A CHARACTERIZATION OF THE FINITE SIMPLE
GROUP $U_4(3)$

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The aim of this paper is to give a characterization of the finite simple
group $U_4(3)$ i.e. the 4-dimensional projective special unitary group over
the field of 9 elements. More precisely, we shall prove the following result.

**Theorem.** Let $t_0$ be an involution in $U_4(3)$. Denote by $H_0$, the centralizer
of $t_0$ in $U_4(3)$.

Let $G$ be a finite group of even order with the following properties:

(a) $G$ has no subgroup of index 2,
(b) $G$ has an involution $t$ such that $H = C_G(t)$, the centralizer of $t$ in $G$
is isomorphic to $H_0$.

Then $G$ is isomorphic to $U_4(3)$.

We shall use the standard notation.

1. Some properties of $H_0$

Let $F_9$ be the finite field with 9 elements. Then the map: $x \rightarrow \bar{x} = x^3$
$(x \in F_9)$ is an automorphism of $F_9$. We extend this map to a map of $GL(4, 9)$
thus: $(a_{ij}) \rightarrow (\overline{a_{ij}}) = (\bar{a}_{ij})$ where $(a_{ij}) \in GL(4, 9)$. The subgroup $SU(4, 9)$
in $GL(4, 9)$ consisting of all matrices with determinant 1 which satisfy the
relation: $(a_{ij}) \cdot (a_{ij})^* = I$ where $(a_{ij})^*$ is the transpose of $(\overline{a_{ij}})$, is known as
4-dimensional special unitary group over $F_9$. Then $U_4(3) (= PSU(4, 9))$
is the factor group $SU(4, 9)/Z(SU(4, 9))$ where $Z(SU(4, 9))$ denotes the
centre of $SU(4, 9)$.

Let $t_0'$ be the matrix

$$t_0' = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$ 

Then $t_0'$ is an involution in $SU(4, 9)$. Now the centre of $SU(4, 9)$ is generated
by the element $c = k^2I$ where $k$ is a fixed primitive element of the multi-

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Denote by $H'_0$, the group of all matrices $(a_{ij})$ in $SU(4, 9)$ which 'commute projectively' with $t'_0$ i.e. which satisfy the relation $(a_{ij})t'_0 = t'_0(a_{ij})c_r$ ($r = 0, 1, 2, 3$). A matrix in $SU(4, 9)$ belongs to $H'_0$ if and only if it has the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $(A)$ and $(B)$ are $2 \times 2$ matrices in $GU(2, 9)$ with det $(A)$ det $(B) = 1$.

Let $L'_1$ be the subgroup of $H'_0$ consisting of matrices of the form

$$\begin{pmatrix} A \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

with $(A) \in SU(2, 9)$. Since $SU(2, 9) \cong SL(2, 3)$, we can easily check that the following matrices generate $L'_1$

$$a'_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad b'_1 = \begin{pmatrix} 0 & k^6 \\ k^6 & 0 \\ 0 & 1 \end{pmatrix};$$

$$\sigma'_1 = \begin{pmatrix} k & k^3 \\ k^5 & k^3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Now we have the matrix $u'$ belongs to $H'_0$

$$u' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and we get

$$u'\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}.$$ 

The matrix $v'$

$$v' = \begin{pmatrix} k^3 & k^3 \\ k^3 & k^5 \\ k & k \\ k & k^5 \end{pmatrix}$$

also belongs to $SU(4, 9)$. We check that $(v')^2 = t'_0c$ and $u'v'u' = (v')^{-1}$. So $\langle u', v' \rangle$ is dihedral of order 16.
Put $a'_2 = u'a'_2u'$, $b'_2 = u'b'_2u'$, $\sigma'_2 = u'\sigma'_2u'$ and $L'_2 = \langle a'_2, b'_2, \sigma'_2 \rangle$. We can now verify that $H'_0 = (L'_1 \times L'_2)\langle u', v' \rangle$. Let $H_0 = H'_0\langle c \rangle$ and in the natural homomorphism from $H'_0$ onto $H_0$, let the images of $t'_0, a'_i, b'_i, \sigma'_i, L'_i, u', v'$ ($i = 1, 2$) be $t_0, a_i, b_i, \sigma_i, L_i, u, v$ respectively. We have then $H_0$ is a non-splitting extension of $L = L_1L_2$ by a four group. More precisely we have the following relations:

$$H_0 = L \cdot F$$

$$L = L_1L_2$$

$$F = \langle u, v \rangle$$

$$L_i = \langle a_i, b_i, \sigma_i a_i^2 = b_i^2 = t_0, b_i^{-1}a_i b_i = a_i^{-1}, \sigma_i^{-1}a_i \sigma_i = b_i, \sigma_i^{-1}b_i \sigma_i = a_i b_i, \sigma_i^3 = 1 \rangle$$

and

$$v^{-1}a_i v = a_i^{-1}, v^{-1}b_i v = b_i a_i, v^{-1} \sigma_i v = \sigma_i^{-1}, v^2 = t_0.$$  

The structure of $H_0$ is now completely determined. Of course, we have to see that the structure of $H_0$ is independent of the choice of $t'_0$ in $SU(4,9)$. This is so because we can check that $U_4(3)$ has only one conjugate class of involutions.

We shall list a few properties of $H_0$, which will be used in the next section.

(1.1) Every element of $H_0$ can be written uniquely in the form

$$a'_1 b'_1 a'_2 b'_2 a^n u^p v^q$$

where $t_1 = a_1 a_2$; $t_2 = b_1 b_2$; $\sigma = \sigma_1 \sigma_2$; $i = 0, 1, 2, 3$; $j = 0, 1$; $k = 0, 1, 2$; $l = 0, 1$; $m = 0, 1, 2$; $p = 0, 1$; $q = 0, 1$. The order of $H_0$ is $2^7 \cdot 3^2$.

(1.2) The group $Q = \langle a_1, a_2, b_1, b_2 \rangle F \subseteq H_0$ is a Sylow 2-subgroup of $H_0$. The centre $Z(Q)$ of $Q$ is $\langle t_0 \rangle$.

(1.3) There are 4 conjugate classes of involutions in $H_0$ with representatives $t_0, t_1, u, uv$. We have the centralizer $C_{H_0}(t_1) = A$ of $t_1$ in $H_0$ is the group $\langle a_1, a_2, t_2, u, v \rangle$, a non-abelian group of order 64. We have the centre $Z(A)$ of $A$ is $\langle t_0, t_1 \rangle$, a four group. The commutator group $A'$ of $A$ is also $\langle t_0, t_1 \rangle$. The centralizer of $u$, $C_{H_0}(u)$ in $H_0$ is $E_1 \langle \sigma \rangle$ where $E_1 = \langle t_0, t_1, t_3, u, v \rangle$, an elementary abelian group of order 16. The centralizer of $uv$, $C_{H_0}(uv)$ in $H_0$ is $E_2 \langle \sigma_1 \sigma_2^{-1} \rangle$ where $E_2$ is $\langle t_0, t_1, t_3, uv \rangle$ ($t_3 = a_1 b_1 b_2$), an elementary abelian group of order 16.

(1.4) Both $E_1$ and $E_2$ are normal in the group $Q$. We have $N_{H_0}(E_1) = Q \langle \sigma \rangle$ and the factor group $N_{H_0}(E_1)/E_1$ is isomorphic to $S_4$, the symmetric group in 4 letters. Similarly we have $N_{H_0}(E_2) = Q \langle \rho \rangle$ ($\rho = \sigma_1 \sigma_2^{-1}$) and the factor group $N_{H_0}(E_2)/E_2$ is isomorphic to $S_4$.

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(1.5) The group $L$ is the smallest normal subgroup of $H_0$ with 2-factor group and $H/L$ is a four-group.

(1.6) A Sylow 3-subgroup $T$ of $H_0$ is $\langle \sigma_1, \sigma_2 \rangle$, an elementary abelian group of order 9. We have $C_{H_0}(T) = \langle t_0 \rangle \times T$ and $N_{H_0}(T) = \langle u, v \rangle T$.

2. Conjugacy of involutions

Let $G$ be a finite group with properties (a) and (b) of the theorem. Since the group $H = C_G(t)$ is isomorphic to $H_0$. We shall identify $H$ with $H_0$. Then we have $t_0 = t$.

(2.1) LEMMA. The Sylow 2-subgroup $Q$ of $H$ is a Sylow 2-subgroup of $G$.

PROOF. This is obvious since $Z(Q) = \langle t \rangle$ is cyclic of order 2.

(2.2) LEMMA. If the involution $u$ is conjugate to $t$ in $G$, then $t_1$ is conjugate to $t$ in $G$.

PROOF. Since by assumption $u$ is conjugate to $t$ in $G$, there exists a Sylow 2-subgroup of $C_G(u)$ properly containing $E_1 = \langle t, t_1, t_2, u \rangle$. Therefore there is an element $x$ in $C_G(u) - H$ which normalizes $E_1$. Let us look more closely at the involutions in $E_1$. We have

$$C_1 = \{t_1, tt_1, t_2, tt_2, t_1t_2, tt_1t_2\}$$

whose elements are conjugate in $H$ and likewise

$$C_2 = \{u, t_1u, t_2u, t_1t_2u, tu, tt_1u, tt_2u, tt_1t_2u\}$$

with elements conjugate in $H$. We see that $C_1 \cup C_2 \cup \{t\} = E_1 - \{1\}$.

Since $x \notin H$, we must have $x^{-1}tx \neq t$. If $x^{-1}tx \in C_1$ or $x^{-1}t_1x \in C_2$, then we are finished. Therefore we may suppose that $x^{-1}tx \in C_2$ and $x^{-1}t_1x \in C_1$. Then we get $x^{-1}tt_1x \in C_2$. Since $tt_1$ is conjugate to $t_1$, the lemma is proved.

(2.3) LEMMA. If the involution $uv$ is conjugate to $t$ in $G$, then $t_1$ is conjugate to $t$ in $G$.

PROOF. As in (2.2) with $E_2$ playing the role of $E_1 - \{1\}$.

For the proof of next lemma, we need an unpublished result of Thompson.

LEMMA (Thompson [7]). Suppose $G$ is a finite group of even order which has no subgroup of index 2. Let $S_2$ be a Sylow 2-subgroup of $G$ and let $M$ be a maximal subgroup of $S_2$. Then for each involution $I$ of $G$, there is an element $B$ of $G$ such that $B^{-1}IB \in M$. 

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(2.4) Lemma. If the involution \( t_1 \) is conjugate to \( t \) in \( G \), then \( G \) has only one conjugate class of involutions.

Proof. We have by (2.1) that \( Q \) is a Sylow 2-subgroup of \( G \). The group \( M = \langle a_1, a_2, b_1, b_2, v \rangle \) is a maximal subgroup of \( Q \). By our assumption, we have one class of involutions in \( M \). The lemma follows from condition (a) of the theorem and Thompson's lemma.

(2.5) Lemma. There is only one class of involutions in \( G \).

Proof. First we want to show that the group \( G \) is not 2-normal. By way of contradiction, suppose that it is 2-normal. Since \( \langle t \rangle \) is the centre of a Sylow 2-subgroup \( Q \) of \( G \). It follows by Hall-Gr"{u}n's theorem [4], that the greatest factor group of \( G \) which is a 2-group is isomorphic to that of \( N_G(Z(Q)) = H \), i.e. by (1.5) isomorphic to \( H/L \) which is of order 4. But this is a contradiction to condition (a) of the theorem. It follows that \( G \) is not 2-normal. This means that there is an element \( z \in G \) such that \( t \in Q \cap z^{-1}Qz \) but \( \langle t \rangle \) is not the centre of \( z^{-1}Qz \).

The centre of \( z^{-1}Qz \) is \( \langle z^{-1}tz \rangle \). So \( z^{-1}tz \neq t \). On the other hand, we have \( t \in z^{-1}Qz \). It follows that \( t \) and \( z^{-1}tz \) commute. Hence \( z^{-1}tz \in H \). Without loss of generality, we may assume that \( z^{-1}tz \in \{ t_1, u, uv \} \). The lemma follows now by (2.2); (2.3) and/or (2.4).

(2.6) Lemma. The group \( G \) is simple.

Proof. Suppose at first that \( O(G) \neq 1 \) where \( O(G) \) denotes the maximal odd-order normal subgroup of \( G \). Then the four group \( \langle t, t_1 \rangle \) acts on \( G \). By the structure of \( H \) and (2.5), we see that \( C_G(w) \) does not have a non-trivial intersection with \( O(G) \) for \( w \in \langle t, t_1 \rangle \). Hence \( \langle t, t_1 \rangle \) acts fixed-point-free on \( O(G) \) which is not possible. Hence we have that \( O(G) = 1 \).

Suppose next that \( N \) is a proper normal subgroup of \( G \) such that \( |G/N| \) is odd. We have then \( H \subseteq N \) since \( H \) does not have a proper odd-order factor group. We have that \( Q \subseteq N \). By Frattini argument, \( G = N \cdot N_G(Q) \). But then \( N_G(Q) \subseteq N_G \langle t \rangle = H \). So \( G = N \), a contradiction.

Lastly suppose that \( G \) is not a simple group. Then \( G \) must have a proper normal subgroup \( K \) such that both \( |K| \) and \( |G/K| \) are even. Since by (2.5), all involutions of \( G \) are in \( K \). This implies that \( Q \subseteq L \) since \( Q \) is generated by its involutions, a contradiction to our assumption. The proof is now complete.

(2.7) Lemma. The group \( N_G(E_1)/E_1 \) is isomorphic to \( A_6 \), the alternating group in 6 letters \( (i = 1, 2) \).

Proof. By (2.5), there is a 2-group in \( C_G(u) \) properly containing \( E_1 \) in which \( E_1 \) is normal. So we get that \( N_G(E_1) \not\subseteq H \). Since \( N_H(E_1)/E_1 \) is
isomorphic to $S_4$, a Sylow 2-subgroup of $N_H(E_1)/E_1$ is dihedral of order 8. Clearly $Q/E_1$ is also a Sylow 2-subgroup of $N_G(E_1)/E_1$. Since we have $C_G(E_1) = E_1$, the group $\mathcal{S} = N_G(E_1)/E_1$ is isomorphic to a subgroup of $GL(4, 2) \cong A_8$ which has order $2^5 \cdot 3^2 \cdot 5 \cdot 7$.

Suppose at first that $O(\mathcal{S}) \neq 1$ where $O(\mathcal{S})$ denotes the maximal odd-order normal subgroup of $\mathcal{S}$. Consider the action of the four-group $\langle a_1 E_1, b_1 E_1 \rangle$ on $O(\mathcal{S})$. Using the facts that all involutions of $\langle a_1 E_1, b_1 E_1 \rangle$ are conjugate in (since $a E_1 \in \mathcal{S}$) and that the centralizer of any involution in $A_8$ has order $2^5 \cdot 3$ or $2^5 \cdot 3$, we get by a result of Brauer-Wielandt [10], that $|O(\mathcal{S})| = 3^3$ or 3. The first case is not possible since $3^3 \nmid |A_8|$. So we have $|O(\mathcal{S})| = 3$. Hence $\langle a_1 E_1, b_1 E_1 \rangle \cdot O(\mathcal{S}) = \langle a_1 E_1, b_1 E_1 \rangle \times O(\mathcal{S})$. We shall rule out this case by considering $N_{\mathcal{S}}(a_1 E_1, b_1 E_1)$. We have $N_G(a_1, b_1, a_2, b_2, u) \subseteq N_G(\mathcal{S})$ since $Z(a_1, b_1, a_2, b_2, u) = \langle b_1 \rangle$. So

$$N_G(a_1, b_1, a_2, b_2, u) \cap N_G(E_1) = Q \cdot \langle a \rangle$$

and it follows $N_G(a_1 E_1, b_1 E_1) \cong S^4$, a contradiction to

$$\langle a_1 E_1, b_1 E_1 \rangle \cdot O(\mathcal{S}) = \langle a_1 E_1, b_1 E_1 \rangle \times O(\mathcal{S})$$

Thus $O(\mathcal{S}) = 1$.

By the structure of $A_8$, the order of $C_{\mathcal{S}}(a_1 E_1)$ is $2^3 \cdot 3$ or $2^3$. Suppose that $|C_{\mathcal{S}}(a_1 E_1)| = 2^3 \cdot 3$. We are now in a position to apply Gorenstein-Walter’s result [3], and get $\mathcal{S} \cong PSL(2, 23); PSL(2, 25); PGL(2, 11); PGL(2, 13)$ or $A_7$. The first four cases are not possible since $|\mathcal{S}| \nmid |A_8|$. If 7 divides the order of $\mathcal{S}$, we would then have an element of order 7 in $N_G(E_1)$ which acts fixed-point-free on $E_1$, a contradiction. Thus we must have $|C_{\mathcal{S}}(a_1 E_1)| = 8$. Let $T$ be a Sylow 2-subgroup of $G$ in $C_G(t_1)$ properly containing $C_G(t_1) \cap H$. Then $Z(T/E_1) \neq \langle a_1 E_1 \rangle$, otherwise we would get $|C_{\mathcal{S}}(a_1 E_1)| > 8$. This means that $\mathcal{S}$ has only one class of involutions. Therefore by Gorenstein-Walter [3], we get $\mathcal{S} \cong PSL(2, 9) \cong A_8$. The proof is finished.

3. Sylow 3-subgroups of $G$ and its normalizers in $G$

We shall determine the structure of a Sylow 3-subgroup of $G$, and the normalizer of this Sylow 3-subgroup in $G$.

We have $T = \langle \sigma_1, \sigma_2 \rangle \subseteq H$ is a Sylow 3-subgroup of $H$ and $C_H(T) = \langle t \rangle \times T, N_H(T) = \langle u, v \rangle T$. By the structure of $H$, clearly a Sylow 2-subgroup of $C_G(T)$ is $\langle t \rangle$. It follows, by a theorem of Burnside [4], that $C_G(T)$ has a normal 2-complement $M \supseteq T$. Since we have $C_G(T) < N_G(T)$, we get by Frattini argument that

$$N_G(T) = (C_G(t) \cap N_G(T))C_G(T) = \langle u, v \rangle M.$$
The normal 2-complement $M$ of $C_G(T)$ is characteristic in $C_G(T)$. Hence $M$ is normal in $N_G(T)$. Thus the four group $\langle t, u \rangle$ acts on $M$. Using the result of Brauer-Wielandt [10] and the fact $C_M(t) = T$; $C_M(t, u) = \langle \sigma \rangle$, we get $|M| = |C_M(u)| |C_M(tu)|$. Since $u$ and $tu$ are conjugate in $N_G(T)$, we have $|C_M(u)| = |C_M(tu)|$. By (2.5), we have $|C_M(u)| = |C_M(tu)| = 3$ or $3^2$. So the order of $M$ is $9$ or $81$.

Suppose that the order of $M$ is $9$. Then we have $T = M$ and so $T$ is a Sylow $3$-subgroup of $G$ with $N_G(T) = \langle u, v \rangle T$. By (2.7), we know that $N_G(E_1)/E_1 \cong A_6$. Let $\bar{T}$ be a Sylow $3$-subgroup of $N_G(E_1)$. By the structure of $A_6$ and our assumption, we have $C_G(\bar{T}) \cap N_G(E_1) = \bar{T}$ or $\langle t' \rangle \times \bar{T}$ where $t'$ is an involution in $E_1$. Suppose we have $C_G(\bar{T}) \cap N_G(E_1) = \langle t' \rangle \times \bar{T}$. Because $C_G(E_1) = E_1$, $\bar{T}$ induces by conjugation on $E_1$ a faithful automorphism of $E_1$ and fixes an involution on $E_1$. Thus we must have $3^2$ dividing $(2^4-2)(2^4-4)(2^4-8) = 2^6 \cdot 3 \cdot 7$, a contradiction. Hence we get $C_G(\bar{T}) \cap N_G(E_1) = \bar{T}$. Now by the structure of $N_G(T)$, and $C_G(\bar{T}) \cap N_G(E_1) = \bar{T}$, we get that $|N_G(\bar{T}) \cap N_G(E_1)| = 3^2$ or $2 \cdot 3^2$. The later case is impossible, since the index of $N_G(\bar{T}) \cap N_G(E_1)$ in $N_G(E_1)$ is $2^4 \cdot 5$ which is not congruent to $1$ modulo $3$. Therefore, we have $|N_G(\bar{T}) \cap N_G(E_1)| = 3^2$. By a transfer theorem of Burnside [4, p. 203], $N_G(E_1)/E_1$ is not simple, a contradiction. So we have shown that the order of $M$ is not $9$.

Thus $M$ is a group of order $81$. We shall show that $M$ is elementary abelian. For this, we need to look at elements of order $3$ in $H$ more closely. There are $3$ conjugate classes of elements of order $3$ in $H$ with representatives $\sigma_1$, $\sigma = \sigma_1 \sigma_2$, $\rho = \sigma_1 \sigma_2^{-1}$ respectively. The centralizer of $\sigma_1$ in $H$ is $T \cdot \langle a_2, b_2 \rangle$ and so a Sylow $2$-subgroup is $C_H(\sigma_1)$ is quaternion of order $8$. The centralizer of $\sigma$ in $H$ is $T \cdot \langle u \rangle T$ and the centralizer of $\rho$ in $H$ is $\langle t, u \rangle \cdot T$. Both $C_H(\sigma)$ and $C_H(\rho)$ have a four group as its Sylow $2$-subgroup and have unique Sylow $3$-subgroup $T$. Let $T_1 = C_M(u)$, $T_2 = C_M(tu)$. We have

\[ M = C_M(t)C_M(u)C_M(tu) = TT_1T_2 \text{ and } T_1 \cap T_2 \cap T = \langle \sigma \rangle. \]

Now we consider $C_G(T_1)$. By (2.5), $T$ is conjugate to $T_1$ in $G$. So we have $C_G(T_1) = \langle u \rangle \times \bar{M}$ where $\bar{M}$ is of order $81$ and $\bar{M}$ is normal in $N_G(T_1)$. We have $\langle t, u \rangle \subseteq N_G(T_1)$ and therefore the four group $\langle t, u \rangle$ acts on $\bar{M}$. So we get $\bar{T} = C_G(t) \cap \bar{M}$, $\bar{T}_2 = C_G(u) \cap \bar{M}$ and $C_G(u) \cap \bar{M} = T_1$, all elementary abelian of order $9$ with $\bar{T} \cap T_1 \cap \bar{T}_2 = \langle \sigma \rangle$. Since we have $\bar{T} \subseteq H \cap C_G(\sigma)$, we must have $\bar{T} = T$. Because $\langle t, u \rangle \subseteq C_G(tu) \cap C_G(\sigma)$, by our remark in last paragraph we get $T_2 = \bar{T}_2$. Thus $M = \bar{M}$. This means that $\langle T, T_1 \rangle \subseteq Z(M)$ and so $M$ is abelian as required.

Thus we have proved the following lemma.

(3.1) Lemma. The centralizer of $T$ in $G$ is a splitting extension of an
elementary abelian group $M$ of order 81 by $\langle t \rangle$. The normalizer of $T$ in $G$ is the group $\langle u, v \rangle M$ where $C_M(t) = T$; $C_M(u) = T_1$; $C_M(tu) = T_2$; $T \cap T_1 \cap T_2 = \langle \sigma \rangle$ and the groups $T$, $T_1$, $T_2$ are elementary abelian of order 9.

Next we take a look at $C_G(\sigma_1)$. By (3.1), we have $M \subseteq C_G(\sigma_1)$. By the structure of $H$, we get $C_G(\sigma_1) \cap H = T \cdot \langle a_2, b_2 \rangle$. Let $U$ be a Sylow 2-subgroup of $C_G(\sigma_1)$ containing $\langle a_2, b_2 \rangle$. If $U$ properly contains $\langle a_2, b_2 \rangle$, we would get that $C_G(\sigma_1) \cap H$, has a Sylow 2-subgroup properly containing $\langle a_2, b_2 \rangle$, a contradiction. Hence a Sylow 2-subgroup of $C_G(\sigma_1)$ is quaternion of order 8. Let $V = O(C_G(\sigma_1))$, the maximum odd-order normal subgroup of $C_G(\sigma_1)$. By Suzuki [9], the factor group $C_G(\sigma_1)/V$ has only one involution $t \cdot V$ and so $\langle t \rangle V$ is normal in $C_G(\sigma_1)$. By the Frattini argument

$$C_G(\sigma_1) = (C_G(\sigma_1) \cap C_G(t)) V = \langle a_2, b_2 \rangle T \cdot V.$$ 

Because $\langle a_2, b_2 \rangle T$ is not 3-closed, it follows that $T \nsubseteq V$ and so $T \cap V = \langle \sigma_1 \rangle$. We get $C_G(\sigma_1) = \langle a_2, b_2, \sigma_2 \rangle V \triangleleft V$ where $L_2 \simeq SL(2, 3)$. Since $C_G(t) \cap V = \langle \sigma_1 \rangle$, it follows that $t$ acts fixed-point-free on $V/\langle \sigma_1 \rangle$. So $V/\langle \sigma_1 \rangle$ is abelian. Hence $V' \subseteq \langle \sigma_1 \rangle \subseteq Z(V)$ and $V$ is nilpotent of class at most 2.

We have therefore proved the following lemma.

(3.2) **Lemma.** The centralizer of the element $\sigma_1$ in $G$ is the group $L_2 V$ where $L_2 = \langle a_2, b_2 \sigma_2 \rangle$ and $V = O(C_G(\sigma_1))$ is odd-order and nilpotent of class at most 2.

The proof of the next lemma is rather involved.

(3.3) **Lemma.** We have that $N_G(M)/M$ is isomorphic to $A_6$, the alternating group in 6 letters.

**Proof.** Since $M$ is characteristic in $N_G(T)$, we get $\langle u, v \rangle \subseteq N_G(M)$. Let $U \supseteq \langle u, v \rangle$ be a Sylow 2-subgroup of $N_G(M)$. If $U \supseteq \langle u, v \rangle$, this would imply that $C_G(t) \cap U \supseteq \langle u, v \rangle$. Since $C_G(t) \cap M = T$ is normalized by $C_G(t) \cap U$, this would give a contradiction to the structure of $C_G(t)$. Hence $U = \langle u, v \rangle$ and a Sylow 2-subgroup of $N_G(M)$ is dihedral of order 8.

Since the four group $\langle t, u \rangle$ acts on $O(N_G(M))$ and $C_M(\langle t, u \rangle) = \langle \sigma \rangle$, we get $O(N_G(M)) = M$. Now suppose that $N_G(M) = N_G(T)$, then $M$ is a Sylow 3-subgroup of $G$. The groups $T$ and $T_1$, being conjugate in $G$, should be conjugate in $N_G(M)$, by a theorem of Burnside [4], a contradiction. So we get that $N_G(M) \supset N_G(T)$.

By (3.2), $C_G(T) = \langle t \rangle M$ and so $C_G(M) = M$. Hence $N_G(M)/M$ is isomorphic to a subgroup of $GL(4, 3)$. Since $C_G(t) \cap N_G(M) = \langle u, v \rangle T$. we get $C(tM) \cap (N_G(M)/M) = \langle u, v \rangle M/M$. We are now in a position to use the result of Gorenstein-Walter [3], giving $N_G(M)/M \cong A_7$; $PSL(2, 7)$; $PSL(2, 9)$; $PGL(2, 3)$ or $PGL(2, 5)$. Because 7 does not divide $|GL(4, 3)|$,
we have $N_G(M)/M$ is isomorphic to $PSL(2, 9)$; $PGL(2, 3)$ or $PGL(2, 5)$. Suppose that $N_G(M)/M$ is isomorphic to $PGL(2, 3)$ or $PGL(2, 5)$. Let $K$ be a subgroup of index 2 in $N_G(M)$. Then a Sylow 2-subgroup of $K$ is either $\langle t, u \rangle$ or $\langle t, uv \rangle$. First suppose that it is $\langle t, u \rangle$. We have then $F/M$ is isomorphic to $A_4$ or $A_5$. In either case, there exists an element $\mu$ of 3-power order in $F$ such that $N_G(t, u) \cap F = \langle \sigma, \mu \rangle$ where $\langle \sigma, \mu \rangle$ is a group of order 9 and $\mu^{-1}tu = u, \mu^{-1}u\mu = t, \mu^{-1}u\mu = t$. The group $\langle \sigma, \mu \rangle$ is either elementary abelian or cyclic of order 9. Since $C_G(t, u) = E_1 \langle \sigma \rangle$, and $E_1$ is characteristic in $C_G(t, u)$, we have $E_1 \triangleleft N_G(t, u)$. By (2.7), a Sylow 3-subgroup of $N_G(E_1)$ is elementary abelian of order 9. Hence we have shown that $\mu$ is an element of order 3 and $\langle t, u \rangle\langle \mu \rangle \cong A_4$.

Put $\mathcal{M} = M \langle \mu \rangle$. It follows that

$$T_1 = M \cap C_G(u) = T^\mu \quad \text{and} \quad T_2 = M \cap C_G(tu) = T^{\mu^2}.$$  

Let $\rho = \sigma_1\sigma_2^{-1}$. Then

$$T = \langle \sigma, \rho \rangle, \quad T_1 = \langle \sigma, \rho^\mu \rangle, \quad T_2 = \langle \sigma, \rho^{\mu^2} \rangle.$$  

So every element of $M$ can be written uniquely in the form $\sigma^\alpha \rho^\beta \rho_1^\gamma \rho_2^\delta$ where $\rho_1 = \rho^\mu; \rho_2 = \rho^{\mu^2}; \alpha, \beta, \gamma, \delta = 0, 1 \text{ or } -1$. Therefore the structure of $\mathcal{M}$ is completely determined. Since $\mathcal{M}$ is non-abelian, we have

$$Z(\mathcal{M}) = C_M(\mu) = \langle \sigma, \rho \rho_1 \rho_2 \rangle.$$  

An easy computation shows that $\mathcal{M}' = \langle \rho \rho_1 \rho_2, \rho_1^{-1} \rangle$, which is elementary abelian of order 9. Since $Z(\mathcal{M}) \neq \mathcal{M}'$, we get $C_G(\mathcal{M}) = M$ and therefore $M \triangleleft N_G(\mathcal{M})$. This gives $N_G(\mathcal{M}) \subseteq N_G(M)$ and in particular $\mathcal{M}$ is a Sylow 3-subgroup of $G$.

Let $M_1 = M \cap V$. Suppose that $V$ has a characteristic subgroup $X$ of order $\geq 9$ contained in $M$. Then $X \triangleleft C_G(\sigma_1)$ and so $C_G(X) \cap C_G(\sigma_1)$ is normal in $C_G(\sigma_1)$. Suppose that $t \in C_G(X)$. Then $X \subseteq C_G(t) \cap V = \langle \sigma_1 \rangle$, a contradiction to our assumption. Thus $\langle \sigma_2 \rangle = C_G(X) \cap L_2$, which would imply that $\langle \sigma_2 \rangle$ is normal in $L_2$, a contradiction. Hence $V$ does not have any characteristic subgroup of order $\geq 9$ contained in $M_1$. It follows that $M_1$ is not a Sylow 3-subgroup of $V$. Let $M_2 \supseteq M_1$ be a Sylow 3-subgroup of $V$. Then $[M_2 : M_1] = 3$ and so $\langle M_2, \sigma_2 \rangle$ is a Sylow 3-subgroup of $G$. If $M_2$ were abelian, then $C_G(M_1) \supseteq \langle M_2, \sigma_2 \rangle$ and so $M_1 \subseteq Z(\langle M_2, \sigma_2 \rangle)$, which contradicts $|Z(\mathcal{M})| = 9$. Hence $M_2$ is non-abelian and so $\langle \sigma_1 \rangle \subseteq Z(M_2) \subseteq M_1$. Thus we get $Z(M_2) = \langle \sigma_1 \rangle$ and also $M_2 = \langle \sigma_1 \rangle$. Since $M_2$ is a 3-group of class at most 2, it follows that $M_2$ is regular (in the sense of P. Hall). If $M_2$ were not of exponent 3, then $M_1$ would be characteristic in $M_2$, a contradiction. It follows that the Frattini group $\phi(M_2) = \langle \sigma_1 \rangle$ and so $M_2/\langle \sigma_1 \rangle$ is a 'vector space' of dimension 3 over the field of 3 elements $F_3$.
For any two elements \( \bar{x} = x<\sigma_1> \), \( \bar{y} = y<\sigma_1> \) of \( M_2/<\sigma_1> \) where \( x, y \in M_2 \), define \([\bar{x}, \bar{y}] = c\) where \( c \in F_3 \) and \([x, y] = x^{-1} y^{-1} xy = \sigma_1^c\). Then \([\bar{x}, \bar{y}]\) is a non-singular bilinear skew symmetric form defined on \( M_2/<\sigma_1> \) with values in \( F_3 \) [5]. But then the dimension of \( M_2/<\sigma_1> \) must be even by [1], a contradiction.

An identical proof applies when a Sylow 2-subgroup of \( K \) is \( \langle t, uv \rangle \). Therefore we have shown that \( N_G(M)/M \) is isomorphic to \( A_6 \).

We shall now begin the determination of the structure of a Sylow 3-subgroup of \( G \). But first, we look at the structure of \( N_G(M) \) more closely. Since the normalizer of a four group in \( A_6 \) is of order 24, there exists an element \( \mu \) of 3-power order such that \( N_G(t, u) \cap N_G(M) = \langle u, v \rangle \cdot \langle \sigma, \mu \rangle \). By the same reasoning as in (3.3), we conclude that \( \mu \) is of order 3 and we have \( \mu^{-1} tu = u, \mu^{-1} u\mu = tu \).

Let \( \mathcal{S} \) be the isomorphism of \( N_G(M)/M \) onto \( A_6 \). Without loss of generality, we may suppose that \( (vM)\mathcal{S} = (1324)(56), (uM)\mathcal{S} = (13)(24) \) and choosing \( \mu \) in \( N_G(t, u) \) suitably, we may assume that \( (\mu M)\mathcal{S} = (132) \). Let \( z \in N_G(M) \) such that \( (zM)\mathcal{S} = (12)(45) \). Then we have

\[
\begin{align*}
(\mu M)\mathcal{S} &= (132) = e_1; \\
(tM)\mathcal{S} &= (12)(34) = e_2; \\
(zM)\mathcal{S} &= (12)(45) = e_3; \\
(\mu u\mu M)\mathcal{S} &= (12)(56) = e_4, \ldots.
\end{align*}
\]

By Moore, we have \( A_6 = \langle e_1, e_2, e_3, e_4 \rangle \). Next we may represent \( N_G(M)/M \) as linear transformations on the vector space, \( M \) over the field of 3 elements in term of the basis \( \sigma, \rho, \rho_1 = \rho^\mu, \rho_2 = \rho^{\mu^2} \). The representation is faithful since \( C_G(M) = M \). Hence we get

\[
\begin{align*}
\mu M &\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \\
t M &\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix};
\end{align*}
\]

\[
\begin{align*}
u M &\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

From the relations \( v^2 = t, v^{-1} uv = tu \), we get \( v \) is represented by the matrix

\[
\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}
\]
interchanging \( v \) by \( v^{-1} \) if necessary.
Let \((zM)\) be represented by \((\alpha_{ij}) \in GL(4, 3)\). Then from the relation \((\mu z M)^2 = M\), we get that \(z\) is represented by

\[
\begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{12} & \alpha_{12} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\
\alpha_{21} & \alpha_{23} & \alpha_{24} & \alpha_{22} \\
\alpha_{21} & \alpha_{24} & \alpha_{23} & \alpha_{22}
\end{pmatrix}
\]

and from \((zM)^2 = M\), we get

\[
(\ast \ast)
\begin{pmatrix}
\alpha_{11}^2 & \alpha_{12} \cdot s & \alpha_{12} \cdot s & \alpha_{12} \cdot s \\
\alpha_{21} \cdot s & g + h_1 & g + h_2 & g + h_2 \\
\alpha_{21} \cdot s & g + h_2 & g + h_1 & g + h_2 \\
\alpha_{21} \cdot s & g + h_2 & g + h_2 & g + h_1
\end{pmatrix}
= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}
\]

where

\[
s = \alpha_{11} + \alpha_{22} + \alpha_{23} + \alpha_{24},
\]

\[
g = \alpha_{12} \alpha_{21},
\]

\[
h_1 = \alpha_{22}^2 + \alpha_{23}^2 + \alpha_{24}^2,
\]

\[
h_2 = \alpha_{22} \alpha_{23} + \alpha_{23} \alpha_{24} + \alpha_{24} \alpha_{22}.
\]

We have \((z \cdot tuv M) \rightarrow (456)\). Therefore the group \(M \langle \mu, ztu v \rangle\) is a Sylow 3-subgroup of \(N_G(M)\). As before, put \(\mathcal{M} = M \langle \mu \rangle\). By the proof in (3.3), we have \(Z(\mathcal{M}) = \langle \alpha, \rho_1 \rho_2 \rangle\); \(\mathcal{M}' = \langle \rho_1 \rho_2, \rho_1^{-1} \rangle\). Hence \(Z(\mathcal{M}) \cap \mathcal{M}' = \langle \rho_1 \rho_2 \rangle\) is characteristic in \(\mathcal{M}\) and so \(\langle \rho_1 \rho_2 \rangle\) is normal in \(M \langle \mu, ztu v \rangle\). Therefore we have \(\rho_1 \rho_2\) centralized by \(\lambda = ztu v\).

Now \(\lambda\) is represented by the matrix

\[
\begin{pmatrix}
-x_{11} & \alpha_{12} & \alpha_{12} & \alpha_{12} \\
-x_{21} & \alpha_{22} & \alpha_{24} & \alpha_{23} \\
-x_{21} & \alpha_{23} & \alpha_{22} & \alpha_{24} \\
-x_{21} & \alpha_{24} & \alpha_{23} & \alpha_{22}
\end{pmatrix}
\]

From \((\lambda M)^3 = M\), we get \(x_{11} = -1\). Since \(\lambda\) commute with \(\rho_1 \rho_2\), we obtain \(x_{22} + x_{23} + x_{24} = 1\). Since \((tz)^3 \in M\), this implies that \(\alpha_{12} \alpha_{21} (1 + \alpha_{22}) = -1\) (by working at the \(1, 1\) entry of the representation of \(tz\)). Therefore \(\alpha_{12} \alpha_{21} \neq 0\). First suppose that \(\alpha_{12} \alpha_{21} = 1\). Then we have \(x_{22} = 1\). By \((\ast \ast)\), we get \(h_2 = -1\). So we obtain \(x_{24} = -x_{23} \neq 0\). Hence \(tz\) is represented by the matrix

\[
tz M \rightarrow
\begin{pmatrix}
-1 & \alpha_{12} & \alpha_{12} & \alpha_{12} \\
\alpha_{12} & 1 & \alpha_{23} & -\alpha_{23} \\
-\alpha_{12} & -\alpha_{23} & \alpha_{23} & -1 \\
-\alpha_{12} & \alpha_{23} & -1 & -\alpha_{23}
\end{pmatrix}
\]

and we check that \((tz)^3 \notin M\), a contradiction.
Thus we must have $\alpha_{12} \alpha_{21} = -1$. Then $\alpha_{22} = 0$, from $\alpha_{22} + \alpha_{23} + \alpha_{24} = 1$, we get $\alpha_{23} = \alpha_{24} = -1$. Hence we have $z$ represented by

$$zM \mapsto \begin{pmatrix} -1 & \alpha_{12} & \alpha_{12} & \alpha_{12} \\ -\alpha_{12} & 0 & -1 & -1 \\ -\alpha_{12} & -1 & -1 & 0 \\ -\alpha_{12} & -1 & 0 & -1 \end{pmatrix}$$

and

$$\lambda M \mapsto \begin{pmatrix} -1 & \alpha_{12} & \alpha_{12} & \alpha_{12} \\ -\alpha_{12} & 0 & -1 & -1 \\ -\alpha_{12} & -1 & 0 & -1 \\ -\alpha_{12} & -1 & 1 & 0 \end{pmatrix}.$$  

Interchanging $\lambda$ by $\lambda^{-1}$, if necessary, we may suppose that $\alpha_{12} = -1$.

Now $M\langle \lambda, \mu \rangle$ is a Sylow 3-subgroup of $N_G(M)$ and by the structure of $A_6$, the commutator $[\lambda, \mu] \in M$. Since $M$ is abelian, and $M\langle \lambda, \mu \rangle$ is not, we get $Z(M\langle \lambda, \mu \rangle) = C_M\langle \lambda, \mu \rangle = \langle \rho_1 \rho_2 \rangle$. An easy computation shows that the commutator group of $M\langle \lambda, \mu \rangle$ contains $\langle \sigma, \rho_1 \rho_2, \rho_1^{-1} \rangle$ and is contained in $M$. Since $Z(M\langle \lambda, \mu \rangle) \neq (M\langle \lambda, \mu \rangle)'$, we see that $M$ is characteristic in $M\langle \lambda, \mu \rangle$. So we have $N_G(M\langle \lambda, \mu \rangle) \subseteq N_G(M)$. Hence $M\langle \lambda, \mu \rangle$ is a Sylow 3-subgroup of $G$ and moreover, by the structure of $A_6$, the normalizer of $M\langle \lambda, \mu \rangle$ is a splitting extension of $M\langle \lambda, \mu \rangle$ by a group of order 4.

Next we check that we have $z' = (\mu^2 t z)^3$ such that $(z')\rho_1 \rho_2 z' = \sigma p$. Let $\mu' = (z')^{-1} \mu z$ and $\lambda' = (z')^{-1} \lambda z'$, we see that $\langle \lambda', \mu' \rangle \subseteq C_G(\sigma_1)$. and that $\mu', \lambda'$ are represented by the following matrices.

$$\mu' M \mapsto \begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}; \quad \lambda' M \mapsto \begin{pmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}.$$  

Therefore we have

$$(\mu')^{-1} \lambda' M \mapsto \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad (\mu')^{-1} (x')^{-1} M \mapsto \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}.$$  

The group $M\langle \lambda', \mu' \rangle$ is contained in $C_G(\sigma_1)$. We turn our attention back to $C_G(\sigma_1)$. Let $U_1 \subseteq V$ be the Sylow 3-subgroup of $V$. We have $M \cap V = M_1$ is elementary abelian of order 27. Suppose that $\rho_1 = \rho^* \notin M_1$.  

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Then we have $\rho_1 = \sigma_1^j m$ for some fixed $j = 1$ or $-1$ and $m \in M_1$. Now $t$ acts fixed-point-free on $V/\langle \sigma_1 \rangle$. Therefore we get

$$t\rho_1 t = \sigma_2^j m^{-1} \sigma_1^j = \rho_1^{-1} = \sigma_2^{-j} m^{-1} \sigma_1^j$$

giving $\sigma_2^j = \sigma_1^j$, a contradiction. Similarly we can show that $\rho_2 = \rho_3^k \in M_1$.

Let $\langle \rho_2^a, \rho_3^b \rangle = \langle \rho_3, \rho_4 \rangle \subseteq U_1$. By way of contradiction, suppose that $\langle \rho_3, \rho_4 \rangle \cap M_1$ is non-empty. Then there exists an element $\rho_2^i \rho_4^j \in M_1$ for fixed $i,j$ not both zero. Since $\sigma_2$ centralize $M_1$, we would then get $b_2^{-1} \rho_2^i \rho_4^j b_2 = a_2^{-1} \rho_2^i \rho_4^j a_2$. This is a contradiction, since $C_G(t) \cap V = \langle \sigma_1 \rangle$. Thus $\langle \rho_3, \rho_4 \rangle \subseteq M_1$. Since a Sylow 3-subgroup of $G$ is of order $3^6$ we must have $U_1 = \langle \sigma_1, \rho_1, \rho_2, \rho_3, \rho_4 \rangle$.

The group $U_1/\langle \sigma_1 \rangle$ is abelian and so is elementary abelian of order 81. We may then represent the group $L_2 = \langle a_2, b_2, \sigma_2 \rangle$ as linear transformations on the 'vector space' $U_1/\langle \sigma_1 \rangle$ over the field of 3 elements. We get in terms of the basis $\rho_1\langle \sigma_1 \rangle, \rho_2\langle \sigma_1 \rangle, \rho_3\langle \sigma_1 \rangle, \rho_4\langle \sigma_1 \rangle$, the representation of $a_2$

$$a_2 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have shown that $v^{-1}\rho_1 v = \rho_2, v^{-1}\rho_2 v = \rho_1^{-1}$. Therefore with the relation $v^{-1}a_2 v = a_2^{-1}$, we get

$$v \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $\sigma_2$ be represented by the matrix

$$\sigma_2 \mapsto \begin{pmatrix} I & C \\ 0 & D \end{pmatrix}$$

where $(C)$ and $(D)$ are $2 \times 2$ matrices. From the relation $(a_2 \sigma_2)^3 = 1$, we get that $(C) = (-D^{-1})$. Since $(D)$ is non-singular, we have $\det(D) = \pm 1$. Suppose $\det(D) = -1$, then using the relation $v^{-1}\sigma_2 v = \sigma_2^{-1}$, we obtain a contradiction. Hence $\det(D) = 1$. Again by the relation $v^{-1}\sigma_2 v = \sigma_2^{-1}$, we obtain that $(D) = \text{identity matrix}$. Hence $\sigma_2$ is represented by

$$\sigma_2 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It follows that \( \langle \rho_3, \rho_4 \rangle \subseteq N_G(M) \cap C_G(\sigma_1) - M \). So comparing the action of the group \( \langle \lambda', \mu' \rangle \) on \( M \), we conclude that \( \rho_3 M = (\mu')^{-1} \lambda' M \) and \( \rho_4 M = (\mu')^{-1} (\lambda')^{-1} M \). We have

\[
N_G(P) = U_1(N_G(\sigma_2) \cap L_2\langle v \rangle) = U_1\langle \sigma_2 \rangle \langle v \rangle = P \langle v \rangle
\]

where \( P = U_1\langle \sigma_2 \rangle \). Thus we have proved the following lemma.

(3.4) Lemma. The group \( P = M \langle \rho_3, \rho_4 \rangle \) is a Sylow 3-subgroup of \( G \) and has the following structure:

\[
M = TT_1 T_2,
\]

an elementary abelian group of order 81 where

\[
T = C_M(t) = \langle \sigma, \rho \rangle,
\]

\[
T_1 = C_M(u) = \langle \sigma, \rho_1 \rangle,
\]

\[
T_2 = C_M(tu) = \langle \sigma, \rho_2 \rangle,
\]

elementary abelian of order 9 and

\[
\rho_3^{-1} \sigma_1 \rho_3 = \sigma_1; \quad \rho_3^{-1} \sigma_2 \rho_3 = \sigma_2 \rho_3 \sigma_1; \quad \rho_3^{-1} \rho_1 \rho_3 = \rho_1 \sigma_1^{-1}; \quad \rho_3^{-1} \rho_2 \rho_3 = \rho_2;
\]

\[
\rho_4^{-1} \sigma_1 \rho_4 = \sigma_1; \quad \rho_4^{-1} \sigma_2 \rho_4 = \sigma_2 \rho_4 \sigma_1; \quad \rho_4^{-1} \rho_1 \rho_4 = \rho_1; \quad \rho_4^{-1} \rho_2 \rho_4 = \rho_2 \sigma_1^{-1}.
\]

Moreover \( N_G(P) = P \cdot \langle v \rangle \) where

\[
v^{-1} \rho_1 v = \rho_2, \quad v^{-1} \rho_2 v = \rho_1^{-1}, \quad v^{-1} \rho_3 v = \rho_4^{-1}, \quad v^{-1} \rho_4 v = \rho_3.
\]

4. Final characterization

Using the informations already found, we shall now prove that \( G \) is isomorphic to \( U_4(3) \). The following preliminary lemmas are required.

(4.1) Lemma. The group \( P \) and its conjugate \( t_1 P t_1 \) have trivial intersection.

Proof. We have \( P \subseteq C_G(\sigma_1) \). Therefore

\[
P \cap t_1 P t_1 \subseteq C_G(\sigma_1) \cap C_G(\sigma_1^2) \subseteq C_G(\sigma_1) \cap C_G(a_1 b_1) = \langle \sigma_2 \rangle.
\]

The group \( P \cap t_1 P t_1 \) is normalized by \( t_1 \). So it follows that \( P \cap t_1 P t_1 = 1 \).

(4.2) Lemma. We have the following relations:

\[
(a_2 \sigma_2)^3 = (ut \rho_3)^3 = (u \rho_4)^3 = (v \rho_3 \rho_4)^3 = (tv \rho_3 \rho_4)^3 = 1.
\]

Proof. Using our representation, of \( N_G(M) \) as linear transformation on the vector space \( M \), we compute that \( (ut \rho_3)^3 \in M \). Since \( \rho_3 = a_2^{-1} \rho_1 a_2 \), we have \( ut \rho_3 \in C_G(ut_1) \). So \( (ut \rho_3)^3 \in M \cap C_G(ut_1) \subseteq P \cap C_G(ut_1) = \langle \rho_2 \rangle \). Therefore we get \( (ut \rho_3)^3 = 1 \).
Next we have \( u\rho_4 = v(ut\rho_3)v^{-1} \). So we get \((u\rho_4)^3 = 1\). Again from our representations of \( uv \) and \( \rho_3^{-1}\rho_4 \), we verify that \((uv\rho_3^{-1}\rho_4)^3 \in M\). Also we have \( u\rho_3^{-1}\rho_4 \in C_G(uvt_1)\). Hence

\[
(u\rho_3^{-1}\rho_4)^3 \subseteq M \cap C_G(uvt_1) \subseteq P \cap C_G(uvt_1) = \langle \rho_3^{-1}\rho_4 \rangle.
\]

Showing that \((u\rho_3^{-1}\rho_4)^3 = 1\). By (3.4) we have \((tvu\rho_3^{-1}\rho_4)^{-1} = v^{-1}(uwv_3^{-1}\rho_4)v\).

By the structure of \( H \), we know that \((a_2\sigma_3)^3 = 1\).

The assertions of this lemma are completely proved.

(4.3) **Lemma.** The group \( W = N_G\langle v \rangle \langle v \rangle \) is generated by the involutions \( r_1 = a_2\langle v \rangle \) and \( r_2 = u\langle v \rangle \) and is dihedral of order 8.

**Proof.** Obvious from the structure of \( H \).

Put \( B = N_G(P) \) and \( N = N_G\langle v \rangle \). We want to show that the set of elements in \( BNB \) forms a subgroup of \( G \). For any \( w \in W \), define \( l(w) = l \) to be the smallest positive integer such that \( w = r_1 r_2 \cdots r_t \) where \( r_i \in \{ r_1, r_2 \} \). Let \( \omega(r_1) = a_2, \omega(r_2) = u \). For any \( w \in W \), and \( w = r_1 \cdots r_t \), define \( \omega(w) = \omega(r_1) \cdots \omega(r_t) \). We shall denote \( BwB \) to mean \( B \omega(w)B \).

(4.4) **Lemma.** The set of elements in \( B \cup Br_iB \ (i = 1, 2) \) forms a subgroup of \( G \).

**Proof.** Let \( g = b\omega(r_i)b' \in Br_i B \) where \( b, b' \in B \). Then the element \( g' = (b')^{-1}\omega(r_i)(\omega(r_i)^{-2}b^{-1}) \in Br_i B \) and is an inverse of \( g \).

Let \( G_1 = B \cup Br_1 B = B \cup Ba_2 B \). Clearly to show that \( G_1 \) is closed with respect to multiplication, we need only to show that \( a_2\sigma_2^\delta a_2 \in G_1 \) \( (\delta = 0, 1, -1) \); since \( B \) has the form \( \langle \sigma_2 \rangle \langle \langle v \rangle \langle \sigma_1, \rho_1, \rho_2, \rho_3, \rho_4 \rangle \rangle \) and \( \langle v \rangle \langle \sigma_1, \rho_1, \rho_2, \rho_3, \rho_4 \rangle \) is normalized by \( a_2 \). If \( \delta = 0 \), then \( a_2\sigma_2^\delta a_2 = t \in B \). If \( \delta = 1 \), then by (4.3), \( a_2\sigma_2^\delta a_2 = \sigma_2^{-2}a_2(t\sigma_2^{-1}) \in Ba_2 B \). Similarly of \( \delta = -1 \), we get \( a_2\sigma_2^{-1}a_2 = t\sigma_2 a_2 \sigma_2 t \in Ba_2 B \). Hence we have shown that \( G_1 \) is a subgroup of \( G \).

Next to show that \( G_2 = B \cup Br_2 B \) is a subgroup of \( G \), we need to show that \( u\rho_3^{i} u\rho_4^{j} \in G_2 \ (i, j = 0, 1, -1) \). By using (4.3), and similar reasoning as in the last case, this is in fact true.

(4.5) **Lemma.** For any \( i \) and \( w \in W \), if \( l(r_i w) \geq l(w) \), then \( r_i Bw \subseteq Br_i wB \).

**Proof.** First of all, we construct table I showing the action of \( a_2 \) and \( u \) on \( P \) by conjugation.

<table>
<thead>
<tr>
<th>( \sigma_1 )</th>
<th>( \sigma_2 )</th>
<th>( \rho_1 )</th>
<th>( \rho_2 )</th>
<th>( \rho_3 )</th>
<th>( \rho_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_2 )</td>
<td>( \sigma_2 )</td>
<td>( \rho_2 )</td>
<td>( \rho_3^{-1} )</td>
<td>( \rho_4 )</td>
<td></td>
</tr>
<tr>
<td>( u )</td>
<td>( \sigma_1 )</td>
<td>( \rho_1 )</td>
<td>( \rho_2^{-1} )</td>
<td>( \rho_3 )</td>
<td>( \rho_4 )</td>
</tr>
</tbody>
</table>

Table I
To prove this lemma, we construct table II, showing $l(r_iw)$ and $l(w)$ for all $i$ and $w \in W$. Clearly we need only to see that $r_i \sigma_2 w \subseteq Br_1 w B$ and $r_2 \rho_3 \rho_4 w \subseteq Br_2 w B$ ($i, j = 0, 1, -1$). It is easily verified that for those $w \in W$ such that $l(r_2w) \geq l(w)$, we can always get $r_1 \sigma_2 w \in Br_1 w \sigma_1$ and $r_2 \rho_3 \rho_4 w \in Br_2 \rho_3 \rho_4 y_2$, using the informations in table I. Hence the lemma is completely proved.

### Table II

<table>
<thead>
<tr>
<th>$w$</th>
<th>$l(w)$</th>
<th>$l(r_1w)$</th>
<th>$y_1$</th>
<th>$l(r_1w)$</th>
<th>$y_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$r_1$</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>$\rho_1^{-i} \rho_2^{-j}$</td>
<td></td>
</tr>
<tr>
<td>$r_2$</td>
<td>1</td>
<td>2</td>
<td>$\sigma_1$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$r_1r_2$</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>$\rho_1^{-i} \rho_2$</td>
<td></td>
</tr>
<tr>
<td>$r_2r_1$</td>
<td>2</td>
<td>3</td>
<td>$\sigma_1$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$r_2r_1r_2$</td>
<td>3</td>
<td>2</td>
<td>$\rho_3^{-i} \rho_4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r_2r_1r_2$</td>
<td>3</td>
<td>4</td>
<td>$\sigma_3$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$r_1r_2r_1r_2$</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(4.6) Lemma. The set of elements $G_0 = BNB$ is a subgroup of $G$ if we have $Bw_1B = Bw_2B$, then $w_1 = w_2$.

Proof. It follows from (4.4), (4.5) and Tits [8].

We shall next compute the order of $G_0$. Define for any $w \in W$, the group $B_w$ generated by elements $x \in P$ such that $\omega(w) \times \omega(w)^{-1} \in t_1 Pt_1$. The groups $B_w$ for all $w \in W$ are shown in the next table.

### Table III

<table>
<thead>
<tr>
<th>$w$</th>
<th>1</th>
<th>$r_1$</th>
<th>$r_2$</th>
<th>$r_1r_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_w$</td>
<td>1</td>
<td>$\langle \sigma_2 \rangle$</td>
<td>$\langle \rho_3, \rho_4 \rangle$</td>
<td>$\langle \sigma_1, \rho_3, \rho_4 \rangle$</td>
</tr>
<tr>
<td>$\langle B_w \rangle'$</td>
<td>$P$</td>
<td>$\langle \sigma_1, \rho_1, \rho_2, \rho_3, \rho_4 \rangle$</td>
<td>$M$</td>
<td>$\langle \sigma_2, \rho_1, \rho_3 \rangle$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$r_2r_1$</th>
<th>$r_1r_2r_1$</th>
<th>$r_2r_1r_2$</th>
<th>$r_1r_2r_1r_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle \sigma_2, \rho_1, \rho_3 \rangle$</td>
<td>$M$</td>
<td>$\langle \sigma_1, \rho_1, \rho_3, \rho_4 \rangle$</td>
<td>$P$</td>
</tr>
<tr>
<td>$\langle \sigma_1, \rho_3, \rho_4 \rangle$</td>
<td>$P$</td>
<td>$\langle \rho_3, \rho_4 \rangle$</td>
<td>$\langle \sigma_2 \rangle$</td>
</tr>
</tbody>
</table>

We observe that for every $B_w$, there exists the subgroup $(B_w)'$ such that $B_w(B_w)' = P$ and $B_w \cap (B_w)' = 1$ (see (4.1)).

(4.7) Lemma. The order of $G_0$ is $2^7 \cdot 3^6 \cdot 5 \cdot 7$.  

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PROOF. We show first that every element of $G_0$ can be written in the ‘normal’ form $h p \omega(w) p_w$ where $h \in \langle v \rangle$, $p \in P$ and $p_w \in B_w$. By (4.6), every element $x$ in $G_0$ has the form $x = b_1 \omega(w) b_2$ where $b_1, b_2 \in B$. Since we have $P = B_w (B_w)'$ we may write $b_2 = h p' p_2$ where $h \in \langle v \rangle$, $p_2 \in B_w$ and $p' \in (B_w)'$. From the facts $\omega(w) \omega(w)^{-1} \in \langle v \rangle$ and $\omega(w) p'_2 \omega(w)^{-1} \in P$, we get $x = b_0 \omega(w) p_2$ showing the existence of the ‘normal’ form.

To show the uniqueness of the ‘normal’ form, suppose that $b_0 \omega(w) b_w = b' \omega(w') b' w$. By (4.6), we have $w = w'$. Therefore we get

$$(b')^{-1} b = \omega(w) b_w (b_w')^{-1} \omega(w)^{-1}.$$  

Since $(b')^{-1} b \in B$ and $\omega(w) b_w (b_w')^{-1} \omega(w)^{-1} \in P^4$, we obtain

$$(b')^{-1} b \in B \cap P^4 \subseteq P.$$  

The uniqueness follows by (4.1).

By (4.1), the 8 double cosets in $B N B$ are distinct, therefore we have

$$|G_0| = |B| \sum_w |B_w| = 2^7 \cdot 3^6 \cdot 5 \cdot 7.$$  

To conclude the proof of the theorem, we require the following result of Thompson.

LEMMA (Thompson). Let $\mathcal{M}$ be a subgroup of $\mathcal{G}$ such that

(a') $|\mathcal{M}|$ is even.

(b') $\mathcal{M}$ contains the centralizer of each of its involutions.

(c') $\bigcap_{x \in \mathcal{G}} \mathcal{M}^x$ is of odd order.

Let $\mathcal{S}$ be a $S_2$-subgroup of $\mathcal{M}$ and let $I$ be an involution in $Z(\mathcal{S})$. We have

(d') $N(\mathcal{S}) \subseteq \mathcal{M}$.

Then

(i) $i(\mathcal{M}) = 1$ (the number of conjugate classes of involution in $\mathcal{M}$)

(ii) $\mathcal{M}$ contains a subgroup $\mathcal{M}_0$ of odd order such that $\mathcal{M} = \mathcal{M}_0 C_{\mathcal{M}}(I)$.

Using the informations of our tables (I, II, III), (4.2) and the structures of $P$ and $\langle v \rangle$, we can multiply any two elements of $G_0$ in the ‘normal’ form to get the product uniquely in the ‘normal’ form. Now if $X$ is any finite group satisfying properties (a) and (b) of the theorem, then $X$ contains a subgroup $X_0$ of order $|U_4(3)|$ with uniquely determined multiplication table. Hence taking $X$ to be $U_4(3)$, we see that $X_0 = U_4(3)$ and so $G_0 \cong U_4(3)$.
Consequently \( G_0 \) satisfies conditions (a'), (b') and (d') of Thompson lemma. Suppose the (c') is also fulfilled, then we obtain that \( G_0 \) contains a subgroup \( M_0 \) of odd order such that \( G_0 = M_0 CG(t) = M_0 H \).

Suppose that \(|M_0 \cap H| = 3^2\), then we have \(|M_0| = 3^6 \cdot 5 \cdot 7\). Let \( S_3 \) be a Sylow 3-subgroup of \( M_0 \). By (3.4) we get \( N_{M_0}(S_3) = S_3 \). This is a contradiction since \(|M_0 : N_{M_0}(S_3)| = 5 \cdot 7 \not\equiv 1 \pmod{3}\). Hence we must have \(|M_0| = 3^4 \cdot 5 \cdot 7 \) or \( 3^5 \cdot 5 \cdot 7 \). Now \( M_0 \) is soluble and so by P. Hall (4), there exists a subgroup of order \( 5 \cdot 7 \) in \( M_0 \). Clearly \( K \) is abelian. Let \( S_7 \) be the Sylow 7-subgroup of \( K \). By Sylow's Theorem, we get that \( S_7 \) is normal in \( M_0 \). Applying Sylow's theorem again, we obtain that \( N_{G_0}(S_7) \) is \( 2^4 \cdot 3^6 \cdot 5 \cdot 7 \), \( 2^5 \cdot 3^4 \cdot 5 \cdot 7 \), \( 2^2 \cdot 3^4 \cdot 5 \cdot 7 \) or \( 2 \cdot 3^6 \cdot 5 \cdot 7 \). The first 3 cases are not possible, since this would then imply that an involution of \( G_0 \) is centralized by elements of order 7, a contradiction of structure of \( H \). Thus we have \(|N_{G_0}(S_7)| = 2 \cdot 3^6 \cdot 5 \cdot 7\). Now a Sylow 2-subgroup of \( N_{G_0}(S_7) \) is cyclic of order 2. Therefore, by Burnside [4], there is a subgroup of order \( 3^6 \cdot 5 \cdot 7 \) in \( N_{G_0}(S_7) \) and this gives a contradiction as before.

Thus we must get \( \bigcap_{g \in G} G_0^g \) is even. By (2.6), the group \( G \) is simple. Hence \( G = G_0 \cong U_4(3) \), proving our theorem.

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**References**


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