ON THE MEASURE OF SETS OF PARALLEL LINEAR SUBSPACES IN AFFINE SPACE

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1. Introduction. Let E_n be the n-dimensional euclidean real space and $\mathfrak{A}$ the group of unimodular affine transformations which operates on it. It is known that the sets of linear h-spaces L_h $(0 < h < n)$ have no invariant measure with respect to $\mathfrak{A}$ **(5)**. We wish now to consider sets of elements

$$H(L_{h_1}, L_{h_2}, \ldots, L_{h_q}) \tag{1.1}$$

composed by q parallel subspaces of dimensions $h_1, h_2, \ldots, h_q$ which transform transitively by $\mathfrak{A}$. We prove the following:

THEOREM 1. *In order that sets of elements H composed by q parallel linear subspaces of dimensions $h_1, h_2, \ldots, h_q$, which transform transitively by the unimodular affine group $\mathfrak{A}$ have an invariant measure with respect to $\mathfrak{A}$, it is necessary and sufficient that the dimensions h_i be all equal,*

$$h_1 = h_2 = h_3 = \ldots = h_q = h \tag{1.2}$$

and that

$$q = n + 1 - h. \tag{1.3}$$

In § 4 we find the explicit form of this measure together with its metrical significance and in § 5 we indicate some applications to the theory of convex bodies.

2. The Unimodular affine group. (See **2**). Each unimodular affine transformation in E_n can be defined by the position of an n-frame composed of an origin P and n independent vectors $\mathbf{I}_i$ which satisfy the condition

$$|\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_n| = 1 \tag{2.1}$$

where the left-hand side represents the determinant formed by the components of the vectors $\mathbf{I}_i$ with respect to an orthogonal frame of reference.

The relative components of the unimodular affine group $\mathfrak{A}$ are the pfaffian forms ω_{ij} defined by the relations

$$dP = \omega_{0i}\mathbf{I}_i, \qquad d\mathbf{I}_k = \omega_{ki}\mathbf{I}_i \tag{2.2}$$

where the summation convention is used, as will be done throughout.

From (2.2) and (2.1) we deduce

$$\omega_{0i} = |\mathbf{I}_1\mathbf{I}_2 \ldots \mathbf{I}_{i-1}dP\mathbf{I}_{i+1} \ldots \mathbf{I}_n|, \quad \omega_{ki} = |\mathbf{I}_1 \ldots \mathbf{I}_{i-1}d\mathbf{I}_k\mathbf{I}_{i+1} \ldots \mathbf{I}_n|. \tag{2.3}$$

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By exterior differentiation of (2.2) we obtain the equations of structure

(2.4) $$d\omega_{0i} = \omega_{0m} \wedge \omega_{mi} \qquad d\omega_{ki} = \omega_{km} \wedge \omega_{mi}$$

and by exterior differentiation of (2.1), having into account (2.3), we get

(2.5) $$\omega_{11} + \omega_{22} + \ldots + \omega_{nn} = 0.$$

3. Measure of sets of parallel linear subspaces. Let H denote a set of q independent parallel linear subspaces

$$L_{h_1}, L_{h_2}, \ldots , L_{h_q}$$

of dimensions $h_1, h_2, \ldots , h_q$ respectively. We assume that each pair of elements H transform transitively by $\mathfrak{A}$ and that the dimensions h_i are ordered in the following way

(3.1) $$n > h_1 \geqslant h_2 \geqslant \ldots \geqslant h_q \geqslant 1.$$

To each H we may associate an n-frame $(P; \mathbf{I}_1, \mathbf{I}_2, \ldots , \mathbf{I}_n)$ such that the following relations hold:

(3.2)
L_{h_1} = subspace spanned by $\mathbf{I}_1, \mathbf{I}_2, \ldots , \mathbf{I}_{h_1}$;
L_{h_2} = subspace which passes through the endpoint of $\mathbf{I}_{h_1+1}$ and is parallel to $\mathbf{I}_1, \mathbf{I}_2, \ldots , \mathbf{I}_{h_2}$;
L_{h_3} = subspace which passes through the endpoint of $\mathbf{I}_{h_1+2}$ and is parallel to $\mathbf{I}_1, \mathbf{I}_2, \ldots , \mathbf{I}_{h_3}$;
. . .
L_{h_q} = subspace which passes through the endpoint of $\mathbf{I}_{h_1+q-1}$ and is parallel to $\mathbf{I}_1, \mathbf{I}_2, \ldots , \mathbf{I}_{h_q}$.

The assumed transitivity for the elements H with respect to $\mathfrak{A}$ gives the condition

(3.3) $$h_1 + q - 1 \leqslant n.$$

In order to see if sets of elements H have an invariant measure with respect to $\mathfrak{A}$ we follow the general method (**3**; **5**). According to (3.2) and (2.2) the completely integrable system whose integral varieties correspond to the elements H is the following

(3.4)
$$\begin{array}{ll}
\omega_{0s_1} = 0 & (s_1 = h_1 + 1, \ldots , n) \\
\omega_{i_1m_1} = 0 & (i_1 = 1, \ldots , h;\ m_1 = h_1 + 1, \ldots , n) \\
\omega_{0s_2} + \omega_{h_1+1,s_2} = 0 & (s_2 = h_2 + 1, \ldots , h_1) \\
\omega_{h_1+1,s'_2} = 0 & (s'_2 = h_1 + 1, \ldots , n) \\
\omega_{i_2m_2} = 0 & (i_2 = 1, \ldots , h_2;\ m_2 = h_2 + 1, \ldots , h_1) \\
\omega_{0s_3} + \omega_{h_1+2,s_3} = 0 & (s_3 = h_3 + 1, \ldots , h_1) \\
\omega_{h_1+2,s'_3} = 0 & (s'_3 = h_1 + 1, \ldots , n) \\
\omega_{i_3m_3} = 0 & (i_3 = 1, \ldots , h_3;\ m_3 = h_3 + 1, \ldots , h_2) \\
\ldots & \\
\omega_{0s_q} + \omega_{h_1+q-1,s_q} = 0 & (s_q = h_q + 1, \ldots , h_1) \\
\omega_{h_1+q-1,s'_q} = 0 & (s'_q = h_1 + 1, \ldots , n) \\
\omega_{i_qm_q} = 0 & (i_q = 1, \ldots , h_q;\ m_q = h_q + 1, \ldots , h_{q-1}).
\end{array}$$

Note that the number of equations is

$$N = n(h_1 + q) - \sum_{i=1}^{q} h_i(h_i + 1) + \sum_{i=1}^{q-1} h_i h_{i+1} \tag{3.5}$$

and coincides with the number of parameters on which H depends, as it should.

The exterior product Π of all the relative components (3.4) is an exterior differential form of order N. The integral of Π will be a measure for sets of elements H, invariant with respect to $\mathfrak{A}$, if and only if the exterior differential $d\Pi$ vanishes when the equations (2.4) and (2.5) are taken into account (see **3**; **5**). Since the system (3.4) is completely integrable, the theorem of Frobenius (**2**, p. 193) says that in the structure equations (2.4) applied to the forms (3.4), at least one of the differential forms of each term of the sum of the right belongs to (3.4). Thus, up to the sign, which is immaterial for us since we will always take the measures in absolute value, we have

$$d\Pi = \Pi \wedge \Phi \tag{3.6}$$

where

$$\Phi = \sum_{i=1}^{h_1} \omega_{ii} + \sum_{i=1}^{h_2} \omega_{ii} + \ldots + \sum_{i=1}^{h_q} \omega_{ii} - \sum_{i=h_1+1}^{n} \omega_{ii} - \sum_{i=h_2+1}^{n} \omega_{ii} - \ldots - 2 \sum_{i=h_q+1}^{n} \omega_{ii}. \tag{3.7}$$

The relative components of the set (3.4) which have equal indices are

$$\omega_{h_1+1,h_1+1},\ \omega_{h_1+2,h_1+2},\ \ldots,\ \omega_{h_1+q-1,h_1+q-1}. \tag{3.8}$$

Since the relative components are only related by the equation (2.5), the condition $d\Pi = 0$ can hold only if (3.7) is equivalent to the left side of (2.5), up to a linear combination of the forms (3.8). This is possible if and only if $h_q = h_1$ and $h_1 + q - 1 = n$. Taking into account (3.1) these relations prove the stated theorem 1.

4. Metrical interpretation of the measure. If the equations (1.2) and (1.3) are satisfied, the measure for sets of elements H composed by $q = n+1-h$ parallel linear h-spaces is given, up to a constant factor, by the integral of the form Π obtained by exterior multiplication of all the pfaffian forms (3.4). The form Π is called the density, invariant with respect to $\mathfrak{A}$, for sets of elements H.

We wish now to give a metrical interpretation of Π.

Let us put

$$\Pi = \Pi_0 \wedge \Pi_1 \wedge \ldots \wedge \Pi_n \tag{4.1}$$

where

$$\Pi_i = \omega_{i,h+1} \wedge \omega_{i,h+2} \wedge \ldots \wedge \omega_{i,n} \qquad (i = 0, 1, \ldots, n). \tag{4.2}$$

Let $(P_0 \cdot \mathbf{e}_i)$ be an orthogonal frame composed of n perpendicular unit vectors $\mathbf{e}_i$ of origin P_0, such that $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_h$ lie on the subspace L_h determined by $P \equiv P_0, \mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_h$. We may write

$$(4.3) \qquad \mathbf{I}_\alpha = \lambda_{\alpha\beta}\mathbf{e}_\beta, \qquad \mathbf{I}_\xi = \lambda_{\xi i}\mathbf{e}_i$$

where we agree to use the following ranges of indices

(4.4) $\alpha, \beta, \gamma, \ldots = 1, 2, \ldots, h; \qquad \xi, \eta, \zeta, \ldots = h+1, h+2, \ldots, n;$ $i, j, k, \ldots = 1, 2, \ldots, n.$

If we put

$$(4.5) \qquad d\mathbf{e}_i = \theta_{ij}\mathbf{e}_j$$

we will have

$$(4.6) \qquad d\mathbf{I}_\alpha = d\lambda_{\alpha\beta}\mathbf{e}_\beta + \lambda_{\alpha\beta}d\mathbf{e}_\beta = \phi_{\alpha\beta}\mathbf{e}_\beta + \psi_{\alpha\xi}\mathbf{e}_\xi$$

$$(4.7) \qquad d\mathbf{I}_\xi = d\lambda_{\xi i}\mathbf{e}_i + \lambda_{\xi i}d\mathbf{e}_i = \sigma_{\xi i}\mathbf{e}_i$$

where

$$(4.8) \qquad \phi_{\alpha\beta} = d\lambda_{\alpha\beta} + \lambda_{\alpha\gamma}\theta_{\gamma\beta}, \qquad \sigma_{\xi i} = d\lambda_{\xi i} + \lambda_{\xi j}\theta_{ji}, \qquad \psi_{\alpha\xi} = \lambda_{\alpha\beta}\theta_{\beta\xi}.$$

Let us note that, according to (4.7), the volume element at the endpoint of $\mathbf{I}_\eta$ is $\sigma_{\eta 1} \wedge \sigma_{\eta 2} \wedge \ldots \wedge \sigma_{\eta n}$ and the element of $(n\text{-}h)$-dimensional volume in the $(n\text{-}h)$-space spanned by $\mathbf{e}_{h+1}, \mathbf{e}_{h+2}, \ldots, \mathbf{e}_n$ at the orthogonal projection on it of the endpoint of $\mathbf{I}_\eta$ is

$$(4.9) \qquad dP_\eta = \sigma_{\eta,h+1} \wedge \sigma_{\eta,h+2} \wedge \ldots \wedge \sigma_{\eta,n}.$$

The first relation (2.2) may be written

$$(4.10) \qquad dP = \omega_{0i}\mathbf{I}_i = dx_i\mathbf{e}_i$$

and from (2.3) and (4.4) we deduce

$$(4.11) \qquad \omega_{0\xi} = |\mathbf{I}_1\mathbf{I}_2 \ldots \mathbf{I}_{\xi-1}(dx_i\mathbf{e}_i)\mathbf{I}_{\xi+1} \ldots \mathbf{I}_n| = \Lambda_{\xi\eta}dx_\eta$$

where $\Lambda_{\xi\eta}$ means the algebraic complement of $\lambda_{\xi\eta}$ in the determinant

$$(4.12) \qquad |\mathbf{I}_1\mathbf{I}_2 \ldots \mathbf{I}_n| = \begin{vmatrix} \lambda_{11}\lambda_{12} \ldots \lambda_{1h} & 0 \ldots 0 \\ \lambda_{21}\lambda_{22} \ldots \lambda_{2h} & 0 \ldots 0 \\ \ldots & \\ \lambda_{h1}\lambda_{h2} \ldots \lambda_{hh} & 0 \ldots 0 \\ \lambda_{h+1,1}\lambda_{h+1,2} \ldots \lambda_{h+1,h} & \lambda_{h+1,h+1} \ldots \lambda_{h+1,n} \\ \ldots & \\ \lambda_{n1}\lambda_{n2} \ldots \lambda_{nh} & \lambda_{n,h+1} \ldots \lambda_{nn} \end{vmatrix} = 1.$$

Therefore, by exterior multiplication of the forms (4.11), taking into account a well-known property on adjoint determinants (**4**, p. 73) we obtain

$$(4.13) \qquad \Pi_0 = |\Lambda_{\xi\eta}|dx_{h+1} \wedge dx_{h+2} \wedge \ldots \wedge dx_n = D\, dP_0$$

where we have put

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$$(4.14) \qquad D = \begin{vmatrix} \lambda_{11}\lambda_{12} \dots \lambda_{1h} \\ \lambda_{21}\lambda_{22} \dots \lambda_{2h} \\ \dots \\ \lambda_{h1}\lambda_{h2} \dots \lambda_{hh} \end{vmatrix}$$

and $dP_0 = dx_{h+1} \wedge \dots \wedge dx_n$ = element of $(n\text{-}h)$-dimensional volume on the $(n\text{-}h)$-space spanned by $\mathbf{e}_{h+1}, \mathbf{e}_{h+2}, \dots, \mathbf{e}_n$ at the point P_0.

From (2.3) and (4.7) we have

$$(4.15) \qquad \omega_{\eta\xi} = |\mathbf{I}_1\mathbf{I}_2 \dots \mathbf{I}_{\xi-1}d\mathbf{I}_\eta\mathbf{I}_{\xi+1} \dots \mathbf{I}_n|$$

and therefore

$$(4.16) \qquad \Pi_\eta = |\Lambda_\xi|\sigma_{\eta,h+1} \wedge \dots \wedge \sigma_{\eta n} = D\,dP_\eta.$$

Finally, we have,

$$(4.17) \qquad \omega_{\alpha\eta} = |\mathbf{I}_1\mathbf{I}_2 \dots \mathbf{I}_{\eta-1}d\mathbf{I}_\alpha\mathbf{I}_{\eta+1} \dots \mathbf{I}_n|$$

and therefore

$$(4.18) \qquad \Pi_\alpha = |\Lambda_{\eta\xi}|\psi_{\alpha,h+1} \wedge \dots \wedge \psi_{\alpha n} = D\psi_{\alpha,h+1} \wedge \dots \wedge \psi_{\alpha n}.$$

If we introduce the density dL_{n-h} invariant with respect to rotations about P_0 (metrical density, see **(6)**), for the linear $(n\text{-}h)$-spaces through P_0 spanned by $\mathbf{e}_{h+1}, \dots, \mathbf{e}_n$, that is,

$$(4.19) \qquad dL_{n-h} = (\theta_{1,h+1} \wedge \theta_{2,h+1} \wedge \dots \wedge \theta_{h,h+1}) \wedge (\theta_{1,h+2} \wedge \dots \wedge \theta_{h,h+2}) \wedge \dots \wedge (\theta_{1,n} \wedge \theta_{2,n} \dots \wedge \theta_{h,n}),$$

we have, from (4.18) and (4.8)

$$(4.20) \qquad \Pi_1 \wedge \Pi_2 \wedge \dots \wedge \Pi_h = D^n dL_{n-h}.$$

Therefore, we have,

$$(4.21) \qquad \Pi = D^{2n-h+1}\,dP_0 \wedge dP_{h+1} \wedge dP_{h+2} \wedge \dots \wedge dP_n \wedge dL_{n-h}.$$

Let us now observe that the volume S of the $(n\text{-}h)$-dimensional simplex of vertices $P_0, P_{h+1}, P_{h+2}, \dots, P_n$, taking into account (4.3), is given by

$$(4.22) \qquad S = \frac{1}{(n-h)!} \begin{vmatrix} \lambda_{h+1,h+1} \dots \lambda_{h+1,n} \\ \lambda_{h+2,h+1} \dots \lambda_{h+2,n} \\ \dots \\ \lambda_{n,h+1} \quad \dots \lambda_{n,n} \end{vmatrix}.$$

From (4.12) and (4.22) we get

$$D = \frac{1}{(n-h)!S}$$

and (4.21) may be written in the definitive form

$$(4.24) \qquad \Pi = \frac{dP_0 \wedge dP_{h+1} \wedge \dots \wedge dP_n \wedge dL_{n-h}}{[(n-h)!S]^{2n-h+1}}.$$

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Let us summarize the meaning of the terms in (4.24). Given $n - h + 1$ parallel h-spaces, we cut them by an orthogonal $(n\text{-}h)$-space L_{n-h} through a fixed origin; let $P_0, P_{h+1}, \ldots, P_n$ be the intersection points. Then, S is the volume of the simplex of vertices $P_0, P_{h+1}, \ldots, P_n$; each dP_i $(i = 0, h + 1, \ldots, n)$ is the element of volume at P_i of L_{n-h}, and dL_{n-h} represents the metrical density for sets of $(n\text{-}h)$-spaces through the origin **(6)**.

5. Application to convex bodies. Let K be a convex body in E_n. It is well known that the measure of sets of linear h-spaces, invariant with respect to the group of motions, which intersect K, gives rise up to a constant factor to the metrical invariants W_h^n (= hth mixed volume of K with the unit sphere; $h = 1, 2, \ldots, n - 1$; see **(6)**).

This result is not straightforwardly generalizable to the affine geometry, because the linear subspaces of dimension $h > 0$ have no invariant measure with respect to the unimodular affine group **(5)** .However, if we consider sets of parallel h-spaces in the sense of § 3, we find that the measure of sets of elements H composed of $n - h + 1$ parallel linear h-spaces whose convex cover $C(H)$ contains K in its interior, will give an affine invariant for K. It has the form

(5.1) $M_h^n(K) =$
$$\int \Pi = [(n-h)!]^{h-2n-1} \int_{K\subset C(H)} \frac{dP_0 \wedge dP_1 \wedge \ldots \wedge dP_{n-h} \wedge dL_{n-h}}{S^{2n-h+1}}$$

where the integral is extended over all L_{n-h} orthogonal to the parallel h-spaces which constitute H, such that $K \subset C(H)$ and dP_i $(i = 0, 1, 2, \ldots, n - h)$ are the volume elements in L_{n-h} at the intersection points of L_{n-h} with H.

For $h = 1, 2, \ldots, n - 1$ we get a set of $n - 1$ affine invariants which may be considered as the affine generalization of the W_h^n of the metrical case. It seems to be an interesting open question to investigate if the affine invariants M_h^n are related by inequalities of the type of those of Minkowski for the metrical invariants W_h^n. For $h = n - 1$, see **(7)**.

Let us consider the cases $n = 2$, $n = 3$.

1. *Case of the plane* $(n = 2)$. According to (1.3) we have the possibility $h = 1$, $q = 2$, that is, the elements H are composed of two parallel lines. Let θ denote the angle of the direction normal to these lines and let p_0, p_1 be their distances to a fixed origin 0. The measure of the set of parallel lines which contain K in its interior gives the following affine invariant for K:

$$M_1^2(K) = \int \frac{dp_0 \wedge dp_1 \wedge d\theta}{|p_1 - p_0|^4} = \frac{1}{6}\int_0^\pi \frac{d\theta}{\Delta^2}, \tag{5.2}$$

where $\Delta = \Delta(\theta)$ denotes the width of K in the direction θ.

2. *Case of the space* $(n = 3)$. According to (1.3) we have two possibilities:

(a) $h = 2$, $q = 2$; (b) $h = 1$, $q = 3$.

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For the case (a) the elements H are pairs of parallel planes. If $d\Omega$ denotes the element of area on the unit sphere corresponding to the direction normal to the planes H and p_0, p_1 are their distances to a fixed origin 0, the measure of the set of pairs of parallel planes which contain a given convex body K gives the following affine invariant for K

$$M_2^3(K) = \int \frac{dp_0 \wedge dp_1 \wedge d\Omega}{|p_0 - p_1|^5} = \frac{1}{12}\int \frac{d\Omega}{\Delta^3}, \tag{5.3}$$

where $\Delta = \Delta(\Omega)$ denotes the width of K in the direction Ω.

For the case (b) the elements H are constituted by three parallel lines. If $d\Omega$ denotes the area element on the unit sphere at the point defined by the direction of these lines and dP_0, dP_1, dP_2 are the elements of area of a plane normal to the parallel lines at the corresponding intersection points, the measure of the set of three parallel lines whose convex cover contains K, gives the following affine invariant for K:

$$M_1^3(K) = \frac{1}{64}\int d\Omega \int \frac{dP_0 \wedge dP_1 \wedge dP_2}{S^6} \tag{5.4}$$

where S denotes the area of the triangle $P_0P_1P_2$. The first integration is extended over all triangles $P_0P_1P_2$ which contain the projection K_Ω of K on the plane normal to the direction Ω. The second integration is extended over half of the unit sphere.

A direct way of obtaining the invariants (5.2) and (5.3) together with certain inequalities between them and the area (volume) of K has been given in **(7)**.

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