $\gamma+\beta$ all belong to Φ^+, $\gamma \leqslant \beta$ implies $\gamma \leqslant \gamma+\beta \leqslant \beta$. By [Pap94], there is a one-to-one correspondence between convex orders on Φ^+ and reduced decompositions of the longest element in the corresponding Weyl group.

A *Kostant partition* of $\alpha \in Q^+$ is a tuple $\lambda = (\lambda_1, \dots, \lambda_r)$ of positive roots $\lambda_1 \geqslant \lambda_2 \geqslant\cdots\geqslant \lambda_r$ such that $\lambda_1+\cdots+\lambda_r = \alpha$. Let $\mathrm{KP}(\alpha)$ denote the set of all Kostant partitions of α, and for λ as above define $\lambda'_m = \lambda_{r-m+1}$. Now we have a *bilexicographical partial order* on $\mathrm{KP}(\alpha)$, also denoted by $\leqslant$, i.e. if $\lambda = (\lambda_1,\dots,\lambda_r), \mu = (\mu_1,\dots,\mu_s) \in \mathrm{KP}(\alpha)$ then $\lambda < \mu$ if and only if the following two conditions are satisfied:

- $\lambda_1 = \mu_1, \dots, \lambda_{l-1} = \mu_{l-1}$ and $\lambda_l < \mu_l$ for some l;
- $\lambda'_1 = \mu'_1, \dots, \lambda'_{m-1} = \mu'_{m-1}$ and $\lambda'_m > \mu'_m$ for some m.

To every $\lambda \in \mathrm{KP}(\alpha)$, McNamara [Mcn15] (cf. [KR11, Theorem 7.2]) associates an absolutely irreducible finite-dimensional $\circledast$-self-dual H_α-module $L(\lambda)$ so that $\{L(\lambda) \mid \lambda \in \mathrm{KP}(\alpha)\}$ is a complete irredundant set of irreducible H_α-modules, up to isomorphism and degree shift. Since $L(\lambda)$ is $\circledast$-self-dual, its formal character is bar-invariant. The key special case is where $\lambda = (\alpha)$ for $\alpha \in \Phi^+$, in which case $L(\lambda) = L(\alpha)$ is called a *cuspidal irreducible module*. For $m \in \mathbb{Z}_{>0}$, we write (α^m) for the Kostant partition $(\alpha,\dots,\alpha) \in \mathrm{KP}(m\alpha)$, where α appears m times. The cuspidal modules have the following nice property.

Lemma 2.5 [Mcn15, Lemma 3.4] (cf. [KR11, Lemma 6.6]). *For any $\alpha \in \Phi^+$ and $m \in \mathbb{Z}_{>0}$, we have $L(\alpha^m) \simeq L(\alpha)^{\circ m}$.*

If $\lambda = (\lambda_1,\dots,\lambda_r) \in \mathrm{KP}(\alpha)$, the *reduced standard module* is defined to be

$$\bar\Delta(\lambda) := q^{s(\lambda)} L(\lambda_1)\circ\cdots\circ L(\lambda_r) \tag{2.6}$$

for a specific degree shift $s(\lambda)$, whose description will not be important. Note that the Grothendieck group of finite-dimensional graded H_α-modules can be considered as a $\mathbb{Z}[q,q^{-1}]$-module with $q[V] = [qV]$. By [Mcn15, Theorem 3.1] (cf. [KR11, 7.2, 7.4]), the H_α-module $\bar\Delta(\lambda)$ has simple head $L(\lambda)$, and in the Grothendieck group we have

$$[\bar\Delta(\lambda)] = [L(\lambda)] + \sum_{\mu<\lambda} d_{\lambda,\mu}[L(\mu)] \tag{2.7}$$

for some coefficients $d_{\lambda,\mu} \in \mathbb{Z}[q,q^{-1}]$, called the *(graded) decomposition numbers*. The decomposition numbers depend on the characteristic of the ground field $\Bbbk$.

https://doi.org/10.1112/S0010437X16008204 Published online by Cambridge University Press

Let $P(\lambda)$ denote a projective cover of $L(\lambda)$ in H_α-mod. For $V \in H_\alpha$-mod we define the (graded) composition multiplicity

$$[V : L(\lambda)]_q := \dim_q \mathrm{Hom}(P(\lambda), V) \in \mathbb{Z}((q)).$$

The *standard module* $\Delta(\lambda)$ is defined as the largest quotient of $P(\lambda)$ all of whose composition factors are of the form $L(\mu)$ with $\mu \leqslant \lambda$; see [Kat14, Corollary 4.13], [BKM14, Corollary 3.16] and [Kle15b, (4.2)]. We note that while the irreducible modules $L(\lambda)$ are all finite dimensional, the standard modules $\Delta(\lambda)$ are always infinite dimensional. The standard modules have the usual nice properties.

Theorem 2.8 [BKM14, § 3]. *Let $\alpha \in Q^+$ and $\lambda, \mu \in \mathrm{KP}(\alpha)$. Then the following properties hold.*

(i) *$\Delta(\lambda)$ has simple head $L(\lambda)$, and $[\Delta(\lambda) : L(\mu)]_q \neq 0$ implies $\mu \leqslant \lambda$.*
(ii) *We have $\mathrm{Hom}_{H_\alpha}(\Delta(\lambda), \Delta(\mu)) = 0$ unless $\lambda \leqslant \mu$.*
(iii) *For $m \geqslant 1$, we have $\mathrm{Ext}^m_{H_\alpha}(\Delta(\lambda), \Delta(\mu)) = 0$ unless $\lambda < \mu$.*
(iv) *The module $P(\lambda)$ has a finite filtration $P(\lambda) = P_0 \supset P_1 \supset \cdots \supset P_N = 0$ such that $P_0/P_1 \cong \Delta(\lambda)$, and for $r = 1, 2, \ldots, N-1$ we have $P_r/P_{r+1} \simeq \Delta(\mu^{(r)})$ for some $\mu^{(r)} > \lambda$.*
(v) *Denoting the graded multiplicities of the factors in a Δ-filtration of $P(\lambda)$ by $(P(\lambda) : \Delta(\mu))_q$, we have $(P(\lambda) : \Delta(\mu))_q = d_{\mu,\lambda}(q)$.*

To construct the standard modules more explicitly, let us first assume that $\alpha \in \Phi^+$ and explain how to construct the *cuspidal standard module* $\Delta(\alpha)$. Put

$$q_\alpha := q^{\alpha\cdot\alpha/2}.$$

By [BKM14, Lemma 3.2], for each $n \in \mathbb{Z}_{>0}$ there exists a unique, up to isomorphism, indecomposable H_α-module $\Delta_n(\alpha)$ such that there are short exact sequences

$$0 \to q_\alpha^{2(n-1)} L(\alpha) \to \Delta_n(\alpha) \to \Delta_{n-1}(\alpha) \to 0,$$
$$0 \to q_\alpha^2 \Delta_{n-1}(\alpha) \to \Delta_n(\alpha) \to L(\alpha) \to 0,$$

where we are using the convention that $\Delta_0(\alpha) = 0$. This yields an inverse system

$$\cdots \to \Delta_2(\alpha) \to \Delta_1(\alpha) \to \Delta_0(\alpha),$$

and we have $\Delta(\alpha) \cong \varprojlim \Delta_n(\alpha)$; see [BKM14, Corollary 3.16].

Let $m \in \mathbb{Z}_{>0}$. An explicit endomorphism $e_m \in \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})^{\mathrm{op}}$ is defined in [BKM14, § 3.2], and then

$$\Delta(\alpha^m) \cong q_\alpha^{m(m-1)/2} \Delta(\alpha)^{\circ m} e_m. \tag{2.9}$$

Finally, for an arbitrary $\alpha \in Q^+$ and $\lambda \in \mathrm{KP}(\alpha)$, gather together the equal parts of λ to write $\lambda = (\lambda_1^{m_1}, \ldots, \lambda_s^{m_s})$, with $\lambda_1 > \cdots > \lambda_s$. Then, by [BKM14, (3.5)],

$$\Delta(\lambda) \cong \Delta(\lambda_1^{m_1}) \circ \cdots \circ \Delta(\lambda_s^{m_s}). \tag{2.10}$$

Thus, cuspidal standard modules are building blocks for arbitrary standard modules. We will need some of their additional properties. Let $\alpha \in \Phi^+$. If $\lambda \in \mathrm{KP}(\alpha)$ is minimal such that $\lambda > (\alpha)$, then by [BKM14, Lemma 2.6], $\lambda = (\beta, \gamma)$ for positive roots $\beta > \alpha > \gamma$. In this case, (β, γ) is called a *minimal pair* for α and we write $\mathrm{mp}(\alpha)$ for the set of all such pairs. The following result proved in [BKM14, §§ 3.1 and 4.3] describes some of the important properties of $\Delta(\alpha)$.

Theorem 2.11. *Let $\alpha \in \Phi^+$. Then the following properties hold.*

(i) $[\Delta(\alpha) : L(\alpha)]_q = 1/(1-q_\alpha^2)$ *and* $[\Delta(\alpha) : L(\lambda)]_q = 0$ *for* $\lambda \neq (\alpha)$.

(ii) *Let* $\mathbf{C}_\alpha$ *be the category of all modules in* H_α-mod *all of whose composition factors are* $\simeq L(\alpha)$. *Any* $V \in \mathbf{C}_\alpha$ *is a finite direct sum of copies of the indecomposable modules* $\simeq \Delta_n(\alpha)$ *and* $\simeq \Delta(\alpha)$. *Moreover,* $\Delta(\alpha)$ *is a projective cover of* $L(\alpha)$ *in* $\mathbf{C}_\alpha$. *Furthermore,* $\mathrm{Ext}^m_{H_\alpha}(\Delta(\alpha), V) = 0$ *for* $m \geqslant 1$ *and* $V \in \mathbf{C}_\alpha$.

(iii) $\mathrm{End}_{H_\alpha}(\Delta(\alpha)) \cong \Bbbk[x]$ *for* x *in degree* $2d_\alpha$.

(iv) *There is a short exact sequence* $0 \to q_\alpha^2\Delta(\alpha) \to \Delta(\alpha) \to L(\alpha) \to 0$.

(v) *For* $(\beta,\gamma) \in \mathrm{mp}(\alpha)$ *there is a short exact sequence*

$$0 \to q^{-\beta\cdot\gamma}\Delta(\beta)\circ\Delta(\gamma) \xrightarrow{\varphi} \Delta(\gamma)\circ\Delta(\beta) \to [p_{\beta,\gamma}+1]\Delta(\alpha) \to 0,$$

where $p_{\beta,\gamma}$ *is the largest integer* p *such that* $\beta - p\gamma$ *is a root.*

Corollary 2.12. *Let $\alpha \in \Phi^+$. The dimensions of the graded components $\Delta(\alpha)_d$ are bounded above by some $N > 0$ independent of d, and are zero for $d \ll 0$.*

Proof. By Theorem 2.11(i), we have $\dim_q \Delta(\alpha) = (\dim_q L(\alpha))/(1-q_\alpha^2)$, which implies the result since $L(\alpha)$ is finite dimensional. □

2.3 Endomorphisms of standard modules

We shall denote by x_α the degree-$2d_\alpha$ endomorphism of $\Delta(\alpha)$ which corresponds to x under the algebra isomorphism $\mathrm{End}_{H_\alpha}(\Delta(\alpha)) \cong \Bbbk[x]$ in Theorem 2.11(iii).

Lemma 2.13. *Let $\alpha \in \Phi^+$. Then every non-zero H_α-endomorphism of $\Delta(\alpha)$ is injective, and every submodule of $\Delta(\alpha)$ is equal to $x_\alpha^s(\Delta(\alpha)) \cong q_\alpha^{2s}\Delta(\alpha)$ for some $s \in \mathbb{Z}_{\geqslant 0}$.*

Proof. It follows from the construction of x_α in [BKM14, Theorem 3.3] that x_α is injective and $x_\alpha(\Delta(\alpha)) \cong q_\alpha^2\Delta(\alpha)$. This in particular implies the first statement.

Let $V \subseteq \Delta(\alpha)$ be a submodule and $f : V \to \Delta(\alpha)$ the natural inclusion. First, assume that V is indecomposable. By Theorem 2.11(ii), up to degree shift, V is isomorphic to $\Delta(\alpha)$ or $\Delta_n(\alpha)$ for some $n \geqslant 1$. If $V \simeq \Delta_n(\alpha)$ then $\Delta(\alpha)/V$ is infinite dimensional and has simple head, so by Theorem 2.11(ii) again, $\Delta(\alpha)/V \simeq \Delta(\alpha)$. Then the short exact sequence

$$0 \to V \to \Delta(\alpha) \to \Delta(\alpha)/V \to 0$$

splits by projectivity in Theorem 2.11(ii), contradicting indecomposability of $\Delta(\alpha)$. If instead $V \simeq \Delta(\alpha)$, consider the composition

$$\Delta(\alpha) \xrightarrow{\sim} V \xrightarrow{f} \Delta(\alpha).$$

This produces a graded endomorphism of $\Delta(\alpha)$, so that $V = x_\alpha^s(\Delta(\alpha))$ for some $s \geqslant 0$. Since there are inclusions $\Delta(\alpha) \supset x_\alpha\Delta(\alpha) \supset x_\alpha^2\Delta(\alpha) \supset \cdots$, the general case follows from the case where V is indecomposable. □

Again let $\alpha \in \Phi^+$. We next consider the standard modules of the form $\Delta(\alpha^m)$. By functoriality, the endomorphism $\mathrm{id}^{\otimes(r-1)} \otimes x_\alpha \otimes \mathrm{id}^{\otimes(m-r)}$ of the $H_\alpha^{\otimes m}$-module $\Delta(\alpha)^{\boxtimes m}$ induces an endomorphism X_r of the $H_{m\alpha}$-module $\Delta(\alpha)^{\circ m}$. The endomorphisms

$$X_1, \ldots, X_m \in \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})$$

https://doi.org/10.1112/S0010437X16008204 Published online by Cambridge University Press

commute. Moreover, in [BKM14, § 3.2], some additional endomorphisms $\partial_1, \dots, \partial_{m-1} \in \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})$ are constructed, and it is proved in [BKM14, Lemmas 3.7–3.9] that the algebra $\mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})^{\mathrm{op}}$ is isomorphic to the nil-Hecke algebra NH_m, with $\partial_1, \dots, \partial_{m-1}$ and (appropriately scaled) $X_1, \dots, X_m$ corresponding to the standard generators of NH_m. The element e_m used in (2.9) is an explicit idempotent in NH_m. Consider the algebra of symmetric functions

$$\Lambda_{\alpha,m} := \Bbbk[X_1, \dots, X_m]^{S_m} = Z(NH_m),$$

with the variables X_r in degree $2d_\alpha$. Note that $\dim_q \Lambda_{\alpha,m} = 1/\prod_{r=1}^m (1 - q_\alpha^{2r})$. It is known (see, e.g., [KLM13, Theorem 4.4(iii)]) that

$$e_m NH_m e_m = e_m \Lambda_{\alpha,m} \cong \Lambda_{\alpha,m}. \tag{2.14}$$

THEOREM 2.15. *Let $\alpha \in \Phi^+$ and $m \in \mathbb{Z}_{>0}$. Then the following properties hold.*

(i) *For any $\lambda \in \mathrm{KP}(m\alpha)$, we have $[\Delta(\alpha^m) : L(\lambda)]_q = \delta_{\lambda,(\alpha^m)}/\prod_{r=1}^m (1 - q_\alpha^{2r})$.*
(ii) *The module $\Delta(\alpha^m)$ is a projective cover of $L(\alpha^m)$ in the category of all modules in H_α-mod all of whose composition factors are $\simeq L(\alpha^m)$.*
(iii) $\mathrm{End}_{H_\alpha}(\Delta(\alpha^m)) \cong \Lambda_{\alpha,m}$.
(iv) *Every submodule of $\Delta(\alpha^m)$ is isomorphic to $q^d\Delta(\alpha^m)$ for some $d \in \mathbb{Z}_{\geqslant 0}$, and every non-zero $H_{m\alpha}$-endomorphism of $\Delta(\alpha^m)$ is injective.*

Proof. Assertion (i) is [BKM14, Lemma 3.10], and (ii) follows from [Kle15b, Lemma 4.11], since (α^m) is minimal in $\mathrm{KP}(\alpha)$ by convexity. By (i) and (ii), we have that $\dim_q \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha^m)) = 1/\prod_{r=1}^m (1 - q_\alpha^{2r})$.

(iii) We have that $NH_m = \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})^{\mathrm{op}}$ acts naturally on $\Delta(\alpha)^{\circ m}$ on the right, and so $\Lambda_{\alpha,m} = Z(NH_m)$ acts naturally on $\Delta(\alpha^m) = \Delta(\alpha)^{\circ m} e_m$. This defines an embedding $\Lambda_{\alpha,m} \to \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha^m))$. This embedding must be an isomorphism because of the dimensions.

(iv) In view of Lemma 2.13, every non-zero

$$f \in \Bbbk[X_1, \dots, X_m] \subseteq NH_m = \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})^{\mathrm{op}}$$

acts as an injective linear operator on $\Delta(\alpha)^{\circ m}$. The result now follows from (2.14) and (ii). □

Finally, we consider a general case. Let $\alpha \in Q^+$ and $\lambda = (\lambda_1^{m_1}, \dots, \lambda_s^{m_s}) \in \mathrm{KP}(\alpha)$ with $\lambda_1 > \dots > \lambda_s$. By functoriality of induction, we have a natural embedding

$$\Lambda_{\lambda_1,m_1} \otimes \dots \otimes \Lambda_{\lambda_s,m_s} \to \mathrm{End}_{H_\alpha}(\Delta(\lambda)), \quad f_1 \otimes \dots \otimes f_s \mapsto f_1 \circ \dots \circ f_s. \tag{2.16}$$

THEOREM 2.17. *Let $\alpha \in Q^+$ and $\lambda = (\lambda_1^{m_1}, \dots, \lambda_s^{m_s}) \in \mathrm{KP}(\alpha)$ with $\lambda_1 > \dots > \lambda_s$. Then*

$$\mathrm{End}_{H_\alpha}(\Delta(\lambda)) \cong \Lambda_{\lambda_1,m_1} \otimes \dots \otimes \Lambda_{\lambda_s,m_s}$$

via (2.16), and every non-zero H_α-endomorphism of $\Delta(\lambda)$ is injective.

Proof. It is easy to see from Theorem 2.15(iv) that every non-zero endomorphism in the image of the embedding (2.16) is injective. To see that there are no other endomorphisms, we first use adjointness of End and Res to show that $\mathrm{End}_{H_\alpha}(\Delta(\lambda))$ is isomorphic to

$$\mathrm{Hom}_{H_{m_1\lambda_1} \otimes \dots \otimes H_{m_s\lambda_s}}(\Delta(\lambda_1^{m_1}) \boxtimes \dots \boxtimes \Delta(\lambda_s^{m_s}), \mathrm{Res}^\alpha_{m_1\lambda_1,\dots,m_s\lambda_s} \Delta(\lambda)),$$

and then note that by the Mackey theorem, as in [Mcn15, Lemma 3.3] for instance, we have $\mathrm{Res}^\alpha_{m_1\lambda_1,\dots,m_s\lambda_s} \Delta(\lambda) \cong \Delta(\lambda_1^{m_1}) \boxtimes \dots \boxtimes \Delta(\lambda_s^{m_s})$. □

https://doi.org/10.1112/S0010437X16008204 Published online by Cambridge University Press

3. Proof of Theorem A

We give a proof of Theorem A based on the recent work of Kashiwara and Park [KP15]. Our original proof was different and relied on some unpleasant computations for non-simply laced types. For simply laced types, however, our original proof is very simple and elementary, so we give it later in this section, too.

3.1 Proof of Theorem A modulo a hypothesis

The following hypothesis concerns a fundamental property of cuspidal standard modules and is probably true beyond finite Lie types.

Hypothesis 3.1. Let α be a positive root of height n and let $1 \leqslant r \leqslant n$. Then upon restriction to the subalgebra $\Bbbk[x_r] \subseteq H_\alpha$, the module $\Delta(\alpha)$ is free of finite rank.

The goal of this subsection is to prove Theorem A assuming the hypothesis. In § 3.2 the hypothesis will be proved using results of Kashiwara and Park, while in § 3.3 we will give a more elementary proof for simply laced types.

Lemma 3.2. *Hypothesis 3.1 is equivalent to the property that $x_1, \ldots, x_n$ act by injective linear operators on $\Delta(\alpha)$.*

Proof. The forward direction is clear. For the converse, assume that x_r acts injectively on $\Delta(\alpha)$. We construct a finite basis for $1_{\boldsymbol{i}}\Delta(\alpha)$ as a $\Bbbk[x_r]$-module for every $\boldsymbol{i} \in I^\alpha$. Let $m := \deg(x_r 1_{\boldsymbol{i}})$. For every $a = 0, 1, \ldots, m-1$, let d_a be a minimal integer with $d_a \equiv a \pmod m$ and $1_{\boldsymbol{i}}\Delta(\alpha)_{d_a} \neq 0$. Pick a linear basis of $\bigoplus_{a=0}^{m-1} 1_{\boldsymbol{i}}\Delta(\alpha)_{d_a}$ and note that the $\Bbbk[x_r]$-module generated by the elements of this basis is free. Factor out this $\Bbbk[x_r]$-submodule, and repeat. The process will stop after finitely many steps, thanks to Corollary 2.12. □

While Hypothesis 3.1 claims that every $\Bbbk[x_r]$ acts freely on $\Delta(\alpha)$, no $\Bbbk[x_r, x_s]$ does.

Lemma 3.3. *Let $\alpha \in \Phi^+$ be a root of height $n > 1$. Then, for every vector $v \in \Delta(\alpha)$ and distinct $r, s \in \{1, \ldots, n\}$, there is a non-zero polynomial $f \in \Bbbk[x, y]$ such that $f(x_r, x_s)v = 0$.*

Proof. We may assume that v is a homogenous weight vector. By Corollary 2.12, the dimensions of the graded components of $\Delta(\alpha)$ are uniformly bounded. The result then follows, as the number of linearly independent degree-d monomials in x and y grows without bound. □

One can say more about the polynomial f in the lemma; see, for example, Proposition 3.14.

Now let $\alpha \in Q^+$ be arbitrary of height n, and let $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_l) \in \mathrm{KP}(\alpha)$. Setting

$$S_\lambda := S_{\mathrm{ht}(\lambda_1)} \times \cdots \times S_{\mathrm{ht}(\lambda_l)} \subset S_n,$$

integers $r, s \in \{1, \ldots, n\}$ are said to be *λ-equivalent*, written $r \sim_\lambda s$, if they belong to the same orbit of the action of S_λ on $\{1, \ldots, n\}$. Finally, recalling the idempotents (2.4), we set

$$1_\lambda := 1_{\lambda_1, \ldots, \lambda_l}.$$

Lemma 3.4. *Let $\alpha \in Q^+$ and $n = \mathrm{ht}(\alpha)$, and let $\lambda \not\geqslant \mu$ be elements of $\mathrm{KP}(\alpha)$. If $w \in S_n$ satisfies $1_\lambda \tau_w 1_\mu \neq 0$, then there exists some $1 \leqslant r < n$ such that $r \sim_\lambda r+1$ but $w^{-1}(r) \not\sim_\mu w^{-1}(r+1)$.*

https://doi.org/10.1112/S0010437X16008204 Published online by Cambridge University Press

Proof. Write $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_l)$ and $\mu = (\mu_1 \geqslant \cdots \geqslant \mu_m)$. The assumption $1_\lambda \tau_w 1_\mu \neq 0$ implies that $\boldsymbol{i}^\lambda = w \cdot \boldsymbol{i}^\mu$ for some $\boldsymbol{i}^\lambda \in I^{\lambda_1,\dots,\lambda_l}$ and $\boldsymbol{i}^\mu \in I^{\mu_1,\dots,\mu_m}$. Write $\boldsymbol{i}^\lambda := \boldsymbol{i}_1^\lambda \cdots \boldsymbol{i}_l^\lambda$ with $\boldsymbol{i}_a^\lambda \in I^{\lambda_a}$ for all a, and $\boldsymbol{i}^\mu := \boldsymbol{i}_1^\mu \cdots \boldsymbol{i}_m^\mu$ with $\boldsymbol{i}_b^\mu \in I^{\mu_b}$ for all b. Assume for a contradiction that for every $1 \leqslant r < n$ we have that $r \sim_\lambda r+1$ implies $w^{-1}(r) \sim_\mu w^{-1}(r+1)$. Then there is a partition $\{1,\dots,l\} = \bigsqcup_{b=1}^m A_b$ such that $\mu_b = \sum_{a \in A_b} \lambda_a$ for all $b = 1,\dots,m$. By convexity (cf. [BKM14, Lemma 2.4]), we have

$$\min\{\lambda_a \mid a \in A_b\} \leqslant \mu_b \leqslant \max\{\lambda_a \mid a \in A_b\}.$$

This implies $\lambda \geqslant \mu$. $\square$

THEOREM 3.5. *Let $\alpha \in Q^+$ and $\lambda, \mu \in \mathrm{KP}(\alpha)$. If $\lambda \neq \mu$, then*

$$\mathrm{Hom}_{H_\alpha}(\Delta(\lambda), \Delta(\mu)) = 0.$$

Proof. Let $n = \mathrm{ht}(\alpha)$ and write $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_l)$ and $\mu = (\mu_1 \geqslant \cdots \geqslant \mu_m)$. It suffices to prove that

$$\mathrm{Hom}_{H_\alpha}(\Delta(\lambda_1) \circ \cdots \circ \Delta(\lambda_l), \Delta(\mu_1) \circ \cdots \circ \Delta(\mu_m)) = 0.$$

Suppose not, and let φ be a non-zero homomorphism. By Theorem 2.8(ii), we may assume that $\lambda < \mu$. Using Lemma 3.3, pick a generator $v \in \Delta(\lambda_1) \circ \cdots \circ \Delta(\lambda_l)$ such that $v = 1_\lambda v$ and, for any $r \sim_\lambda r+1$, there is a non-zero polynomial $f \in \Bbbk[x,y]$ with $f(x_r, x_{r+1})v = 0$. Then $f(x_r, x_{r+1})\varphi(v) = 0$ as well.

Denote by S^μ the set of shortest-length coset representatives for S_n/S_μ. Then we can write $\varphi(v) = \sum_{w \in S^\mu} \tau_w \otimes v_w$ for some $v_w \in \Delta(\mu_1) \otimes \cdots \otimes \Delta(\mu_m)$. Since $\varphi(v) = 1_\lambda \varphi(v)$ and $1_\mu v_w = v_w$, we have that $1_\lambda \tau_w 1_\mu \neq 0$ whenever $v_w \neq 0$. In particular, if $u \in S^\mu$ is an element of maximal length such that $v_u \neq 0$, then by Lemma 3.4 we have $r \sim_\lambda r+1$ and $u^{-1}(r) \not\sim_\mu u^{-1}(r+1)$ for some $1 \leqslant r < n$.

Now we have

$$\begin{aligned} f(x_r, x_{r+1})\varphi(v) &= f(x_r, x_{r+1}) \sum_{w \in S^\mu} \tau_w \otimes v_w \\ &= f(x_r, x_{r+1})\tau_u \otimes v_u + \sum_{w \neq u} f(x_r, x_{r+1})\tau_w \otimes v_w \\ &= \tau_u \otimes f(x_{u^{-1}(r)}, x_{u^{-1}(r+1)})v_u + (*), \end{aligned}$$

where $(*)$ is a sum of elements of the form $\tau_w \otimes v'_w$ with $v'_w \in \Delta(\mu_1) \otimes \cdots \otimes \Delta(\mu_m)$ and $w \in S^\mu \backslash \{u\}$. The last equality holds because in H_α, for all $1 \leqslant t \leqslant n$ and $w \in S_n$ we have that $x_t \tau_w = \tau_w x_{w^{-1}(t)} + (**)$, where $(**)$ is a linear combination of elements of the form τ_y with $y \in S_n$ being Bruhat smaller than w.

Since $u^{-1}(r) \not\sim_\mu u^{-1}(r+1)$, there are distinct integers $a, b \in \{1,\dots,m\}$ and integers $1 \leqslant c \leqslant \mathrm{ht}(\mu_a)$ and $1 \leqslant d \leqslant \mathrm{ht}(\mu_b)$ such that for any pure tensor $v = v^1 \otimes \cdots \otimes v^m \in \Delta(\mu_1) \otimes \cdots \otimes \Delta(\mu_m)$ and any $s, t \in \mathbb{Z}_{\geqslant 0}$, we have

$$x_{u^{-1}(r)}^s x_{u^{-1}(r+1)}^t v = v^1 \otimes \cdots \otimes x_c^s v^a \otimes \cdots \otimes x_d^t v^b \otimes \cdots \otimes v^m.$$

By Hypothesis 3.1, $f(x_{u^{-1}(r)}, x_{u^{-1}(r+1)})v_u \neq 0$. Hence $f(x_r, x_{r+1})\varphi(v) \neq 0$, which gives a contradiction. $\square$

https://doi.org/10.1112/S0010437X16008204 Published online by Cambridge University Press

3.2 Proof of Hypothesis 3.1 using the Kashiwara–Park lemma

We begin with a key lemma which follows immediately from the results of [KP15].

Lemma 3.6. *Let $\alpha \in \Phi^+$, $n = \mathrm{ht}(\alpha)$ and $i \in I$. Define*

$$\mathfrak{p}_{i,\alpha} := \sum_{\boldsymbol{i} \in I^\alpha} \Big(\prod_{r \in [1,n], i_r = i} x_r \Big) 1_{\boldsymbol{i}}.$$

Then $\mathfrak{p}_{i,\alpha}\Delta(\alpha) \neq 0$.

Proof. This follows from [KP15, Definition 2.2(b)] and [KP15, Proposition 3.5]. □

Theorem 3.7. *Let $\alpha \in \Phi^+$ have height n. Then $x_r^m v \neq 0$ for all $1 \leqslant r \leqslant n$, $m \in \mathbb{Z}_{\geqslant 0}$ and non-zero $v \in \Delta(\alpha)$. In particular, Hypothesis 3.1 holds.*

Proof. The 'in particular' statement follows from Lemma 3.2.

We may assume that v is a weight vector of some weight $\boldsymbol{i}$. Let $i = i_r$. The element $\mathfrak{p}_{i,\alpha}$ defined in Lemma 3.6 is central by Theorem 2.3. By Lemma 3.6 and Theorem 2.17, the multiplication with $\mathfrak{p}_{i,\alpha}$ on $\Delta(\alpha)$ is injective, so multiplication with $\mathfrak{p}_{i,\alpha}^m$ is also injective. But $\mathfrak{p}_{i,\alpha}$ involves $x_r 1_{\boldsymbol{i}}$, so $0 \neq \mathfrak{p}_{i,\alpha}^m v = h x_r^m v$ for some $h \in H_\alpha$, and the theorem follows. □

3.3 Elementary proof of Hypothesis 3.1 for simply laced types

Throughout this subsection, we assume that the root system Φ is of (finite) ADE type. Let $\alpha = a_1\alpha_1 + \cdots + a_l\alpha_l \in Q^+$ and $n = \mathrm{ht}(\alpha) = a_1 + \cdots + a_l$. Pick a permutation $(i_1, \ldots, i_l)$ of $(1, \ldots, l)$ with $a_{i_1} > 0$, and define $\boldsymbol{i} := i_1^{a_{i_1}} \cdots i_l^{a_{i_l}} \in I^\alpha$. Then the stabilizer of $\boldsymbol{i}$ in S_n is the standard parabolic subgroup

$$S_{\boldsymbol{i}} := S_{a_{i_1}} \times \cdots \times S_{a_{i_l}}.$$

Let $S^{\boldsymbol{i}}$ be a set of coset representatives for $S_n/S_{\boldsymbol{i}}$. Then by Theorem 2.3, the element

$$z = z_{\boldsymbol{i}} := \sum_{w \in S^{\boldsymbol{i}}} (x_{w(1)} + \cdots + x_{w(a_{i_1})}) 1_{w \cdot \boldsymbol{i}} \tag{3.8}$$

is central of degree 2 in H_α. For any $1 \leqslant r \leqslant n$, note that

$$a_{i_1} x_r = z - \sum_{w \in S^{\boldsymbol{i}}} ((x_{w(1)} - x_r) + \cdots + (x_{w(a_{i_1})} - x_r)) 1_{w \cdot \boldsymbol{i}}. \tag{3.9}$$

Let H'_α be the subalgebra of H_α generated by

$$\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^\alpha\} \cup \{\tau_r \mid 1 \leqslant r < n\} \cup \{x_r - x_{r+1} \mid 1 \leqslant r < n\}.$$

For the reader's convenience, we reprove a result from [BK12, Lemma 3.1].

Lemma 3.10. *Let α, $\boldsymbol{i}$ and z be as above. Then:*

(i) $\{(x_1 - x_2)^{m_1} \cdots (x_{n-1} - x_n)^{m_{n-1}} \tau_w 1_{\boldsymbol{i}} \mid m_r \in \mathbb{Z}_{\geqslant 0}, w \in S_n, \boldsymbol{i} \in I^\alpha\}$ *is a basis for* H'_α;

(ii) *if $a_{i_1} \cdot 1_{\Bbbk} \neq 0$ in $\Bbbk$, then there is an algebra isomorphism*

$$H_\alpha \cong H'_\alpha \otimes \Bbbk[z]. \tag{3.11}$$

https://doi.org/10.1112/S0010437X16008204 Published online by Cambridge University Press

Proof. In view of the basis (2.2), assertion (i) follows upon checking that the span of the given monomials is closed under multiplication, which follows from the defining relations. For (ii), note by using (3.9) that the natural multiplication map $\Bbbk[z] \otimes H'_\alpha \to H_\alpha$ is surjective. It remains to observe that the two algebras have the same graded dimension. $\square$

Now let α be a positive root. Then one can always find an index i_1 with $a_{i_1} \cdot 1_\Bbbk \neq 0$, so in this case we always have (3.11) for an appropriate choice of $\boldsymbol{i}$. We always assume that this choice has been made. Following [BK12], we can now present another useful description of the cuspidal standard module $\Delta(\alpha)$. Denote by $L'(\alpha)$ the restriction of the cuspidal irreducible module $L(\alpha)$ from H_α to H'_α.

LEMMA 3.12. *Let $\alpha \in \Phi^+$. Then:*

(i) *$L'(\alpha)$ is an irreducible H'_α-module;*
(ii) *$\Delta(\alpha) \cong H_\alpha \otimes_{H'_\alpha} L'(\alpha)$;*
(iii) *the element z acts on $\Delta(\alpha)$ freely.*

Proof. Note that z acts as zero on $L(\alpha)$, which implies (i) in view of (3.11). Moreover, it is now easy to see that $H_\alpha \otimes_{H'_\alpha} L'(\alpha)$ has a filtration with the subfactors isomorphic to $q^{2d}L(\alpha)$ for $d = 0, 1, \ldots$. Furthermore, by Frobenius reciprocity and (i), the module $H_\alpha \otimes_{H'_\alpha} L'(\alpha)$ has simple head $L(\alpha)$. Now (ii) follows from Theorem 2.11(ii). Finally, (iii) follows from (ii) and (3.11). $\square$

Using the description of $\Delta(\alpha)$ from Lemma 3.12(ii), we can now establish Hypothesis 3.1.

THEOREM 3.13. *Let $\alpha \in \Phi^+$ and let $\{v_1, \ldots, v_N\}$ be a $\Bbbk$-basis of $L'(\alpha)$. Then the $\Bbbk[x_r]$-module $\Delta(\alpha)$ is free with basis $\{1 \otimes v_1, \ldots, 1 \otimes v_N\}$. In particular, Hypothesis 3.1 holds for simply laced types.*

Proof. By (3.9), we can write $x_r = (1/a_{i_1})z + (*)$, where $(*)$ is an element of H'_α. For each $1 \leqslant m \leqslant N$, we have

$$x_r^b(1 \otimes v_m) = \left(\frac{1}{a_{i_1}}\right)^b z^b \otimes v_m + (**),$$

where $(**)$ is a linear combination of terms of the form $z^c \otimes v_t$ with $c < b$. So $\{1 \otimes v_1, \ldots, 1 \otimes v_N\}$ is a basis of the free $\Bbbk[x_r]$-module $\Delta(\alpha)$. $\square$

The following strengthening of Lemma 3.3 is not needed for the proof of Theorem A, but we include it for completeness.

PROPOSITION 3.14. *Let $\alpha \in \Phi^+$ and $n = \mathrm{ht}(\alpha)$. For any $1 \leqslant r, s \leqslant n$, there is $d \in \mathbb{Z}_{>0}$ such that $(x_r - x_s)^d$ annihilates $\Delta(\alpha)$.*

Proof. Pick d such that $(x_r - x_s)^d$ annihilates $L(\alpha)$. Since $\Delta(\alpha) = H_\alpha \otimes_{H'_\alpha} L'(\alpha)$ is spanned by vectors of the form $z^m \otimes v'$ with $m \in \mathbb{Z}_{\geqslant 0}$ and $v' \in L'(\alpha)$, it suffices to note that $(x_r - x_s)^d (z^m \otimes v') = z^m \otimes (x_r - x_s)^d v' = 0$. $\square$

https://doi.org/10.1112/S0010437X16008204 Published online by Cambridge University Press

4. Reduction modulo p

Let p be a fixed prime number, and let F be the prime field of characteristic p. We will use the p-modular system (F, R, K) with $R = \mathbb{Z}_p$ and $K = \mathbb{Q}_p$. Note that $R/pR = F$.

From now on, we will work with different ground rings, so our notation needs to become more elaborate. Recall that the KLR algebra H_α is defined over an arbitrary commutative unital ring $\Bbbk$, and to emphasize which $\Bbbk$ we are working with, we will use the notation $H_{\alpha,\Bbbk}$ or $H_{\alpha;\Bbbk}$. In all our notation we will now use the corresponding index.

For example, let $\Bbbk$ be a field. We now denote the irreducible cuspidal modules over $H_{\alpha,\Bbbk}$ by $L(\alpha)_\Bbbk$. We now write $\dim^\Bbbk V$ for the dimension of a $\Bbbk$-vector space V, and $\dim_q^\Bbbk V$ for the graded dimension of a graded $\Bbbk$-vector space V.

If V is a finitely generated R-module, we write

$$\mathrm{d}^R V := \dim^{R/pR}(V/pV),$$

which, by Nakayama's lemma, equals the number of generators in any minimal generating set of V. If V is a graded R-module with finitely generated graded components V_m such that $V_m = 0$ for $m \ll 0$, we set

$$\mathrm{d}_q^R V := \sum_{m\in\mathbb{Z}} (\mathrm{d}^R V_m) q^m \in \mathbb{Z}((q)).$$

Let $\Bbbk \in \{F, R, K\}$, and let B be a Noetherian connected positively graded $\Bbbk$-algebra, so that $B/B_{>0} \cong \Bbbk$. If V is a finitely generated graded B-module, we define

$$\mathrm{d}_q^B V := \mathrm{d}_q^{B/B_{>0}}(V/B_{>0}V) \in \mathbb{Z}[q, q^{-1}].$$

By Nakayama's lemma, if $\{v_1, \dots, v_r\}$ is a minimal set of homogeneous generators of the B-module V, then $\mathrm{d}_q^B V = q^{\deg(v_1)} + \dots + q^{\deg(v_r)}$.

4.1 Changing scalars

In this subsection we develop a usual formalism of modular representation theory for KLR algebras. There will be nothing surprising here, but we need to exercise care since we work with infinite-dimensional algebras and often with infinite-dimensional modules.

Recall from § 2 that for a left Noetherian graded algebra H, we denote by H-mod the category of finitely generated graded H-modules, for which we have the groups $\mathrm{ext}^i_H(V, W)$ and $\mathrm{Ext}^i_H(V, W)$. To deal with change of scalars in Ext groups, we will use the following version of the universal coefficient theorem.

Theorem 4.1 (Universal coefficient theorem). *Let V_R and W_R be modules in $H_{\alpha,R}$-mod, free as R-modules, and let $\Bbbk$ be an R-algebra. Then for every $j \in \mathbb{Z}_{\geqslant 0}$ there is an exact sequence of (graded) $\Bbbk$-modules*

$$\begin{aligned} 0 \to \mathrm{Ext}^j_{H_{\alpha,R}}(V_R, W_R) \otimes_R \Bbbk \to \mathrm{Ext}^j_{H_{\alpha,\Bbbk}}(V_R \otimes_R \Bbbk, W_R \otimes_R \Bbbk) \\ \to \mathrm{Tor}_1^R(\mathrm{Ext}^{j+1}_{H_{\alpha,R}}(V_R, W_R), \Bbbk) \to 0. \end{aligned}$$

In particular,

$$\mathrm{Ext}^j_{H_{\alpha,R}}(V_R, W_R) \otimes_R K \cong \mathrm{Ext}^j_{H_{\alpha,K}}(V_R \otimes_R K, W_R \otimes_R K).$$

Proof. The standard proof for the ungraded modules works in our setting. First, apply the functor $\mathrm{Hom}_{H_{\alpha,R}}(-, W_R)$ to a free resolution of V_R to get a complex $C_\bullet$ of free (graded) R-modules with finitely many generators in every graded degree. Then follow the proof of [Rot79, Theorem 8.22]. The second statement follows from the first since K is a flat R-module. □

We need another standard result, whose proof is omitted.

Lemma 4.2. *Let* $\Bbbk = K$ *or* F, *let* $V_R, W_R \in H_{\alpha,R}$-mod *be free as* R-*modules, and let*

$$0 \to W_R \xrightarrow{\iota} E_R \xrightarrow{\pi} V_R \to 0$$

be the extension corresponding to a class $\xi \in \mathrm{Ext}^1_{H_{\alpha,R}}(V_R, W_R)$. *Identifying* $\mathrm{Ext}^1_{H_{\alpha,R}}(V_R, W_R) \otimes_R \Bbbk$ *with a subgroup of* $\mathrm{Ext}^1_{H_{\alpha,\Bbbk}}(V_R \otimes_R \Bbbk, W_R \otimes_R \Bbbk)$, *we have that*

$$0 \to W_R \otimes_R \Bbbk \xrightarrow{\iota \otimes \mathrm{id}_\Bbbk} E_R \otimes_R \Bbbk \xrightarrow{\pi \otimes \mathrm{id}_\Bbbk} V_R \otimes_R \Bbbk \to 0$$

is the extension corresponding to a class $\xi \otimes 1_\Bbbk \in \mathrm{Ext}^1_{H_{\alpha,R}}(V_R, W_R) \otimes_R \Bbbk$.

Let $\Bbbk = K$ or F, and let $V_\Bbbk \in H_{\alpha,\Bbbk}$-mod. We say that $V_R \in H_{\alpha,R}$-mod is an *R-form of* $V_\Bbbk$ if every graded component of V_R is free of finite rank as an R-module and, upon identifying $H_{\alpha,R} \otimes_R \Bbbk$ with $H_{\alpha,\Bbbk}$, we have $V_R \otimes_R \Bbbk \cong V_\Bbbk$ as $H_{\alpha,\Bbbk}$-modules. If $\Bbbk = K$, by a *full lattice* in V_K we mean a (graded) R-submodule V_R of V_K such that every graded component $V_{d,R}$ of V_R is a finite-rank free R-module which generates the graded component $V_{d,K}$ as a K-module. If V_R is an $H_{\alpha,R}$-invariant full lattice in V_K, it is an R-form of V_K. Now we can see that every $V_K \in H_{\alpha,K}$-mod has an R-form: pick $H_{\alpha,K}$-generators $v_1, \dots, v_r$ and define $V_R := H_{\alpha,R} \cdot v_1 + \dots + H_{\alpha,R} \cdot v_1$.

The projective indecomposable modules over $H_{\alpha,F}$ have projective R-forms. Indeed, any $P(\lambda)_F$ is of the form $H_{\alpha,F} e_{\lambda,F}$ for some *degree-zero* idempotent $e_{\lambda,F}$. By the basis theorem, the degree-zero component $H_{\alpha,F,0}$ of $H_{\alpha,F}$ is defined over R; more precisely, we have $H_{\alpha,\Bbbk,0} = H_{\alpha,R,0} \otimes_R \Bbbk$ for $\Bbbk = K$ or F. Since $H_{\alpha,F,0}$ is finite dimensional, by the classical theorem on lifting idempotents [CR81, (6.7)], there exists an idempotent $e_{\lambda,R} \in H_{\alpha,R,0}$ such that $e_{\lambda,F} = e_{\lambda,R} \otimes 1_F$, and

$$P(\lambda)_R := H_{\alpha,R} e_{\lambda,R}$$

is an R-form of $P(\lambda)_F$. The notation $P(\lambda)_R$ will be reserved for this specific R-form of $P(\lambda)_F$. Note that while the $H_{\alpha,R}$-module $P(\lambda)_R$ is indecomposable, it is not in general true that $P(\lambda)_R \otimes_R K \cong P(\lambda)_K$; see Lemma 4.8 for more information.

Let $V_K \in H_{\alpha,K}$-mod and let V_R be an R-form of V_K. The $H_{\alpha,F}$-module $V_R \otimes_R F$ is called a *reduction modulo* p of V_K. Reduction modulo p in general depends on the choice of V_R. However, as usual, we have a result of the following form.

Lemma 4.3. *If* $V_K \in H_{\alpha,K}$-mod *and* V_R *is an* R-*form of* V_K, *then for any* $\lambda \in \mathrm{KP}(\alpha)$ *we have*

$$[V_R \otimes_R F : L(\lambda)_F]_q = \dim_q^K \mathrm{Hom}_{H_{\alpha,K}}(P(\lambda)_R \otimes_R K, V_K).$$

In particular, the composition multiplicities $[V_R \otimes_R F : L(\lambda)_F]_q$ *are independent of the choice of the* R-*form* V_R.

Proof. We have

$$[V_R \otimes_R F : L(\lambda)_F]_q = \dim_q^F \mathrm{Hom}_{H_{\alpha,F}}(P(\lambda)_F, V_R \otimes_R F).$$

By the universal coefficient theorem,

$$\mathrm{Hom}_{H_{\alpha,F}}(P(\lambda)_F, V_R \otimes_R F) \cong \mathrm{Hom}_{H_{\alpha,R}}(P(\lambda)_R, V_R) \otimes_R F.$$

https://doi.org/10.1112/S0010437X16008204 Published online by Cambridge University Press

Moreover, note that $\mathrm{Hom}_{H_{\alpha,R}}(P(\lambda)_R, V_R)$ is R-free with (graded) rank equal to

$$\dim_q^{\Bbbk} \mathrm{Hom}_{H_{\alpha,R}}(P(\lambda)_R, V_R) \otimes_R \Bbbk$$

for $\Bbbk = F$ or K. Now, by the universal coefficient theorem again, we have that

$$\dim_q^{K} \mathrm{Hom}_{H_{\alpha,R}}(P(\lambda)_R, V_R) \otimes_R K = \dim_q^{K} \mathrm{Hom}_{H_{\alpha,K}}(P(\lambda)_R \otimes_R K, V_R \otimes_R K),$$

which completes the proof, since $V_R \otimes_R K \cong V_K$. □

Our main interest is in reduction modulo p of the irreducible $H_{\alpha,K}$-modules $L(\lambda)_K$. Pick a non-zero homogeneous vector $v \in L(\lambda)_K$ and define

$$L(\lambda)_R := H_{\alpha,R} \cdot v.$$

Then $L(\lambda)_R$ is an $H_{\alpha,R}$-invariant full lattice in $L(\lambda)_K$, and upon reducing modulo p we get an $H_{\alpha,F}$-module $L(\lambda)_R \otimes_R F$. In general, $L(\lambda)_R \otimes_R F$ is not $L(\lambda)_F$, although this happens 'often', for example for cuspidal modules, as stated in the following lemma.

Lemma 4.4 [Kle15a, Proposition 3.20]. *Let $\alpha \in \Phi^+$. Then $L(\alpha)_R \otimes_R F \cong L(\alpha)_F$.*

To generalize this lemma to irreducible modules of the form $L(\alpha^m)$, we need to observe that induction and restriction commute with extension of scalars. More precisely, for $\beta_1, \dots, \beta_m \in Q^+$, $\alpha = \beta_1 + \cdots + \beta_m$ and any ground ring $\Bbbk$, we denote by $H_{\beta_1,\dots,\beta_m;\Bbbk}$ the algebra $H_{\beta_1,\Bbbk} \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} H_{\beta_m,\Bbbk}$ identified as usual with a (non-unital) subalgebra of $H_{\alpha,\Bbbk}$. Now the following lemma is immediate.

Lemma 4.5. *Let $V_R \in H_{\beta_1,\dots,\beta_m;R}$-mod and $W_R \in H_{\alpha,R}$-mod. Then for any R-algebra $\Bbbk$, there are natural isomorphisms of $H_{\alpha,\Bbbk}$-modules*

$$(\mathrm{Ind}^{\alpha}_{\beta_1,\dots,\beta_m} V_R) \otimes_R \Bbbk \cong \mathrm{Ind}^{\alpha}_{\beta_1,\dots,\beta_m} (V_R \otimes_R \Bbbk)$$

and of $H_{\beta_1,\dots,\beta_m;\Bbbk}$-modules

$$(\mathrm{Res}^{\alpha}_{\beta_1,\dots,\beta_m} W_R) \otimes_R \Bbbk \cong \mathrm{Res}^{\alpha}_{\beta_1,\dots,\beta_m} (W_R \otimes_R \Bbbk).$$

Let $\alpha \in \Phi^+$ and $m \in \mathbb{Z}_{>0}$. If $\Bbbk$ is a field, then by Lemma 2.5 we have $L(\alpha^m)_\Bbbk \simeq L(\alpha)_\Bbbk^{\circ m}$. By Lemma 4.5,

$$L(\alpha^m)_R := (L(\alpha)_R)^{\circ m}$$

satisfies $L(\alpha^m)_R \otimes_R \Bbbk \simeq L(\alpha^m)_\Bbbk$ for $\Bbbk = K$ or F. Taking into account Lemmas 4.3 and 4.4, we get the next result.

Lemma 4.6. *Let $\alpha \in \Phi^+$ and $m \in \mathbb{Z}_{>0}$. Then the reduction modulo p of $L(\alpha^m)_K$ is $L(\alpha^m)_F$.*

It was conjectured in [KR11, Conjecture 7.3] that the reduction modulo p of $L(\lambda)_K$ is always $L(\lambda)_F$, but counterexamples are given in [Wil14] (see also [BKM14, Example 2.16]). Still, it is important to understand when we have $L(\lambda)_R \otimes_R F \cong L(\lambda)_F$.

Problem 4.7. Let $\alpha \in Q^+$.

(i) If $\lambda \in \mathrm{KP}(\alpha)$, determine when $L(\lambda)_R \otimes_R F \cong L(\lambda)_F$.

(ii) We say that *the James conjecture has positive solution* (for α) if the isomorphism in (i) holds for all $\lambda \in \mathrm{KP}(\alpha)$. Determine the minimal lower bound p_α on $p = \mathrm{char}\, F$ such that the James conjecture has positive solution for α for all $p \geqslant p_\alpha$. Determine all p values for which the James conjecture has positive solution for α.

At least, we always have the following property.

LEMMA 4.8. *Let $\alpha \in Q^+$ and $\lambda \in \mathrm{KP}(\alpha)$. Then in the Grothendieck group of finite-dimensional $H_{\alpha,F}$-modules we have*

$$[L(\lambda)_R \otimes_R F] = [L(\lambda)_F] + \sum_{\mu<\lambda} a_{\lambda,\mu}[L(\mu)_F] \tag{4.9}$$

for some bar-invariant Laurent polynomials $a_{\lambda,\mu} \in \mathbb{Z}[q, q^{-1}]$. Moreover,

$$P(\lambda)_R \otimes_R K \cong P(\lambda)_K \oplus \bigoplus_{\mu>\lambda} a_{\mu,\lambda} P(\mu)_K.$$

Proof. Let $\Bbbk = K$ or F, and consider the reduced standard module $\bar{\Delta}(\lambda)_\Bbbk$; see (2.6). In view of (2.7), we can write

$$[L(\lambda)_\Bbbk] := [\bar{\Delta}(\lambda)_\Bbbk] + \sum_{\mu<\lambda} f^\Bbbk_{\lambda,\mu}[\bar{\Delta}(\mu)_\Bbbk]$$

for some $f^\Bbbk_{\lambda,\mu} \in \mathbb{Z}[q, q^{-1}]$. Using Lemmas 4.4 and 4.5 induction on the bilexicographical order on $\mathrm{KP}(\lambda)$, we deduce that (4.9) holds for some, not necessarily bar-invariant, coefficients $a_{\lambda,\mu} \in \mathbb{Z}[q, q^{-1}]$. Then we also have

$$\mathrm{ch}_q\,(L(\lambda)_R \otimes_R F) = \mathrm{ch}_q\,(L(\lambda)_F) + \sum_{\mu<\lambda} a_{\lambda,\mu}\mathrm{ch}_q\,(L(\mu)_F).$$

Since reduction modulo p preserves formal characters, the left-hand side is bar-invariant. Moreover, every $\mathrm{ch}_q\,(L(\mu)_F)$ is bar-invariant. This implies that the coefficients $a_{\lambda,\mu}$ are also bar-invariant, since by [KL09, Theorem 3.17] the formal characters $\{\mathrm{ch}_q\, L(\nu)_F \mid \nu \in \mathrm{KP}(\alpha)\}$ are linearly independent.

Finally, for any $\mu \in \mathrm{KP}(\lambda)$ we have

$$a_{\mu,\lambda} = \dim_q^K \mathrm{Hom}_{H_{\alpha,K}}(P(\lambda)_R \otimes_R K, L(\mu)_K),$$

thanks to Lemma 4.3. This implies the second statement. $\square$

Remark 4.10. For $\Bbbk = K$ or F, denote by $d^\Bbbk_{\lambda,\mu}$ the corresponding decomposition numbers, as in (2.7), and consider the *decomposition matrices* $D^\Bbbk := (d^\Bbbk_{\lambda,\mu})_{\lambda,\mu\in\mathrm{KP}(\alpha)}$. Setting $A := (a_{\lambda,\mu})_{\lambda,\mu\in\mathrm{KP}(\alpha)}$, we have $D^F = D^K A$. So the matrix A plays the role of the *adjustment matrix* in the classical James conjecture [Jam90]. Note that the James conjecture has positive solution in the sense of Problem 4.7 if and only if A is the identity matrix.

4.2 Integral forms of standard modules

Our next goal is to construct some special R-forms of standard modules. We call an $H_{\alpha,R}$-module $\Delta(\lambda)_R$ a *universal R-form of a standard module* if it is an R-form for both $\Delta(\lambda)_K$ and $\Delta(\lambda)_F$. We show how to construct these for all λ.

By Theorem 2.8(i), for any field $\Bbbk$ and $\beta \in R^+$, the standard module $\Delta(\beta^m)_\Bbbk$ has simple head $L(\beta^m)_\Bbbk$. Pick a homogeneous generator $v \in \Delta(\beta^m)_K$ and consider the R-form $\Delta(\beta^m)_R := H_{m\beta,R} \cdot v$ of $\Delta(\beta^m)_K$. Further, for any $\alpha \in Q^+$ and $\lambda = (\lambda_1^{m_1}, \dots, \lambda_s^{m_s}) \in \mathrm{KP}(\alpha)$ with $\lambda_1 > \cdots > \lambda_s$, we define the following R-form of $\Delta(\lambda)_K$ (cf. Lemma 4.5):

$$\Delta(\lambda)_R := \Delta(\lambda_1^{m_1})_R \circ \cdots \circ \Delta(\lambda_s^{m_s})_R.$$

Let

$$1_{(\lambda),R} := 1_{m_1\lambda_1,\dots,m_s\lambda_s;R}.$$

Then, for an appropriate set $S^{(\lambda)}$ of coset representatives in a symmetric group, we have that $\{\tau_w 1_{(\lambda),R} \mid w \in S^{(\lambda)}\}$ is a basis of $H_{\alpha,R}1_{(\lambda),R}$ considered as a right $H_{m_1\lambda_1,\dots,m_s\lambda_s;R}$-module. So

$$\Delta(\lambda)_R = \bigoplus_{w \in S^{(\lambda)}} \tau_w 1_{(\lambda),R} \otimes \Delta(\lambda_1^{m_1})_R \otimes \cdots \otimes \Delta(\lambda_s^{m_s})_R.$$

In particular, choosing $v_t \in \Delta(\lambda_t^{m_t})_K$ with $\Delta(\lambda_t^{m_t})_R = H_{m_t\lambda_t,R} \cdot v_t$ for all $1 \leqslant t \leqslant s$ and setting $v := 1_{(\lambda),K} \otimes v_1 \otimes \cdots \otimes v_s$, we have

$$\Delta(\lambda)_R = H_{\alpha,R} \cdot v. \tag{4.11}$$

Now we show that $\Delta(\lambda)_R$ is a universal R-form.

Lemma 4.12. *Let $\alpha \in Q^+$ and $\lambda \in \mathrm{KP}(\alpha)$. Then $\Delta(\lambda)_R \otimes_R F \cong \Delta(\lambda)_F$.*

Proof. In view of (2.10) and Lemma 4.5, we may assume that λ is of the form (β^m) for a positive root β so that $\alpha = m\beta$. By Lemma 4.3, for any $\mu \in \mathrm{KP}(\alpha)$ we have

$$[\Delta(\beta^m)_R \otimes_R F : L(\mu)_F]_q = \dim_q^K \mathrm{Hom}_{H_{\alpha,K}}(P(\mu)_R \otimes_R K, \Delta(\beta^m)_K).$$

By convexity, (β^m) is a minimal element of $\mathrm{KP}(\alpha)$. So Lemma 4.8 implies that all composition factors of $\Delta(\beta^m)_R \otimes_R F$ are $\simeq L(\beta^m)_F$. Moreover,

$$[\Delta(\beta^m)_R \otimes_R F : L(\beta^m)_F]_q = [\Delta(\beta^m)_K : L(\beta^m)_K]_q = [\Delta(\beta^m)_F : L(\beta^m)_F]_q.$$

By construction, $\Delta(\beta^m)_R$ is cyclic, hence so is $\Delta(\beta^m)_R \otimes_R F$. Therefore $\Delta(\beta^m)_R \otimes_R F$ is a module with simple head and belongs to the category of all modules in $H_{\alpha,F}$-mod with composition factors $\simeq L(\beta^m)_F$. Since (β^m) is minimal in $\mathrm{KP}(\alpha)$, we have that $\Delta(\beta^m)_F$ is the projective cover of $L(\beta^m)_F$ in this category; see [Kle15b, Lemma 4.11]. So there is a surjective homomorphism from $\Delta(\beta^m)_F$ onto $\Delta(\beta^m)_R \otimes_R F$. This has to be an isomorphism since we have proved that the two modules have the same composition multiplicities. $\square$

From now on, the notation $\Delta(\lambda)_R$ will be reserved for a *universal* R-form.

Proposition 4.13. *Let $\alpha \in Q^+$ and $\lambda, \mu \in \mathrm{KP}(\alpha)$.*

(i) *If $\lambda \neq \mu$, then $\mathrm{Hom}_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R) = 0$.*

https://doi.org/10.1112/S0010437X16008204 Published online by Cambridge University Press

(ii) *For any R-algebra $\Bbbk$, we have*

$$\mathrm{End}_{H_{\alpha,R}}(\Delta(\lambda)_R) \otimes_R \Bbbk \cong \mathrm{End}_{H_{\alpha,\Bbbk}}(\Delta(\lambda)_R \otimes_R \Bbbk).$$

(iii) *If $\lambda \not< \mu$, then $\mathrm{Ext}^j_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R) = 0$ for all $j \geqslant 1$.*

Proof. By the universal coefficient theorem, for any $j \geqslant 0$ we can embed

$$\mathrm{Ext}^j_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R) \otimes_R F$$

into $\mathrm{Ext}^j_{H_{\alpha,F}}(\Delta(\lambda)_F, \Delta(\mu)_F)$. So assertion (i) follows from Theorem A, and (iii) follows from Theorem 2.8(iii). Now statement (ii) also follows from the universal coefficient theorem, since

$$\mathrm{Ext}^1_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\lambda)_R) = 0$$

by (iii), which makes the Tor_1 term trivial. □

Given an R-module V, denote by $V^{\mathtt{Tors}}$ its *torsion submodule*. Torsion in Ext groups

$$\mathrm{Ext}^j_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R)^{\mathtt{Tors}}$$

is of importance for Problem 4.7; see Remark 4.17. The following result was surprising to us.

Theorem 4.14. *Let $\alpha \in Q^+$ and $\lambda, \mu \in \mathrm{KP}(\alpha)$. Then the R-module*

$$\mathrm{Ext}^1_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R)$$

is torsion-free.

Proof. By Proposition 4.13(iii), we may assume that $\lambda \neq \mu$. By the universal coefficient theorem, there is an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R) \otimes_R F &\to \mathrm{Hom}_{H_{\alpha,F}}(\Delta(\lambda)_F, \Delta(\mu)_F) \\ &\to \mathrm{Tor}_1^R(\mathrm{Ext}^1_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R), F) \to 0. \end{aligned}$$

By Theorem A, the middle term vanishes; hence the third term also vanishes, which implies the theorem. □

We will need the following generalization.

Corollary 4.15. *Let $\alpha \in Q^+$ and $\mu \in \mathrm{KP}(\alpha)$, and let V be an $H_{\alpha,R}$-module with a finite Δ-filtration, all of whose subfactors are of the form $\simeq \Delta(\lambda)_R$ for $\lambda \neq \mu$. Then the R-module $\mathrm{Ext}^1_{H_{\alpha,R}}(V, \Delta(\mu)_R)$ is torsion-free.*

Proof. Apply induction on the length of the Δ-filtration, the induction base coming from Theorem 4.14. If the filtration has length greater than 1, we have an exact sequence

$$0 \to V_1 \to V \to V_2 \to 0,$$

such that the inductive assumption applies to V_1 and V_2. Then we get a long exact sequence

$$\begin{aligned} \mathrm{Hom}_{H_{\alpha,R}}(V_1, \Delta(\mu)_R) &\to \mathrm{Ext}^1_{H_{\alpha,R}}(V_2, \Delta(\mu)_R) \\ \to \mathrm{Ext}^1_{H_{\alpha,R}}(V, \Delta(\mu)_R) &\to \mathrm{Ext}^1_{H_{\alpha,R}}(V_1, \Delta(\mu)_R). \end{aligned}$$

By Proposition 4.13(i), the first term vanishes. By the inductive assumption, the second and fourth terms are torsion-free. Hence so is the third term. □

https://doi.org/10.1112/S0010437X16008204 Published online by Cambridge University Press

While we have just proved that there is no torsion in $\mathrm{Ext}^1_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R)$, the following result reveals the importance of torsion in Ext^2-groups.

COROLLARY 4.16. *Let $\alpha \in Q^+$ and $\lambda, \mu \in \mathrm{KP}(\alpha)$. We have*

$$\begin{aligned}&\dim_q^F \mathrm{Ext}^1_{H_{\alpha,F}}(\Delta(\lambda)_F, \Delta(\mu)_F)\\ &\quad = \dim_q^K \mathrm{Ext}^1_{H_{\alpha,K}}(\Delta(\lambda)_K, \Delta(\mu)_K) + \mathrm{d}_q^R \mathrm{Ext}^2_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R)^{\mathtt{Tors}}.\end{aligned}$$

In particular,

$$\dim_q^F \mathrm{Ext}^1_{H_{\alpha,F}}(\Delta(\lambda)_F, \Delta(\mu)_F) = \dim_q^K \mathrm{Ext}^1_{H_{\alpha,K}}(\Delta(\lambda)_K, \Delta(\mu)_K)$$

if and only if the R-module $\mathrm{Ext}^2_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R)$ is torsion-free.

Proof. By the universal coefficient theorem, there is an exact sequence

$$\begin{aligned}0 \to \mathrm{Ext}^1_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R) \otimes_R F \to \mathrm{Ext}^1_{H_{\alpha,F}}(\Delta(\lambda)_F, \Delta(\mu)_F)\\ \to \mathrm{Tor}_1^R(\mathrm{Ext}^2_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R), F) \to 0\end{aligned}$$

and an isomorphism

$$\mathrm{Ext}^1_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R) \otimes_R K \cong \mathrm{Ext}^1_{H_{\alpha,K}}(\Delta(\lambda)_K, \Delta(\mu)_K).$$

The last isomorphism and Theorem 4.14 imply

$$\dim_q^K \mathrm{Ext}^1_{H_{\alpha,K}}(\Delta(\lambda)_K, \Delta(\mu)_K) = \mathrm{d}_q^R \mathrm{Ext}^1_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R).$$

On the other hand,

$$\mathrm{d}_q^R \mathrm{Ext}^2_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R)^{\mathtt{Tors}} = \dim_q^F \mathrm{Tor}_1^R(\mathrm{Ext}^2_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R), F),$$

so the result now follows from the exactness of the first sequence. □

Remark 4.17. By Theorem 4.14, lack of torsion in $\mathrm{Ext}^2_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R)$ is equivalent to the fact that the extension groups $\mathrm{Ext}^1_{H_\alpha}(\Delta(\lambda), \Delta(\mu))$ have the same graded dimension in characteristic 0 and characteristic p. This is relevant for Problem 4.7. However, we do not understand the *precise* connection between Problem 4.7 and lack of torsion in the groups $\mathrm{Ext}^2_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R)$. For example, we do not know if such lack of torsion for all λ and μ implies (or is equivalent to) the James conjecture having positive solution. In the next section we establish a different statement of that nature. Set

$$\Delta_{\Bbbk} := \bigoplus_{\lambda \in \mathrm{KP}(\alpha)} \Delta(\lambda)_{\Bbbk}.$$

By the universal coefficient theorem, all groups $\mathrm{Ext}^j_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R)$ are torsion-free if and only if the dimension of the $\Bbbk$-algebras $\mathrm{Ext}^\bullet_{H_{\alpha,\Bbbk}}(\Delta_{\Bbbk}, \Delta_{\Bbbk})$ is the same for $\Bbbk = K$ and $\Bbbk = F$, and

$$\mathrm{Ext}^\bullet_{H_{\alpha,\Bbbk}}(\Delta_{\Bbbk}, \Delta_{\Bbbk}) \cong \mathrm{Ext}^\bullet_{H_{\alpha,R}}(\Delta_R, \Delta_R) \otimes_R \Bbbk$$

for $\Bbbk = K$ or F. We do not know if the James conjecture has positive solution under the assumption that *all* groups $\mathrm{Ext}^j_{H_{\alpha,R}}(\Delta(\lambda)_R, \Delta(\mu)_R)$ are torsion-free.

https://doi.org/10.1112/S0010437X16008204 Published online by Cambridge University Press

4.3 Integral forms of projective modules in characteristic zero

Recall that by lifting idempotents, we have constructed projective R-forms $P(\lambda)_R$ of the projective indecomposable modules $P(\lambda)_F$. Our next goal is to construct some interesting R-forms of the projective modules $P(\lambda)_K$. As we cannot denote them by $P(\lambda)_R$, we will have to use the notation $Q(\lambda)_R$. We will construct $Q(\lambda)_R$ using the usual 'universal extension procedure' applied to universal R-forms of the standard modules, but in our 'infinite-dimensional integral' situation we need to be rather careful. We begin with some lemmas.

Lemma 4.18. *Let $\Bbbk$ be a field and suppose that $V \in H_{\alpha,\Bbbk}$-mod has the following properties:*

(i) *V is indecomposable;*

(ii) *V has a finite Δ-filtration with the top factor $\Delta(\lambda)_\Bbbk$;*

(iii) *$\mathrm{Ext}^1_{H_{\alpha,\Bbbk}}(V, \Delta(\mu)_\Bbbk) = 0$ for all $\mu \in \mathrm{KP}(\alpha)$.*

Then $V \cong P(\lambda)_\Bbbk$.

Proof. We have a short exact sequence $0 \to M \to P \to V \to 0$, where P is a finite direct sum of indecomposable projective modules. By [Kle15b, Corollary 7.10(i)], M has a finite Δ-filtration. Now, by property (iii), the short exact sequence splits. Hence V is projective. As it is indecomposable, it must be of the form $q^d P(\mu)$. By property (ii), we must have $\lambda = \mu$ and $d = 0$. □

For $\lambda \in \mathrm{KP}(\alpha)$ and $\Bbbk \in \{F, K, R\}$, we consider the endomorphism algebra

$$B_{\lambda,\Bbbk} := \mathrm{End}_{H_{\alpha,\Bbbk}}(\Delta(\lambda)_\Bbbk)^{\mathrm{op}}.$$

By Proposition 4.13(ii), we have $B_{\lambda,F} \cong B_{\lambda,R} \otimes F$ and $B_{\lambda,K} \cong B_{\lambda,R} \otimes K$. Note that $\Delta(\lambda)_\Bbbk$ is naturally a right $B_{\lambda,\Bbbk}$-module. We need to know that this $B_{\lambda,\Bbbk}$-module is finitely generated. In fact, we will prove that it is free of finite rank. First of all, this is known over a field.

Lemma 4.19. *Let $\lambda \in \mathrm{KP}(\alpha)$ and let $\Bbbk$ be a field. Then the following properties hold.*

(i) *$B_{\lambda,\Bbbk}$ is a commutative polynomial algebra in finitely many variables of positive degrees.*

(ii) *Let $N_{\lambda,\Bbbk}$ be the ideal in $B_{\lambda,\Bbbk}$ spanned by all monomials of positive degree, and let $M := \Delta(\lambda)_\Bbbk N_{\lambda,\Bbbk}$. Then $\Delta(\lambda)_\Bbbk / M \cong \bar{\Delta}(\lambda)_\Bbbk$; see the notation (2.6).*

(iii) *Let $v_1, \dots, v_N \in \Delta(\lambda)_\Bbbk$ be such that $\{v_1 + M, \dots, v_N + M\}$ is a $\Bbbk$-basis of $\Delta(\lambda)_\Bbbk/M$; then $\{v_1, \dots, v_N\}$ is a basis of $\Delta(\lambda)_\Bbbk$ as a $B_{\lambda,\Bbbk}$-module.*

Proof. For (i) see Theorem 2.17. For (ii) and (iii), see [Kle15b, Proposition 5.7]. □

The following general lemma, whose proof is omitted, will help us to transfer the result of Lemma 4.19 from K and F to R.

Lemma 4.20. *Let B_R be an R-algebra and V_R a B_R-module. Assume that B_R and V_R are free as R-modules. If $v_1, \dots, v_N \in V_R$ are such that $\{v_1 \otimes 1_\Bbbk, \dots, v_N \otimes 1_\Bbbk\}$ is a basis of $V_R \otimes_R \Bbbk$ as a $B_R \otimes_R \Bbbk$-module for $\Bbbk = K$ and F, then $\{v_1, \dots, v_N\}$ is a basis of V_R as a B_R-module.*

Lemma 4.21. *Let $\lambda \in \mathrm{KP}(\alpha)$. As a $B_{\lambda,R}$-module, $\Delta(\lambda)_R$ is free of finite rank.*

https://doi.org/10.1112/S0010437X16008204 Published online by Cambridge University Press

Proof. Let $\lambda = (\lambda_1^{m_1}, \dots, \lambda_s^{m_s})$ for positive roots $\lambda_1 > \cdots > \lambda_s$. Choose $v = 1_{(\lambda),K} \otimes v_1 \otimes \cdots \otimes v_s$ as in (4.11). There is a submodule $M \subset \Delta(\lambda)_K$ with $\Delta(\lambda)_K/M \cong \bar{\Delta}(\lambda)_K$. Pick $h_1, \dots, h_N \in H_{\alpha,R}$ such that $\{h_1 v + M, \dots, h_N v + M\}$ is an R-basis of $\bar{\Delta}(\lambda)_R = H_{\alpha,R} \cdot (v + M)$. By Lemma 4.19,

$$\{h_1 v \otimes 1_{\Bbbk}, \dots, h_N v \otimes 1_{\Bbbk}\}$$

is a $B_{\lambda,\Bbbk}$-basis of $\Delta(\lambda)_R \otimes_R \Bbbk$ for $\Bbbk = K$ or F. Now apply Proposition 4.13(ii) and Lemma 4.20. $\square$

Corollary 4.22. *Let $\Bbbk \in \{F, K, R\}$, $V \in H_{\alpha,\Bbbk}$-mod, $\lambda \in \mathrm{KP}(\alpha)$ and $j \in \mathbb{Z}_{\geqslant 0}$. Then, as a $B_{\lambda,\Bbbk}$-module, $\mathrm{Ext}^j_{H_{\alpha,\Bbbk}}(V, \Delta(\lambda)_{\Bbbk})$ is finitely generated.*

Proof. Since $H_{\alpha,\Bbbk}$ is Noetherian, V has a resolution by finite-rank free modules over $H_{\alpha,\Bbbk}$. Applying $\mathrm{Hom}_{H_{\alpha,\Bbbk}}(-, \Delta(\lambda)_{\Bbbk})$ to this resolution, we get a complex with terms that are finite direct sums of modules $\simeq \Delta(\lambda)_{\Bbbk}$, which are finite-rank free over $B_{\lambda,\Bbbk}$, thanks to Lemmas 4.19 and 4.21. As $B_{\lambda,\Bbbk}$ is Noetherian, the cohomology groups of the complex are finitely generated $B_{\lambda,\Bbbk}$-modules. $\square$

Remark 4.23. It is a more subtle issue to determine whether $\mathrm{Ext}^j_{H_{\alpha,\Bbbk}}(\Delta(\lambda)_{\Bbbk}, V)$ is finitely generated as a $B_{\lambda,\Bbbk}$-module. We do not know if this is always true.

Lemma 4.24 (Universal extension procedure). *Let $\Bbbk \in \{F, K, R\}$ and $\mu \in \mathrm{KP}(\alpha)$, and let $V_{\Bbbk}$ be an indecomposable $H_{\alpha,\Bbbk}$-module with a finite Δ-filtration, all of whose subfactors are of the form $\simeq \Delta(\lambda)_{\Bbbk}$ for $\lambda \not\geqslant \mu$. If $\Bbbk = R$, assume in addition that $V_R \otimes_R K$ is indecomposable. Let*

$$r(q) := \mathrm{d}_q^{B_{\mu,\Bbbk}} \mathrm{Ext}^1_{H_{\alpha,\Bbbk}}(V_{\Bbbk}, \Delta(\mu)_{\Bbbk}) \in \mathbb{Z}[q, q^{-1}].$$

Then there exists an $H_{\alpha,\Bbbk}$-module $E(V_{\Bbbk}, \Delta(\mu)_{\Bbbk})$ with the following properties:

(i) *$E(V_{\Bbbk}, \Delta(\mu)_{\Bbbk})$ is indecomposable;*
(ii) *$\mathrm{Ext}^1_{H_{\alpha,\Bbbk}}(E(V_{\Bbbk}, \Delta(\mu)_{\Bbbk}), \Delta(\mu)_{\Bbbk}) = 0$;*
(iii) *there is a short exact sequence*

$$0 \to \overline{r(q)}\Delta(\mu)_{\Bbbk} \to E(V_{\Bbbk}, \Delta(\mu)_{\Bbbk}) \to V_{\Bbbk} \to 0.$$

Proof. In this proof we drop $H_{\alpha,\Bbbk}$ from the indices and write Ext^1 for $\mathrm{Ext}^1_{H_{\alpha,\Bbbk}}$ etc. Also, when it is unlikely to cause confusion, we drop $\Bbbk$ from the indices. Let $\xi_1, \dots, \xi_r$ be a minimal set of homogeneous generators of $\mathrm{Ext}^1(V, \Delta(\mu))$ as a B_μ-module, and let $d_s := \deg(\xi_s)$ for $s = 1, \dots, r$ so that $r(q) = \sum_s q^{d_s}$. The extension

$$0 \to q^{-d_1}\Delta(\mu) \to E_1 \to V \to 0,$$

corresponding to ξ_1, yields the long exact sequence

$$\mathrm{Hom}(q^{-d_1}\Delta(\mu), \Delta(\mu)) \xrightarrow{\varphi} \mathrm{Ext}^1(V, \Delta(\mu)) \xrightarrow{\psi} \mathrm{Ext}^1(E_1, \Delta(\mu)) \to 0.$$

Here we have used the fact that $\mathrm{Ext}^1(q^{-d_1}\Delta(\mu), \Delta(\mu)) = 0$; see Proposition 4.13(iii). Note that $q^{-d_1}\Delta(\mu) = \Delta(\mu)$ as H_α-modules but with degrees shifted down by d_1. So we can consider the identity map $\mathrm{id} : q^{-d_1}\Delta(\mu) \to \Delta(\mu)$, which has degree d_1. The connecting homomorphism φ maps this identity map to ξ_1. It follows that $\mathrm{Ext}^1(E_1, \Delta(\mu))$ is generated as a B_μ-module by

https://doi.org/10.1112/S0010437X16008204 Published online by Cambridge University Press

the elements $\bar{\xi}_2 := \psi(\xi_2), \dots, \bar{\xi}_r := \psi(\xi_r)$. Repeating the argument $r-1$ more times, we get an extension

$$0 \to q^{-d_1}\Delta(\mu) \oplus \cdots \oplus q^{-d_r}\Delta(\mu) = \overline{r(q)}\Delta(\mu) \to E \to V \to 0$$

such that in the corresponding long exact sequence

$$\begin{aligned}&\mathrm{Hom}(E, \Delta(\mu)) \xrightarrow{\chi} \mathrm{Hom}(\overline{r(q)}\Delta(\mu), \Delta(\mu))\\ &\xrightarrow{\varphi} \mathrm{Ext}^1(V, \Delta(\mu)) \to \mathrm{Ext}^1(E, \Delta(\mu)) \to 0,\end{aligned}$$

for all $s = 1, \dots, r$ we have $\varphi(\pi_s) = \xi_s$, where π_s is the (degree-d_s) projection onto the sth summand. In particular, φ is surjective and $\mathrm{Ext}^1(E, \Delta(\mu)) = 0$.

It remains to prove that E is indecomposable. We first prove this when $\Bbbk$ is a field. In that case, if $E = E' \oplus E''$, then both E' and E'' have finite Δ-filtrations; see [Kle15b, Corollary 7.10]. Since $\mathrm{Ext}^1(\Delta(\mu), \Delta(\lambda)) = 0$ for $\lambda \not> \mu$, there is a partition $J' \sqcup J'' = \{1, \dots, r\}$ such that there are submodules

$$M' \cong \bigoplus_{j \in J'} q^{d_j}\Delta(\mu) \subseteq E', \quad M'' \cong \bigoplus_{j \in J''} q^{d_j}\Delta(\mu) \subseteq E'',$$

and E'/M' and M''/E'' have Δ-filtrations. Since $\mathrm{Hom}(\Delta(\mu), V) = 0$, we deduce that $V \cong E'/M' \oplus E''/M''$. As V is indecomposable, we may assume that $E'/M' = 0$. Then some projection π_s lifts to a homomorphism $E \to \Delta(\mu)$, which shows that this π_s is in the image of χ and hence in the kernel of φ, which is a contradiction.

Now let $\Bbbk = R$. Note that V and E are free as R-modules, since all the $\Delta(\nu)_R$ are. If E_R is decomposable, then so is $E_R \otimes K$; therefore it suffices to prove that $E_R \otimes K$ is indecomposable. In view of Corollary 4.15, the $B_{\mu,K}$-module

$$\mathrm{Ext}^1(V_R, \Delta(\mu)_R) \otimes_R K \cong \mathrm{Ext}^1(V_R \otimes_R K, \Delta(\mu)_K)$$

is minimally generated by $\xi_{1,R} \otimes 1_K, \dots, \xi_{r,R} \otimes 1_K$. It follows, using Lemma 4.2, that $E_R \otimes_R K \cong E_K$, where E_K is constructed using the universal extension procedure starting with the indecomposable module $V_K := V_R \otimes_R K$ as in the first part of the proof of the lemma. By the field case established in the previous paragraph, E_K is indecomposable. $\square$

Let $\lambda \in \mathrm{KP}(\alpha)$. For $\Bbbk \in \{R, K, F\}$, we construct a module $Q(\lambda)_\Bbbk$ by starting with $\Delta(\lambda)_\Bbbk$ and repeatedly applying the universal extension procedure. To simplify notation, we drop some of the indices $\Bbbk$ if this is unlikely to lead to confusion. Given Laurent polynomials $r_0(q), r_1(q), \dots, r_m(q) \in \mathbb{Z}[q, q^{-1}]$ with non-negative coefficients and Kostant partitions $\lambda^0, \lambda^1, \dots, \lambda^m \in \mathrm{KP}(\alpha)$, we will use the notation

$$V = r_0(q)\Delta(\lambda^0) \mid r_1(q)\Delta(\lambda^1) \mid \cdots \mid r_m(q)\Delta(\lambda^m)$$

to indicate that the H_α-module V has a filtration $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{m+1} = (0)$ such that $V_s/V_{s+1} \cong r_s(q)\Delta(\lambda^s)$ for $s = 0, 1, \dots, m$.

If $\mathrm{Ext}^1_{H_\alpha}(\Delta(\lambda), \Delta(\mu)) = 0$ for all $\mu \in \mathrm{KP}(\alpha)$, we set $Q(\lambda)_\Bbbk := \Delta(\lambda)_\Bbbk$. Otherwise, let $\lambda^{1,\Bbbk} \in \mathrm{KP}(\alpha)$ be minimal with $\mathrm{Ext}^1_{H_\alpha}(\Delta(\lambda), \Delta(\lambda^{1,\Bbbk})) \neq 0$. Note that this $\lambda^{1,\Bbbk}$ could indeed depend on the ground ring $\Bbbk$, hence the notation. Also notice that $\lambda^{1,\Bbbk} > \lambda$. Let

$$E(\lambda, \lambda^{1,\Bbbk})_\Bbbk := E(\Delta(\lambda), \Delta(\lambda^{1,\Bbbk}));$$

see Lemma 4.24. By construction, we have

$$E(\lambda, \lambda^{1,\Bbbk})_\Bbbk = \Delta(\lambda) \mid \overline{r_{1,\Bbbk}(q)}\Delta(\lambda^{1,\Bbbk}),$$

https://doi.org/10.1112/S0010437X16008204 Published online by Cambridge University Press

where

$$r_{1,\Bbbk}(q) = \mathrm{d}_q^{B_{\lambda^{1,\Bbbk}}} \mathrm{Ext}^1_{H_\alpha}(\Delta(\lambda), \Delta(\lambda^{1,\Bbbk})).$$

This Laurent polynomial may depend on $\Bbbk$, hence the notation. If

$$\mathrm{Ext}^1_{H_\alpha}(E(\lambda, \lambda^{1,\Bbbk}), \Delta(\mu)) = 0$$

for all $\mu \in \mathrm{KP}(\alpha)$, we set $Q(\lambda)_\Bbbk := E(\lambda, \lambda^{1,\Bbbk})_\Bbbk$. Otherwise, let $\lambda^{2,\Bbbk} \in \mathrm{KP}(\alpha)$ be minimal with $\mathrm{Ext}^1_{H_\alpha}(E(\lambda, \lambda^{1,\Bbbk}), \Delta(\lambda^{2,\Bbbk})) \neq 0$. Note that $\lambda^{2,\Bbbk} > \lambda$ and $\lambda^{2,\Bbbk} \neq \lambda^{1,\Bbbk}$. Let

$$E(\lambda, \lambda^{1,\Bbbk}, \lambda^{2,\Bbbk})_\Bbbk := E(E(\lambda, \lambda^{1,\Bbbk}), \Delta(\lambda^{2,\Bbbk})).$$

By construction, we have

$$E(\lambda, \lambda^{1,\Bbbk}, \lambda^{2,\Bbbk})_\Bbbk = \Delta(\lambda) \mid \overline{r_{1,\Bbbk}(q)}\Delta(\lambda^{1,\Bbbk}) \mid \overline{r_{2,\Bbbk}(q)}\Delta(\lambda^{2,\Bbbk}),$$

where

$$r_{2,\Bbbk}(q) = \mathrm{d}_q^{B_{\lambda^{2,\Bbbk}}} \mathrm{Ext}^1_{H_\alpha}(E(\lambda, \lambda^{1,\Bbbk}), \Delta(\lambda^{2,\Bbbk})).$$

If $\mathrm{Ext}^1_{H_\alpha}(E(\lambda, \lambda^{1,\Bbbk}, \lambda^{2,\Bbbk}), \Delta(\mu)) = 0$ for all $\mu \in \mathrm{KP}(\alpha)$, we set

$$Q(\lambda)_\Bbbk := E(\lambda, \lambda^{1,\Bbbk}, \lambda^{2,\Bbbk}).$$

Since in each step we have to pick $\lambda^{t,\Bbbk} > \lambda$, which does not belong to $\{\lambda, \lambda^{1,\Bbbk}, \dots, \lambda^{t-1,\Bbbk}\}$, the process will stop after finitely many steps, and we will obtain an indecomposable module

$$E(\lambda, \lambda^{1,\Bbbk}, \dots, \lambda^{m_\Bbbk,\Bbbk})_\Bbbk = \Delta(\lambda) \mid \overline{r_{1,\Bbbk}(q)}\Delta(\lambda^{1,\Bbbk}) \mid \cdots \mid \overline{r_{m_\Bbbk,\Bbbk}(q)}\Delta(\lambda^{m_\Bbbk,\Bbbk}),$$

where

$$r_{t,\Bbbk}(q) = \mathrm{d}_q^{B_{\lambda^{t,\Bbbk}}} \mathrm{Ext}^1_{H_{\alpha,\Bbbk}}(E(\lambda, \lambda^{1,\Bbbk}, \dots, \lambda^{t-1,\Bbbk})_\Bbbk, \Delta(\lambda^{t,\Bbbk})_\Bbbk) \tag{4.25}$$

for all $1 \leqslant t \leqslant m_\Bbbk$, such that

$$\mathrm{Ext}^1_{H_{\alpha,\Bbbk}}(E(\lambda, \lambda^{1,\Bbbk}, \dots, \lambda^{m_\Bbbk,\Bbbk})_\Bbbk, \Delta(\mu)_\Bbbk) = 0$$

for all $\mu \in \mathrm{KP}(\alpha)$. We set

$$Q(\lambda)_\Bbbk := E(\lambda, \lambda^{1,\Bbbk}, \dots, \lambda^{m_\Bbbk,\Bbbk})_\Bbbk.$$

THEOREM 4.26. *Let $\alpha \in Q^+$ and $\lambda \in \mathrm{KP}(\alpha)$.*

(i) *For $\Bbbk = K$ or F, we have $Q(\lambda)_\Bbbk \cong P(\lambda)_\Bbbk$.*

(ii) *For $\Bbbk = K$ or F, the Laurent polynomial $r_{t,\Bbbk}(q)$ from (4.25) equals the decomposition number $d^\Bbbk_{\lambda^{t,\Bbbk},\lambda}$ for all $1 \leqslant t \leqslant m_\Bbbk$, and $d^\Bbbk_{\mu,\lambda} = 0$ for $\mu \notin \{\lambda^{t,\Bbbk} \mid 1 \leqslant t \leqslant m_\Bbbk\}$.*

(iii) *We have $m_R = m_K$; setting $m := m_R$, we may choose $\lambda^{1,R} = \lambda^{1,K}, \dots, \lambda^{m,R} = \lambda^{m,K}$ and then $r_{t,R}(q) = r_{t,K}(q)$ for all $1 \leqslant t \leqslant m$.*

(iv) *We have $Q(\lambda)_R \otimes_R K \cong P(\lambda)_K$.*

Proof. Property (i) follows from the construction and Lemma 4.18. Assertion (ii) follows from (i), the construction, and Theorem 2.8(v).

To prove (iii) and (iv), we show by induction on $t = 0, 1, \dots$, that we can choose $\lambda^{t,R} = \lambda^{t,K}$, $r_{t,R}(q) = r_{t,K}(q)$ and

$$E(\lambda, \lambda^{1,R}, \dots, \lambda^{t,R})_R \otimes_R K \cong E(\lambda, \lambda^{1,K}, \dots, \lambda^{t,K})_K. \tag{4.27}$$

https://doi.org/10.1112/S0010437X16008204 Published online by Cambridge University Press

The induction base is simply the statement $\Delta(\lambda)_R \otimes_R K \cong \Delta(\lambda)_K$. For the induction step, assume $t > 0$ and that the claim has been proved for all $s < t$.

Let $\xi_{1,R}, \dots, \xi_{r,R}$ be a minimal set of generators of the $B_{\lambda^{t,R},R}$-module

$$\mathrm{Ext}^1_{H_{\alpha,R}}(E(\lambda, \lambda^{1,R}, \dots, \lambda^{t-1,R})_R, \Delta(\lambda^{t,R})_R),$$

so that

$$r_{t,R}(q) = \deg(\xi_{1,R}) + \dots + \deg(\xi_{r,R}).$$

Using Corollary 4.15 and the universal coefficient theorem, we deduce that $\lambda^{t,K}$ can be chosen to be $\lambda^{t,R}$ and the $B_{\lambda^{t,R},K}$-module

$$\mathrm{Ext}^1(\Delta(\lambda)_R, \Delta(\lambda^{t,R})_R) \otimes_R K \cong \mathrm{Ext}^1(V_R \otimes_R K, \Delta(\lambda^{t,R})_K)$$

is minimally generated by $\xi_{1,R} \otimes 1_K, \dots, \xi_{r,R} \otimes 1_K$, so that $r_{t,K}(q) = r_{t,R}(q)$. Finally, (4.27) comes from Lemma 4.2. $\square$

In view of Theorem 4.26(i), $Q(\lambda)_R$ is not in general an R-form of $Q(\lambda)_F \cong P(\lambda)_F$. For every $\lambda \in \mathrm{KP}(\alpha)$, define the $H_{\alpha,F}$-module

$$X(\lambda) := Q(\lambda)_R \otimes F.$$

Theorem 4.28. *The James conjecture has positive solution for α if and only if one of the following equivalent conditions holds:*

(i) *$X(\lambda)$ is projective;*

(ii) *$X(\lambda) \cong P(\lambda)_F$ for all $\lambda \in \mathrm{KP}(\alpha)$;*

(iii) *$\mathrm{Ext}^1_{H_{\alpha,F}}(X(\lambda), \Delta(\mu)_F) = 0$ for all $\lambda, \mu \in \mathrm{KP}(\alpha)$;*

(iv) *the R-module $\mathrm{Ext}^2_{H_{\alpha,R}}(Q(\lambda)_R, \Delta(\mu)_R)$ is torsion-free for all $\lambda, \mu \in \mathrm{KP}(\alpha)$.*

Proof. Conditions (i) and (ii) are equivalent by an argument involving formal characters and Lemma 4.8. Furthermore, (i) and (iii) are equivalent by Lemma 4.18. Since $\mathrm{Ext}^1_{H_{\alpha,R}}(Q(\lambda)_R, \Delta(\mu)_R) = 0$ for all μ, (iii) is equivalent to (iv) by the universal coefficient theorem.

We now prove that (ii) is equivalent to the James conjecture having positive solution. If $X(\lambda) \cong P(\lambda)_F$ for all λ, then they have the same graded dimension, so the R-modules $Q(\lambda)_R$ and $P(\lambda)_R$ have the same graded R-rank, and hence the K-modules $P(\lambda)_K \cong Q(\lambda)_R \otimes_R K$ and $P(\lambda)_R \otimes_R K$ have the same graded dimension; therefore $P(\lambda)_R \otimes_R K \cong P(\lambda)_K$ for all λ (see Lemma 4.8), and so the James conjecture has positive solution.

Conversely, assume that the James conjecture has positive solution. This means that $d^K_{\mu,\lambda} = d^F_{\mu,\lambda}$ for all $\mu, \lambda \in \mathrm{KP}(\alpha)$. Then, by Theorem 4.26(ii), in every step of our universal extension process, we will have the same dimension of the Ext^1-group over K and F; so, by Theorem 4.26(iii), in every step of our universal extension process, we are also going to have the same rank of the appropriate Ext^1-groups over R and F. Now use Lemma 4.2 as in the proof of Theorem 4.26(iv) to show that $Q(\lambda)_R \otimes_R F \cong P(\lambda)_F$. $\square$

Remark 4.29. We conjecture that $P(\lambda)_F$ has an X-filtration with the top quotient $X(\lambda)$ and $X(\mu)$ appearing $a_{\mu,\lambda}(q)$ times. On the level of Grothendieck groups, this is true thanks to Lemma 4.8. But it seems not obvious even that $X(\lambda)$ is a quotient of $P(\lambda)_F$.

References

BCGM11 M. Bennett, V. Chari, J. Greenstein and N. Manning, *On homomorphisms between global Weyl modules*, Represent. Theory **15** (2011), 733–752.

BK09a J. Brundan and A. Kleshchev, *Blocks of cyclotomic Hecke algebras and Khovanov–Lauda algebras*, Invent. Math. **178** (2009), 451–484.

BK09b J. Brundan and A. Kleshchev, *Graded decomposition numbers for cyclotomic Hecke algebras*, Adv. Math. **222** (2009), 1883–1942.

BK12 J. Brundan and A. Kleshchev, *Homological properties of finite type Khovanov–Lauda–Rouquier algebras*, Preprint (2012), arXiv:1210.6900v1 (this is a first version of [BKM14]).

BKM14 J. Brundan, A. Kleshchev and P. McNamara, *Homological properties of finite type Khovanov–Lauda–Rouquier algebras*, Duke Math. J. **163** (2014), 1353–1404.

BKW11 J. Brundan, A. Kleshchev and W. Wang, *Graded Specht modules*, J. Reine Angew. Math. **655** (2011), 61–87.

CR81 C. W. Curtis and I. Reiner, *Methods of representation theory,* Vol. I (John Wiley & Sons, New York, 1981).

Jam90 G. James, *The decomposition matrices of* $\mathrm{GL}_n(q)$ *for* $n \leqslant 10$, Proc. Lond. Math. Soc. (3) **60** (1990), 225–265.

KP15 M. Kashiwara and E. Park, *Affinizations and R-matrices for quiver Hecke algebras*, Preprint (2015), arXiv:1505.03241.

Kat14 S. Kato, *Poincaré–Birkhoff–Witt bases and Khovanov–Lauda–Rouquier algebras*, Duke Math. J. **163** (2014), 619–663.

KL09 M. Khovanov and A. Lauda, *A diagrammatic approach to categorification of quantum groups I*, Represent. Theory **13** (2009), 309–347.

KL11 M. Khovanov and A. Lauda, *A diagrammatic approach to categorification of quantum groups II*, Trans. Amer. Math. Soc. **363** (2011), 2685–2700.

Kle15a A. Kleshchev, *Representation theory and cohomology of Khovanov–Lauda–Rouquier algebras*, in *Modular representation theory of finite and p-adic groups*, Institute of Mathematical Statistics Lecture Note Series, vol. 30 (World Scientific, Singapore, 2015), 109–164.

Kle15b A. Kleshchev, *Affine highest weight categories and affine quasihereditary algebras*, Proc. Lond. Math. Soc. (3) **110** (2015), 841–882.

KL15 A. Kleshchev and J. Loubert, *Affine cellularity of Khovanov–Lauda–Rouquier algebras of finite types*, Int. Math. Res. Not. IMRN **2015** (2015), 5659–5709.

KLM13 A. Kleshchev, J. W. Loubert and V. Miemietz, *Affine cellularity of Khovanov–Lauda–Rouquier algebras in type A*, J. Lond. Math. Soc. (2) **88** (2013), 338–358.

KR11 A. Kleshchev and A. Ram, *Representations of Khovanov–Lauda–Rouquier algebras and combinatorics of Lyndon words*, Math. Ann. **349** (2011), 943–975.

KX12 S. Koenig and C. Xi, *Affine cellular algebras*, Adv. Math. **229** (2012), 139–182.

Mcn15 P. McNamara, *Finite dimensional representations of Khovanov–Lauda–Rouquier algebras I: finite type*, J. Reine Angew. Math. **707** (2015), 103–124.

Pap94 P. Papi, *A characterization of a special ordering in a root system*, Proc. Amer. Math. Soc. **120** (1994), 661–665.

Rot79 J. Rotman, *An introduction to homological algebra* (Academic Press, New York, 1979).

Rou08 R. Rouquier, *2-Kac-Moody algebras*, Preprint (2008), arXiv:0812.5023.

Rou12 R. Rouquier, *Quiver Hecke algebras and 2-Lie algebras*, Algebra Colloq. **19** (2012), 359–410.

Wil14 G. Williamson, *On an analogue of the James conjecture*, Represent. Theory **18** (2014), 15–27.

Alexander S. Kleshchev klesh@uoregon.edu
Department of Mathematics, University of Oregon, Eugene, OR 97403, USA

David J. Steinberg dsteinbe@uoregon.edu
Department of Mathematics, University of Oregon, Eugene, OR 97403, USA