A CHARACTERIZATION OF THE FINITE SIMPLE GROUP $U_4(3)$

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The aim of this paper is to give a characterization of the finite simple group $U_4(3)$ i.e. the 4-dimensional projective special unitary group over the field of 9 elements. More precisely, we shall prove the following result.

THEOREM. *Let t_0 be an involution in $U_4(3)$. Denote by H_0, the centralizer of t_0 in $U_4(3)$.*

Let G be a finite group of even order with the following properties:

(a) *G has no subgroup of index 2,*

(b) *G has an involution t such that $H = C_G(t)$, the centralizer of t in G is isomorphic to H_0.*

Then G is isomorphic to $U_4(3)$.

We shall use the standard notation.

1. Some properties of H_0

Let F_9 be the finite field with 9 elements. Then the map: $x \to \bar{x} = x^3$ $(x \in F_9)$ is an automorphism of F_9. We extend this map to a map of $GL(4, 9)$ thus: $(\alpha_{ij}) \to \overline{(\alpha_{ij})} = (\bar{\alpha}_{ij})$ where $(\alpha_{ij}) \in GL(4, 9)$. The subgroup $SU(4, 9)$ in $GL(4, 9)$ consisting of all matrices with determinant 1 which satisfy the relation: $(\alpha_{ij}) \cdot (\alpha_{ij})^* = I$ where $(\alpha_{ij})^*$ is the transpose of $\overline{(\alpha_{ij})}$, is known as 4-dimensional special unitary group over F_9. Then $U_4(3)$ $(= PSU(4, 9))$ is the factor group $SU(4, 9)/Z(SU(4, 9))$ where $Z(SU(4, 9))$ denotes the centre of $SU(4, 9)$.

Let t_0' be the matrix

$$t_0' = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Then t_0' is an involution in $SU(4, 9)$. Now the centre of $SU(4, 9)$ is generated by the element $c = k^2 I$ where k is a fixed primitive element of the multiplicative group of F_9. So $Z(SU(4, 9)) = \langle c \rangle$ is cyclic of order 4.

Denote by H_0', the group of all matrices (α_{ij}) in $SU(4, 9)$ which 'commute projectively' with t_0' i.e. which satisfy the relation $(\alpha_{ij})t_0' = t_0'(\alpha_{ij})c_r$ $(r = 0, 1, 2, 3)$. A matrix in $SU(4, 9)$ belongs to H_0' if and only if it has the form

$$\begin{pmatrix} A & \\ & B \end{pmatrix} \text{ or } \begin{pmatrix} & B \\ A & \end{pmatrix}$$

where (A) and (B) are 2×2 matrices in $GU(2, 9)$ with $\det(A)\det(B) = 1$.

Let L_1' be the subgroup of H_0' consisting of matrices of the form

$$\begin{pmatrix} A & & \\ & 1 & 0 \\ & 0 & 1 \end{pmatrix}$$

with $(A) \in SU(2, 9)$. Since $SU(2, 9) \cong SL(2, 3)$, we can easily check that the following matrices generate L_1'

$$a_1' = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}; \quad b_1' = \begin{pmatrix} 0 & k^6 & & \\ k^6 & 0 & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix};$$

$$\sigma_1' = \begin{pmatrix} k & k^3 & & \\ k^5 & k^3 & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}.$$

Now we have the matrice u' belongs to H_0

$$u' = \begin{pmatrix} & & 1 & 0 \\ & & 0 & 1 \\ 1 & 0 & & \\ 0 & 1 & & \end{pmatrix}$$

and we get

$$u' \begin{pmatrix} A & \\ & B \end{pmatrix} = \begin{pmatrix} & B \\ A & \end{pmatrix}.$$

The matrix v'

$$v' = \begin{pmatrix} k^3 & k^3 & & \\ k^3 & k^7 & & \\ & & k & k \\ & & k & k^5 \end{pmatrix}$$

also belongs to $SU(4, 9)$. We check that $(v')^2 = t_0 c$ and $u'v'u' = (v')^{-1}$. So $\langle u', v' \rangle$ is dihedral of order 16.

Put $a_2' = u'a_1'u'$, $b_2' = u'b_1'u'$, $\sigma_2' = u'\sigma_1'u'$ and $L_2' = \langle a_2', b_2', \sigma_2'\rangle$. We can now verify that $H_0' = (L_1' \times L_2')\langle u', v'\rangle$. Let $H_0 = H_0'/\langle c\rangle$ and in the natural homomorphism from H_0' onto H_0, let the images of t_0', a_i', b_i', σ_i', L_i', u', v' $(i = 1, 2)$ be t_0, a_i, b_i, σ_i, L_i, u, v respectively. We have then H_0 is a non-splitting extension of $L = L_1L_2$ by a four group. More precisely we have the following relations:

$$\begin{aligned} H_0 &= L \cdot F \\ L &= L_1L_2 \text{ where } L_1 \cap L_2 = \langle t_0\rangle \text{ and } [L_1, L_2] = 1 \\ F &= \langle u, v\rangle, \text{ a dihedral group of order 8} \\ L_i &= \langle a_i, b_i, \sigma_i | a_i^2 = b_i^2 = t_0,\ b_i^{-1}a_ib_i = a_i^{-1},\ \sigma_i^{-1}a_i\sigma_i = b_i, \\ &\qquad\qquad \sigma_i^{-1}b_i\sigma_i = a_ib_i,\ \sigma_i^3 = 1\rangle \end{aligned}$$

and

$$v^{-1}a_iv = a_i^{-1},\ v^{-1}b_iv = b_ia_i,\ v^{-1}\sigma_iv = \sigma_i^{-1},\ v^2 = t_0.$$

The structure of H_0 is now completely determined. Of course, we have to see that the structure of H_0 is independent of the choice of t_0' in $SU(4, 9)$. This is so because we can check that $U_4(3)$ has only one conjugate class of involutions.

We shall list a few properties of H_0, which will be used in the next section.

(1.1) Every element of H_0 can be written uniquely in the form $a_1^ib_1^j\sigma_1^kt_1^lt_2^m\sigma^nu^pv^q$ where $t_1 = a_1a_2$; $t_2 = b_1b_2$; $\sigma = \sigma_1\sigma_2$; $i = 0, 1, 2, 3$; $j = 0, 1$; $k = 0, 1, 2$; $l = 0, 1$; $m = 0, 1$; $n = 0, 1, 2$; $p = 0, 1$; $q = 0, 1$. The order of H_0 is $2^7 \cdot 3^2$.

(1.2) The group $Q = \langle a_1, a_2, b_1, b_2\rangle F \subseteqq H_0$ is a Sylow 2-subgroup of H_0. The centre $Z(Q)$ of Q is $\langle t_0\rangle$.

(1.3) There are 4 conjugate classes of involutions in H_0 with representatives t_0, t_1, u, uv. We have the centralizer $C_{H_0}(t_1) = A$ of t_1 in H_0 is the group $\langle a_1, a_2, t_2, u, v\rangle$, a non-abelian group of order 64. We have the centre $Z(A)$ of A is $\langle t_0, t_1\rangle$, a four group. The commutator group A' of A is also $\langle t_0, t_1\rangle$. The centralizer of u, $C_{H_0}(u)$ in H_0 is $E_1\langle\sigma\rangle$ where $E_1 = \langle t_0, t_1, t_2, u\rangle$, an elementary abelian group of order 16. The centralizer of uv, $C_{H_0}(uv)$ in H_0 is $E_2\langle\sigma_1\sigma_2^{-1}\rangle$ where E_2 is $\langle t_0, t_1, t_3, uv\rangle$ $(t_3 = a_1b_1b_2)$, an elementary abelian group of order 16.

(1.4) Both E_1 and E_2 are normal in the group Q. We have $N_{H_0}(E_1) = Q\langle\sigma\rangle$ and the factor group $N_{H_0}(E_1)/E_1$ is isomorphic to S_4, the symmetric group in 4 letters. Similarly we have $N_{H_0}(E_2) = Q\langle\rho\rangle$ $(\rho = \sigma_1\sigma_2^{-1})$ and the factor group $N_{H_0}(E_2)/E_2$ is isomorphic to S_4.

(1.5) The group L is the smallest normal subgroup of H_0 with 2-factor group and H/L is a four-group.

(1.6) A Sylow 3-subgroup T of H_0 is $\langle \sigma_1, \sigma_2 \rangle$, an elementary abelian group of order 9. We have $C_{H_0}(T) = \langle t_0 \rangle \times T$ and $N_{H_0}(T) = \langle u, v \rangle T$.

2. Conjugacy of involutions

Let G be a finite group with properties (a) and (b) of the theorem. Since the group $H = C_G(t)$ is isomorphic to H_0. We shall identify H with H_0. Then we have $t_0 = t$.

(2.1) LEMMA. *The Sylow 2-subgroup Q of H is a Sylow 2-subgroup of G.*

PROOF. This is obvious since $Z(Q) = \langle t \rangle$ is cyclic of order 2.

(2.2) LEMMA. *If the involution u is conjugate to t in G, then t_1 is conjugate to t in G.*

PROOF. Since by assumption u is conjugate to ι in G, there exists a Sylow 2-subgroup of $C_G(u)$ properly containing $E_1 = \langle t, t_1, t_2, u \rangle$. Therefore there is an element x in $C_G(u) - H$ which normalizes E_1. Let us look more closely at the involutions in E_1. We have

$$C_1 = \{t_1, tt_1, t_2, tt_2, t_1t_2, tt_1t_2\}$$

whose elements are conjugate in H and likewise

$$C_2 = \{u, t_1u, t_2u, t_1t_2u, tu, tt_1u, tt_2u, tt_1t_2u\}$$

with elements conjugate in H. We see that $C_1 \cup C_2 \cup \{t\} = E_1 - \{1\}$.

Since $x \notin H$, we must have $x^{-1}tx \neq t$. If $x^{-1}tx \in C_1$ or $x^{-1}t_1x \in C_2$, then we are finished. Therefore we may suppose that $x^{-1}tx \in C_2$ and $x^{-1}t_1x \in C_1$. Then we get $x^{-1}tt_1x \in C_2$. Since tt_1 is conjugate to t_1, the lemma is proved.

(2.3) LEMMA. *If the involution uv is conjugate to t in G, then t_1 is conjugate to t in G.*

PROOF. As in (2.2) with E_2 playing the role of $E_1 - \{1\}$.

For the pıoof of next lemma, we need an unpublished result of Thompson.

LEMMA (Thompson [7]). *Suppose $\mathfrak{G}$ is a finite group of even order which has no subgroup of index 2. Let $\mathscr{S}_2$ be a Sylow 2-subgroup of $\mathfrak{G}$ and let $\mathscr{M}$ be a maximal subgroup of $\mathscr{S}_2$. Then for each involution I of $\mathfrak{G}$, there is an element B of $\mathfrak{G}$ such that $B^{-1}IB \in \mathscr{M}$.*

(2.4) LEMMA. *If the involution t_1 is conjugate to t in G, then G has only one conjugate class of involutions.*

PROOF. We have by (2.1) that Q is a Sylow 2-subgroup of G. The group $M = \langle a_1, a_2, b_1, b_2, v\rangle$ is a maximal subgroup of Q. By our assumption, we have one class of involutions in M. The lemma follows from condition (a) of the theorem and Thompson's lemma.

(2.5) LEMMA. *There is only one class of involutions in G.*

PROOF. First we want to show that the group G is not 2-normal. By way of contradiction, suppose that it is 2-normal. Since $\langle t\rangle$ is the centre of a Sylow 2-subgroup Q of G. It follows by Hall-Grün's theorem [4], that the greatest factor group of G which is a 2-group is isomorphic to that of $N_G(Z(Q)) = H$, i.e. by (1.5) isomorphic to H/L which is of order 4. But this is a contradiction to condition (a) of the theorem. It follows that G is not 2-normal. This means that there is an element $z \in G$ such that $t \in Q \cap z^{-1}Qz$ but $\langle t\rangle$ is not the centre of $z^{-1}Qz$.

The centre of $z^{-1}Qz$ is $\langle z^{-1}tz\rangle$. So $z^{-1}tz \neq t$. On the other hand, we have $t \in z^{-1}Qz$. It follows that t and $z^{-1}tz$ commute. Hence $z^{-1}tz \in H$. Without loss of generality, we may assume that $z^{-1}tz \in \{t_1, u, uv\}$. The lemma follows now by (2.2); (2.3) and/or (2.4).

(2.6) LEMMA. *The group G is simple.*

PROOF. Suppose at first that $O(G) \neq 1$ where $O(G)$ denotes the maximal odd-order normal subgroup of G. Then the four group $\langle t, t_1\rangle$ acts on G. By the structure of H and (2.5), we see that $C_G(x)$ does not have a non-trivial intersection with $O(G)$ for $x \in \langle t, t_1\rangle$. Hence $\langle t, t_1\rangle$ acts fixed-point-free on $O(G)$ which is not possible. Hence we have that $O(G) = 1$.

Suppose next that N is a proper normal subgroup of G such that $|G/N|$ is odd. We have then $H \subseteqq N$ since H does not have a proper odd-order factor group. We have that $Q \subseteqq N$. By Frattini argument, $G = N \cdot N_G(Q)$. But then $N_G(Q) \subseteqq N_G\langle t\rangle = H$. So $G = N$, a contradiction.

Lastly suppose that G is not a simple group. Then G must have a proper normal subgroup K such that both $|K|$ and $|G/K|$ are even. Since by (2.5), all involutions of G are in K. This implies that $Q \subseteqq L$ since Q is generated by its involutions, a contradiction to our assumption. The proof is now complete.

(2.7) LEMMA. *The group $N_G(E_i)/E_i$ is isomorphic to A_6, the alternating group in 6 letters ($i = 1, 2$).*

PROOF. By (2.5), there is a 2-group in $C_G(u)$ properly containing E_1 in which E_1 is normal. So we get that $N_G(E_1) \nsubseteq H$. Since $N_H(E_1)/E_1$ is

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isomorphic to S_4, a Sylow 2-subgroup of $N_H(E_1)/E_1$ is dihedral of order 8. Clearly Q/E_1 is also a Sylow 2-subgroup of $N_G(E_1)/E_1$. Since we have $C_G(E_1) = E_1$, the group $\mathscr{S} = N_G(E_1)/E_1$ is isomorphic to a subgroup of $GL(4, 2) \cong A_8$ which has order $2^6 \cdot 3^2 \cdot 5 \cdot 7$.

Suppose at first that $O(\mathscr{S}) \neq 1$ where $O(\mathscr{S})$ denotes the maximal odd-order normal subgroup of $\mathscr{S}$. Consider the action of the four-group $\langle a_1 E_1, b_1 E_1\rangle$ on $O(\mathscr{S})$. Using the facts that all involutions of $\langle a_1 E_1, b_1 E_1\rangle$ are conjugate in (since $\sigma E_1 \in \mathscr{S}$) and that the centralizer of any involution in A_8 has order $2^6 \cdot 3$ or $2^5 \cdot 3$, we get by a result of Brauer-Wielandt [10], that $|O(\mathscr{S})| = 3^3$ or 3. The first case is not possible since $3^3 \nmid |A_8|$. So we have $|O(\mathscr{S})| = 3$. Hence $\langle a_1 E_1, b_1 E_1\rangle \cdot O(\mathscr{S}) = \langle a_1 E_1, b_1 E_1\rangle \times O(\mathscr{S})$. We shall rule out this case by considering $N_{\mathscr{S}}\langle a_1 E_1, b_1 E_1\rangle$. We have $N_G\langle a_1, b_1, a_2, b_2, u\rangle \subseteqq N_G\langle t\rangle$ since $Z\langle a_1, b_1, a_2, b_2, u\rangle = \langle t\rangle$. So

$$N_G\langle a_1, b_1, a_2, b_2, u\rangle \cap N_G(E_1) = Q \cdot \langle \sigma\rangle$$

and it follows $N_{\mathscr{S}}\langle a_1 E_1, b_1 E_1\rangle \cong S^4$, a contradiction to

$$\langle a_1 E_1, b_1 E_1\rangle \cdot O(\mathscr{S}) = \langle a_1, E_1, b_1 E_1\rangle \times O(\mathscr{S}).$$

Thus $O(\mathscr{S}) = 1$.

By the structure of A_8, the order of $C_{\mathscr{S}}(a_1 E_1)$ is $2^3 \cdot 3$ or 2^3. Suppose that $|C_{\mathscr{S}}(a_1 E_1)| = 2^3 \cdot 3$. We are now in a position to apply Gorenstein-Walter's result [3], and get $\mathscr{S} \cong PSL(2, 23)$; $PSL(2, 25)$; $PGL(2, 11)$; $PGL(2, 13)$ or A_7. The first four cases are not possible since $|\mathscr{S}| \nmid |A_8|$. If 7 divides the order of $\mathscr{S}$, we would then have an element of order 7 in $N_G(E_1)$ which acts fixed-point-free on E_1, a contradiction. Thus we must have $|C_{\mathscr{S}}(a_1 E_1)| = 8$. Let T be a Sylow 2-subgroup of G in $C_G(t_1)$ properly containing $C_G(t_1) \cap H$. Then $Z(T/E_1) \neq \langle a_1 E_1\rangle$, otherwise we would get $|C_{\mathscr{S}}(a_1 E_1)| > 8$. This means that $\mathscr{S}$ has only one class of involutions. Therefore by Gorenstein-Walter [3], we get $\mathscr{S} \cong PSL(2, 9) \cong A_6$. The proof is finished.

3. Sylow 3-subgroups of G and its normalizers in G

We shall determine the structure of a Sylow 3-subgroup of G, and the normalizer of this Sylow 3-subgroup in G.

We have $T = \langle \sigma_1, \sigma_2\rangle \subseteqq H$ is a Sylow 3-subgroup of H and $C_H(T) = \langle t\rangle \times T, N_H(T) = \langle u, v\rangle T$. By the structure of H, clearly a Sylow 2-subgroup of $C_G(T)$ is $\langle t\rangle$. It follows, by a theorem of Burnside [4], that $C_G(T)$ has a normal 2-complement $M \supseteqq T$. Since we have $C_G(T) \triangleleft N_G(T)$, we get by Frattini argument that

$$N_G(T) = (C_G(t) \cap N_G(T))C_G(T) = \langle u, v\rangle M.$$

The normal 2-complement M of $C_G(T)$ is characteristic in $C_G(T)$. Hence M is normal in $N_G(T)$. Thus the four group $\langle t, u\rangle$ acts on M. Using the result of Brauer-Wielandt [10] and the fact $C_M(t) = T$; $C_M\langle t, u\rangle = \langle\sigma\rangle$, we get $|M| = |C_M(u)|\,|C_M(tu)|$. Since u and tu are conjugate in $N_G(T)$, we have $|C_M(u)| = |C_M(tu)|$. By (2.5), we have $|C_M(u)| = |C_M(tu)| = 3$ or 3^2. So the order of M is 9 or 81.

Suppose that the order of M is 9. Then we have $T = M$ and so T is a Sylow 3-subgroup of G with $N_G(T) = \langle u, v\rangle T$. By (2.7), we know that $N_G(E_1)/E_1 \cong A_6$. Let $\tilde{T}$ be a Sylow 3-subgroup of $N_G(E_1)$. By the structure of A_6 and our assumption, we have $C_G(\tilde{T}) \cap N_G(E_1) = \tilde{T}$ or $\langle t'\rangle \times \tilde{T}$ where t' is an involution in E_1. Suppose we have $C_G(\tilde{T}) \cap N_G(E_1) = \langle t'\rangle \times \tilde{T}$. Because $C_G(E_1) = E_1$, $\tilde{T}$ induces by conjugation on E_1 a faithful automorphism of E_1 and fixes an involution on E_1. Thus we must have 3^2 dividing $(2^4-2)(2^4-4)(2^4-8) = 2^6 \cdot 3 \cdot 7$, a contradiction. Hence we get $C_G(\tilde{T}) \cap N_G(E_1) = \tilde{T}$. Now by the structure of $N_G(T)$, and $C_G(\tilde{T}) \cap N_G(E_1) = \tilde{T}$, we get that $|N_G(\tilde{T}) \cap N_G(E_1)| = 3^2$ or $2 \cdot 3^2$. The later case is impossible, since the index of $N_G(\tilde{T}) \cap N_G(E_1)$ in $N_G(E_1)$ is $2^6 \cdot 5$ which is not congruent to 1 modulo 3. Therefore, we have $|N_G(\tilde{T}) \cap N_G(E_1)| = 3^2$. By a transfer theorem of Burnside [4, p. 203], $N_G(E_1)/E_1$ is not simple, a contradiction. So we have shown that the order of M is not 9.

Thus M is a group of order 81. We shall show that M is elementary abelian. For this, we need to look at elements of order 3 in H more closely. There are 3 conjugate classes of elements of order 3 in H with representatives σ_1, $\sigma = \sigma_1\sigma_2$, $\rho = \sigma_1\sigma_2^{-1}$ respectively. The centralizer of σ_1 in H is $T \cdot \langle a_2, b_2\rangle$ and so a Sylow 2-subgroup is $C_H(\sigma_1)$ is quaternion of order 8. The centralizer of σ in H is $\langle t, u\rangle T$ and the centralizer of ρ in H is $\langle t, uv\rangle \cdot T$. Both $C_H(\sigma)$ and $C_H(\rho)$ has a four group as its Sylow 2-subgroup and have unique Sylow 3-subgroup T. Let $T_1 = C_M(u)$, $T_2 = C_M(tu)$. We have

$$M = C_M(t)C_M(u)C_M(ut) = TT_1T_2 \text{ and } T_1 \cap T_2 \cap T = \langle\sigma\rangle.$$

Now we consider $C_G(T_1)$. By (2.5), T is conjugate to T_1 in G. So we have $C_G(T_1) = \langle u\rangle \times \tilde{M}$ where $\tilde{M}$ is of order 81 and $\tilde{M}$ is normal in $N_G(T_1)$. We have $\langle t, u\rangle \subseteqq N_G(T_1)$ and therefore the four group $\langle t, u\rangle$ acts on $\tilde{M}$. So we get $\tilde{T} = C_G(t) \cap \tilde{M}$, $\tilde{T}_2 = C_G(tu) \cap \tilde{M}$ and $C_G(u) \cap \tilde{M} = T_1$, all elementary abelian of order 9 with $\tilde{T} \cap T_1 \cap \tilde{T}_2 = \langle\sigma\rangle$. Since we have $\tilde{T} \subseteqq H \cap C_G(\sigma)$, we must have $\tilde{T} = T$. Because $\langle t, u\rangle \subseteqq C_G(tu) \cap C_G(\sigma)$, by our remark in last paragraph we get $T_2 = \tilde{T}_2$. Thus $M = \tilde{M}$. This means that $\langle T, T_1\rangle \subseteqq Z(M)$ and so M is abelian as required.

Thus we have proved the following lemma.

(3.1) Lemma. *The centralizer of T in G is a splitting extension of an*

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elementary abelian group M of order 81 *by* $\langle t\rangle$. *The normalizer of T in G is the group* $\langle u, v\rangle M$ *where* $C_M(t) = T$; $C_M(u) = T_1$; $C_M(tu) = T_2$; $T \cap T_1 \cap T_2 = \langle\sigma\rangle$ *and the groups* T, T_1, T_2 *are elementary abelian of order* 9.

Next we take a look at $C_G(\sigma_1)$. By (3.1), we have $M \subseteqq C_G(\sigma_1)$. By the structure of H, we get $C_G(\sigma_1) \cap H = T \cdot \langle a_2, b_2\rangle$. Let U be a Sylow 2-subgroup of $C_G(\sigma_1)$ containing $\langle a_2, b_2\rangle$. If U properly contains $\langle a_2, b_2\rangle$, we would get that $C_G(\sigma_1) \cap H$, has a Sylow 2-subgroup properly containing $\langle a_2, b_2\rangle$, a contradiction. Hence a Sylow 2-subgroup of $C_G(\sigma_1)$ is quaternion of order 8. Let $V = O(C_G(\sigma_1))$, the maximum odd-order normal subgroup of $C_G(\sigma_1)$. By Suzuki [9], the factor group $C_G(\sigma_1)/V$ has only one involution $t \cdot V$ and so $\langle t\rangle V$ is normal in $C_G(\sigma_1)$. By the Frattini argument

$$C_G(\sigma_1) = (C_G(\sigma_1) \cap C_G(t))V = \langle a_2, b_2\rangle T \cdot V.$$

Because $\langle a_2, b_2\rangle T$ is not 3-closed, it follows that $T \nsubseteq V$ and so $T \cap V = \langle\sigma_1\rangle$. We get $C_G(\sigma_1) = \langle a_2, b_2, \sigma_2\rangle V = L_2 V$ where $L_2 \cong SL(2, 3)$. Since $C_G(t) \cap V = \langle\sigma_1\rangle$, it follows that t acts fixed-point-free on $V/\langle\sigma_1\rangle$. So $V/\langle\sigma_1\rangle$ is abelian. Hence $V' \subseteqq \langle\sigma_1\rangle \subseteqq Z(V)$ and V is nilpotent of class at most 2.

We have therefore proved the following lemma.

(3.2) LEMMA. *The centralizer of the element σ_1 in G is the group L_2V where $L_2 = \langle a_2, b_2\sigma_2\rangle$ and $V = O(C_G(\sigma_1))$ is odd-order and nilpotent of class at most* 2.

The proof of the next lemma is rather involved.

(3.3) LEMMA. *We have that $N_G(M)/M$ is isomorphic to A_6, the alternating group in* 6 *letters.*

PROOF. Since M is characteristic in $N_G(T)$, we get $\langle u, v\rangle \subseteqq N_G(M)$. Let $U \supseteqq \langle u, v\rangle$ be a Sylow 2-subgroup of $N_G(M)$. If $U \supset \langle u, v\rangle$, this would imply that $C_G(t) \cap U \supset \langle u, v\rangle$. Since $C_G(t) \cap M = T$ is normalized by $C_G(t) \cap U$, this would give a contradiction to the structure of $C_G(t)$. Hence $U = \langle u, v\rangle$ and a Sylow 2-subgroup of $N_G(M)$ is dihedral of order 8.

Since the four group $\langle t, u\rangle$ acts on $O(N_G(M))$ and $C_M\langle t, u\rangle = \langle\sigma\rangle$, we get $O(N_G(M)) = M$. Now suppose that $N_G(M) = N_G(T)$, then M is a Sylow 3-subgroup of G. The groups T and T_1, being conjugate in G, should be conjugate in $N_G(M)$, by a theorem of Burnside [4], a contradiction. So we get that $N_G(M) \supset N_G(T)$.

By (3.2), $C_G(T) = \langle t\rangle M$ and so $C_G(M) = M$. Hence $N_G(M)/M$ is isomorphic to a subgroup of $GL(4, 3)$. Since $C_G(t) \cap N_G(M) = \langle u, v\rangle T$, we get $C(tM) \cap (N_G(M)/M) = \langle u, v\rangle M/M$. We are now in a position to use the result of Gorenstein-Walter [3], giving $N_G(M)/M \cong A_7$; $PSL(2, 7)$; $PSL(2, 9)$; $PGL(2, 3)$ or $PGL(2, 5)$. Because 7 does not divide $|GL(4, 3)|$, we have $N_G(M)/M$ is isomorphic to $PSL(2, 9)$; $PGL(2, 3)$ or $PGL(2, 5)$.

Suppose that $N_G(M)/M$ is isomorphic to $PGL(2, 3)$ or $PGL(2, 5)$. Let K be a subgroup of index 2 in $N_G(M)$. Then a Sylow 2-subgroup of K is either $\langle t, u\rangle$ or $\langle t, uv\rangle$. First suppose that it is $\langle t, u\rangle$. We have then F/M is isomorphic to A_4 or A_5. In either case, there exists an element μ of 3-power order in F such that $N_G\langle t, u\rangle \cap F = \langle t, u\rangle\langle\sigma, \mu\rangle$ where $\langle\sigma, \mu\rangle$ is a group of order 9 and $\mu^{-1}t\mu = u$, $\mu^{-1}u\mu = tu$, $\mu^{-1}u\mu = t$. The group $\langle\sigma, \mu\rangle$ is either elementary abelian or cyclic of order 9. Since $C_G\langle t, u\rangle = E_1\langle\sigma\rangle$, and E_1 is characteristic in $C_G\langle t, u\rangle$, we have $E_1 \triangleleft N_G\langle t, u\rangle$. By (2.7), a Sylow 3-subgroup of $N_G(E_1)$ is elementary abelian of order 9. Hence we have shown that μ is an element of order 3 and $\langle t, u\rangle\langle\mu\rangle \cong A_4$.

Put $\mathscr{M} = M\langle\mu\rangle$. It follows that

$$T_1 = M \cap C_G(u) = T^{\mu} \text{ and } T_2 = M \cap C_G(tu) = T^{\mu^2}.$$

Let $\rho = \sigma_1\sigma_2^{-1}$. Then

$$T = \langle\sigma, \rho\rangle,\ T_1 = \langle\sigma, \rho^{\mu}\rangle,\ T_2 = \langle\sigma, \rho^{\mu^2}\rangle.$$

So every element of M can be written uniquely in the form $\sigma^{\alpha}\rho^{\beta}\rho_1^{\gamma}\rho_2^{\delta}$ where $\rho_1 = \rho^{\mu}$; $\rho_2 = \rho^{\mu^2}$; $\alpha, \beta, \gamma, \delta = 0, 1$ or -1. Therefore the structure of $\mathscr{M}$ is completely determined. Since $\mathscr{M}$ is non-abelian, we have

$$Z(\mathscr{M}) = C_M(\mu) = \langle\sigma, \rho\rho_1\rho_2\rangle.$$

An easy computation shows that $\mathscr{M}' = \langle\rho\rho_1\rho_2, \rho\rho_1^{-1}\rangle$, which is elementary abelian of order 9. Since $Z(\mathscr{M}) \neq \mathscr{M}'$, we get $C_{\mathscr{M}}(\mathscr{M}') = M$ and therefore $M \triangleleft N_G(\mathscr{M})$. This gives $N_G(\mathscr{M}) \subseteqq N_G(M)$ and in particular $\mathscr{M}$ is a Sylow 3-subgroup of G.

Let $M_1 = M \cap V$. Suppose that V has a characteristic subgroup X of order $\geqq 9$ contained in M. Then $X \triangleleft C_G(\sigma_1)$ and so $C_G(X) \cap C_G(\sigma_1)$ is normal in $C_G(\sigma_1)$. Suppose that $t \in C_G(X)$. Then $X \subseteqq C_G(t) \cap V = \langle\sigma_1\rangle$, a contradiction to our assumption. Thus $\langle\sigma_2\rangle = C_G(X) \cap L_2$, which would imply that $\langle\sigma_2\rangle$ is normal in L_2, a contradiction. Hence V does not have any characteristic subgroup of order $\geqq 9$ contained in M_1. It follows that M_1 is not a Sylow 3-subgroup of V. Let $M_2 \supset M_1$ be a Sylow 3-subgroup of V. Then $[M_2 : M_1] = 3$ and so $\langle M_2, \sigma_2\rangle$ is a Sylow 3-subgroup of G. If M_2 were abelian, then $C_G(M_1) \supseteqq \langle M_2, \sigma_2\rangle$ and so $M_1 \subseteqq Z\langle M_2, \sigma_2\rangle$, which contradicts $|Z(\mathscr{M})| = 9$. Hence M_2 is non-abelian and so $\langle\sigma_1\rangle \subseteqq Z(M_2) \subseteqq M_1$. Thus we get $Z(M_2) = \langle\sigma_1\rangle$ and also $M_2' = \langle\sigma_1\rangle$. Since M_2 is a 3-group of class at most 2, it follows that M_2 is regular (in the sense of P. Hall). If M_2 were not of exponent 3, then M_1 would be characteristic in M_2, a contradiction. It follows that the Frattini group $\phi(M_2) = \langle\sigma_1\rangle$ and so $M_2/\langle\sigma_1\rangle$ is a 'vector space' of dimension 3 over the field of 3 elements F_3.

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For any two elements $\bar{x} = x\langle\sigma_1\rangle$, $\bar{y} = \langle\sigma_1\rangle$ of $M_2/\langle\sigma_1\rangle$ where $x, y \in M_2$, define $[\bar{x}, \bar{y}] = c$ where $c \in F_3$ and $[x, y] = x^{-1}y^{-1}xy = \sigma_1^c$. Then $[\bar{x}, \bar{y}]$ is a non-singular bilinear skew symmetric form defined on $M_2/\langle\sigma_1\rangle$ with values in F_3 [5]. But then the dimension of $M_2/\langle\sigma_1\rangle$ must be even by [1], a contradiction.

An identical proof applies when a Sylow 2-subgroup of K is $\langle t, uv\rangle$. Therefore we have shown that $N_G(M)/M$ is isomorphic to A_6.

We shall now begin the determination of the structure of a Sylow 3-subgroup of G. But first, we look at the structure of $N_G(M)$ more closely. Since the normalizer of a four group in A_6 is of order 24, there exists an element μ of 3-power order such that $N_G\langle t, u\rangle \cap N_G(M) = \langle u, v\rangle \cdot \langle\sigma, \mu\rangle$. By the same reasoning as in (3.3), we conclude that μ is of order 3 and we have $\mu^{-1}t\mu = u$, $\mu^{-1}u\mu = tu$.

Let $\mathscr{S}$ be the isomorphism of $N_G(M)/M$ onto A_6. Without loss of generality, we may suppose that $(vM)\mathscr{S} = (1324)(56)$, $(uM)\mathscr{S} = (13)(24)$ and choosing μ in $N_G\langle t, u\rangle$ suitably, we may assume that $(\mu M)\mathscr{S} = (132)$. Let $z \in N_G(M)$ such that $(zM)\mathscr{S} = (12)(45)$. Then we have

$$(*) \qquad \begin{aligned} &(\mu M)\mathscr{S} = (132) = e_1; \ (tM)\mathscr{S} = (12)(34) = e_2; \\ &(zM)\mathscr{S} = (12)(45) = e_3; \ (tuvM)\mathscr{S} = (12)(56) = e_4, \cdots. \end{aligned}$$

By Moore, we have $A_6 = \langle e_1, e_2, e_3, e_4\rangle$. Next we may represent $N_G(M)/M$ as linear transformations on the vector space, M over the field of 3 elements in term of the basis $\sigma, \rho, \rho_1 = \rho^\mu, \rho_2 = \rho^{\mu^2}$. The representation is faithful since $C_G(M) = M$. Hence we get

$$\mu M \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad tM \to \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix};$$

$$uM \to \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

From the relations $v^2 = t$, $v^{-1}uv = tu$, we get v is represented by the matrix

$$vM \to \begin{pmatrix} -1 & 0 & & \\ 0 & -1 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}$$

interchanging v by v^{-1} if necessary.

Let (zM) be represented by $(\alpha_{ij}) \in GL(4, 3)$. Then from the relation $(\mu zM)^2 = M$, we get that z is representated by

$$zM \to \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{12} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{21} & \alpha_{23} & \alpha_{24} & \alpha_{22} \\ \alpha_{21} & \alpha_{24} & \alpha_{22} & \alpha_{23} \end{pmatrix}$$

and from $(zM)^2 = M$, we get

$$(**) \qquad \begin{pmatrix} \alpha_{11}^2 & \alpha_{12}\cdot s & \alpha_{12}\cdot s & \alpha_{12}\cdot s \\ \alpha_{21}\cdot s & g+h_1 & g+h_2 & g+h_2 \\ \alpha_{21}\cdot s & g+h_2 & g+h_1 & g+h_2 \\ \alpha_{21}\cdot s & g+h_2 & g+h_2 & g+h_1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdots$$

where

$$s = \alpha_{11}+\alpha_{22}+\alpha_{23}+\alpha_{24}$$
$$g = \alpha_{12}\alpha_{21}$$
$$h_1 = \alpha_{22}^2+\alpha_{23}^2+\alpha_{24}^2$$
$$h_2 = \alpha_{22}\alpha_{23}+\alpha_{23}\alpha_{24}+\alpha_{24}\alpha_{22}.$$

We have $(z \cdot tuvM) \to (456)$. Therefore the group $M\langle \mu, ztuv\rangle$ is a Sylow 3-subgroup of $N_G(M)$. As before, put $\mathscr{M} = M\langle\mu\rangle$. By the proof in (3.3), we have $Z(\mathscr{M}) = \langle \sigma, \rho\rho_1\rho_2\rangle$; $\mathscr{M}' = \langle \rho\rho_1\rho_2, \rho\rho_1^{-1}\rangle$. Hence $Z(\mathscr{M}) \cap \mathscr{M}' = \langle \rho\rho_1\rho_2\rangle$ is characteristic in $\mathscr{M}$ and so $\langle\rho\rho_1\rho_2\rangle$ is normal in $M\langle \mu, ztuv\rangle$. Therefore we have $\rho\rho_1\rho_2$ centralized by $\lambda = ztuv$.

Now λ is represented by the matrix

$$\lambda M \to \begin{pmatrix} -\alpha_{11} & \alpha_{12} & \alpha_{12} & \alpha_{12} \\ -\alpha_{21} & \alpha_{22} & \alpha_{24} & \alpha_{23} \\ -\alpha_{21} & \alpha_{23} & \alpha_{22} & \alpha_{24} \\ -\alpha_{21} & \alpha_{24} & \alpha_{23} & \alpha_{22} \end{pmatrix}.$$

From $(\lambda M)^3 = M$, we get $\alpha_{11} = -1$. Since λ commute with $\rho\rho_1\rho_2$, we obtain $\alpha_{22}+\alpha_{23}+\alpha_{24} = 1$. Since $(tz)^3 \in M$, this implies that $\alpha_{12}\alpha_{21}(1+\alpha_{22}) = -1$ (by working at the (1, 1) entry of the representation of tz). Therefore $\alpha_{12}\alpha_{21} \neq 0$. First suppose that $\alpha_{12}\alpha_{21} = 1$. Then we have $\alpha_{22} = 1$. By (**), we get $h_2 = -1$. So we obtain $\alpha_{24} = -\alpha_{23} \neq 0$. Hence tz is represented by the matrix

$$tzM \to \begin{pmatrix} -1 & \alpha_{12} & \alpha_{12} & \alpha_{12} \\ \alpha_{12} & 1 & \alpha_{23} & -\alpha_{23} \\ -\alpha_{12} & -\alpha_{23} & \alpha_{23} & -1 \\ -\alpha_{12} & \alpha_{23} & -1 & -\alpha_{23} \end{pmatrix}$$

and we check that $(tz)^3 \notin M$, a contradiction.

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Thus we must have $\alpha_{12}\alpha_{21} = -1$. Then $\alpha_{22} = 0$, from $\alpha_{22}+\alpha_{23}+\alpha_{24} = 1$, we get $\alpha_{23} = \alpha_{24} = -1$. Hence we have z represented by

$$zM \to \begin{pmatrix} -1 & \alpha_{12} & \alpha_{12} & \alpha_{12} \\ -\alpha_{12} & 0 & -1 & -1 \\ -\alpha_{12} & -1 & -1 & 0 \\ -\alpha_{12} & -1 & 0 & -1 \end{pmatrix}$$

and

$$\lambda M \to \begin{pmatrix} -1 & \alpha_{12} & \alpha_{12} & \alpha_{12} \\ -\alpha_{12} & 0 & -1 & -1 \\ -\alpha_{12} & -1 & 0 & -1 \\ -\alpha_{12} & -1 & -1 & 0 \end{pmatrix}.$$

Interchanging λ by λ^{-1}, if necessary, we may suppose that $\alpha_{12} = -1$.

Now $M\langle\lambda, \mu\rangle$ is a Sylow 3-subgroup of $N_G(M)$ and by the structure of A_6, the commutator $[\lambda, \mu] \in M$. Since M is abelian, and $M\langle\lambda, \mu\rangle$ is not, we get $Z(M\langle\lambda, \mu\rangle) = C_M\langle\lambda, \mu\rangle = \langle\rho\rho_1\rho_2\rangle$. An easy computation shows that the commutator group of $M\langle\lambda, \mu\rangle$ contains $\langle\sigma, \rho\rho_1\rho_2, \rho\rho_1^{-1}\rangle$ and is contained in M. Since $Z(M\langle\lambda, \mu\rangle) \neq (M\langle\lambda, \mu\rangle)'$, we see that M is characteristic in $M\langle\lambda, \mu\rangle$. So we have $N_G(M\langle\lambda, \mu\rangle) \subseteqq N_G(M)$. Hence $M\langle\lambda, \mu\rangle$ is a Sylow 3-subgroup of G. and moreover, by the structure of A_6, the normalizer of $M\langle\lambda, \mu\rangle$ is a splitting extension of $M\langle\lambda, \mu\rangle$ by a group of order 4.

Next we check that we have $z' = (\mu^2 tz)^3$ such that $(z')\rho\rho_1\rho_2 z' = \sigma\rho$. Let $\mu' = (z')^{-1}\mu z$ and $\lambda' = (z')^{-1}\lambda z'$, we see that $\langle\lambda', \mu'\rangle \subseteqq C_G(\sigma_1)$. and that μ', λ' are represented by the following matrices.

$$\mu' M \to \begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}; \quad \lambda' M \to \begin{pmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}.$$

Therefore we have

$$(\mu')^{-1}\lambda' M \to \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad (\mu')^{-1}(x')^{-1}\cdot M \to \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}.$$

The group $M\langle\lambda', \mu'\rangle$ is contained in $C_G(\sigma_1)$. We turn our attention back to $C_G(\sigma_1)$. Let $U_1 \subseteqq V$ be the Sylow 3-subgroup of V. We have $M \cap V = M_1$ is elementary abelian of order 27. Suppose that $\rho_1 = \rho^\mu \notin M_1$.

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Then we have $\rho_1 = \sigma_2^j m$ for some fixed $j = 1$ or -1 and $m \in M_1$. Now t acts fixed-point-free on $V/\langle\sigma_1\rangle$. Therefore we get

$$t\rho_1 t = \sigma_2^j m^{-1}\sigma_1^i = \rho_1^{-1} = \sigma_2^{-j} m^{-1}\sigma_1^i$$

giving $\sigma_2^j = \sigma_1^i$, a contradiction. Similarly we can show that $\rho_2 = \rho^{\mu^2} \in M_1$.

Let $\langle\rho_1^{a_2}, \rho_2^{a_2}\rangle = \langle\rho_3, \rho_4\rangle \subseteqq U_1$. By way of contradiction, suppose that $\langle\rho_3, \rho_4\rangle \cap M_1$ is non-empty. Then there exists an element $\rho_3^i\rho_4^j \in M_1$ for fixed i, j not both zero. Since σ_2 centralize M_1, we would then get $b_2^{-1}\rho_1^i\rho_2^j b_2 = a_2^{-1}\rho_1^i\rho_2^j a_2$. This is a contradiction, since $C_G(t) \cap V = \langle\sigma_1\rangle$. Thus $\langle\rho_3, \rho_4\rangle \subseteqq M_1$. Since a Sylow 3-subgroup of G is of order 3^6 we must have $U_1 = \langle\sigma_1, \rho_1, \rho_2, \rho_3, \rho_4\rangle$.

The group $U_1/\langle\sigma_1\rangle$ is abelian and so is elementary abelian of order 81. We may then represent the group $L_2 = \langle a_2, b_2, \sigma_2\rangle$ as linear transformations on the 'vector space' $U_1/\langle\sigma_1\rangle$ over the field of 3 elements. We get in terms of the basis $\rho_1\langle\sigma_1\rangle, \rho_2\langle\sigma_1\rangle, \rho_3\langle\sigma_1\rangle, \rho_4\langle\sigma_1\rangle$, the representation of a_2

$$a_2 \to \begin{pmatrix} & & -1 & 0 \\ & & 0 & -1 \\ 1 & 0 & & \\ 0 & 1 & & \end{pmatrix}.$$

We have shown that $v^{-1}\rho_1 v = \rho_2$, $v^{-1}\rho_2 v = \rho_1^{-1}$. Therefore with the relation $v^{-1}a_2 v = a_2^{-1}$, we get

$$v \to \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}.$$

Let σ_2 be represented by the matrix

$$\sigma_2 \to \begin{pmatrix} I & C \\ 0 & D \end{pmatrix}$$

where (C) and (D) are 2×2 matrices. From the relation $(a_2\sigma_2)^3 = 1$, we get that $(C) = (-D^{-1})$. Since (D) is non-singular, we have $\det(D) = \pm 1$. Suppose $\det(D) = -1$, then using the relation $v^{-1}\sigma_2 v = \sigma_2^{-1}$, we obtain a contradiction. Hence $\det(D) = 1$. Again by the relation $v^{-1}\sigma_2 v = \sigma_2^{-1}$, we obtain that $(D) =$ identity matrix. Hence σ_2 is represented by

$$\sigma_2 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

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It follows that $\langle \rho_3, \rho_4 \rangle \subseteqq N_G(M) \cap C_G(\sigma_1) - M$. So comparing the action of the group $\langle \lambda', \mu' \rangle$ on M, we conclude that $\rho_3 M = (\mu')^{-1}\lambda' M$ and $\rho_4 M = (\mu')^{-1}(\lambda')^{-1}M$. We have

$$N_G(P) = U_1(N_G(\sigma_2) \cap L_2\langle v \rangle) = U_1\langle \sigma_2 \rangle \langle v \rangle = P\langle v \rangle$$

where $P = U_1\langle \sigma_2 \rangle$. Thus we have proved the following lemma.

(3.4) LEMMA. *The group* $P = M\langle \rho_3, \rho_4 \rangle$ *is a Sylow* 3*-subgroup of* G *and has the following structure*:

$$M = TT_1T_2,$$

an elementary abelian group of order 81 *where*

$$\begin{aligned} T &= C_M(t) = \langle \sigma, \rho \rangle \\ T_1 &= C_M(u) = \langle \sigma, \rho_1 \rangle \\ T_2 &= C_M(tu) = \langle \sigma, \rho_2 \rangle \end{aligned}$$

elementary abelian of order 9 *and*

$$\begin{array}{llll} \rho_3^{-1}\sigma_1\rho_3 = \sigma_1; & \rho_3^{-1}\sigma_2\rho_3 = \sigma_2\rho_3\sigma_1; & \rho_3^{-1}\rho_1\rho_3 = \rho_1\sigma_1^{-1}; & \rho_3^{-1}\rho_2\rho_3 = \rho_2; \\ \rho_4^{-1}\sigma_1\rho_4 = \sigma_1; & \rho_4^{-1}\sigma_2\rho_4 = \sigma_2\rho_4\sigma_1; & \rho_4^{-1}\rho_1\rho_4 = \rho_1; & \rho_4^{-1}\rho_2\rho_4 = \rho_2\sigma_1^{-1}. \end{array}$$

Moreover $N_G(P) = P \cdot \langle v \rangle$ *where*

$$v^{-1}\rho_1 v = \rho_2, \quad v^{-1}\rho_2 v = \rho_1^{-1}, \quad v^{-1}\rho_3 v = \rho_4^{-1}, \quad v^{-1}\rho_4 v = \rho_3.$$

4. Final characterization

Using the informations already found, we shall now prove that G is isomorphic to $U_4(3)$. The following preliminary lemmas are required.

(4.1) LEMMA. *The group* P *and its conjugate* t_1Pt_1 *have trivial intersection.*

PROOF. We have $P \subseteqq C_G(\sigma_1)$. Therefore

$$P \cap t_1Pt_1 \subseteqq C_G(\sigma_1) \cap C_G(\sigma_1^{t_1}) \subseteqq C_G(\sigma_1) \cap C_G(a_1b_1) = \langle \sigma_2 \rangle.$$

The group $P \cap t_1Pt_1$ is normalized by t_1. So it follows that $P \cap t_1Pt_1 = 1$.

(4.2) LEMMA. *We have the following relations*:

$$(a_2\sigma_2)^3 = (ut\rho_3)^3 = (u\rho_4)^3 = (vu\rho_3^{-1}\rho_4)^3 = (tuv\rho_3^{-1}\rho_4^{-1})^3 = 1.$$

PROOF. Using our representation, of $N_G(M)$ as linear transformation on the vector space M, we compute that $(ut\rho_3)^3 \in M$. Since $\rho_3 = a_2^{-1}\rho_1a_2$, we have $ut\rho_3 \in C_G(utt_1)$. So $(ut\rho_3)^3 \in M \cap C_G(utt_1) \subseteqq P \cap C_G(utt_1) = \langle \rho_3 \rangle$. Therefore we get $(ut\rho_3)^3 = 1$.

Next we have $u\rho_4 = v(ut\rho_3)v^{-1}$. So we get $(u\rho_4)^3 = 1$. Again from our representations of uv and $\rho_3^{-1}\rho_4$, we verify that $(uv\rho_3^{-1}\rho_4)^3 \in M$. Also we have $uv\rho_3^{-1}\rho_4 \in C_G(uvt_1)$. Hence

$$(uv\rho_3^{-1}\rho_4)^3 \subseteqq M \cap C_G(uvt_1) \subseteqq P \cap C_G(uvt_1) = \langle \rho_3^{-1}\rho_4 \rangle.$$

Showing that $(uv\rho_3^{-1}\rho_4)^3 = 1$. By (3.4) we have $(tuv\rho_3^{-1}\rho_4^{-1}) = v^{-1}(uv\rho_3^{-1}\rho_4)v$. Therefore $(tuv\rho_3^{-1}\rho_4^{-1})^3 = 1$.

By the structure of H, we know that $(a_2\sigma_2)^3 = 1$.

The assertions of this lemma are completely proved.

(4.3) LEMMA. *The group $W = N_G\langle v \rangle / \langle v \rangle$ is generated by the involutions $r_1 = a_2\langle v \rangle$ and $r_2 = u\langle v \rangle$ and is dihedral of order* 8.

PROOF. Obvious from the structure of H.

Put $B = N_G(P)$, and $N = N_G\langle v \rangle$. We want to show that the set of elements in BNB forms a subgroup of G. For any $w \in W$, define $l(w) = l$ to be the smallest positive integer such that $w = r_{i_1} r_{i_2} \cdots r_{i_l}$ where $r_{i_j} \in \{r_1, r_2\}$. Let $\omega(r_1) = a_2$, $\omega(r_2) = u$. For any $w \in W$, and $w = r_{i_1} \cdots r_{i_l}$, define $\omega(w) = \omega(r_{i_1}) \cdots \omega(r_{i_l})$. We shall denote BwB to mean $B\omega(w)B$.

(4.4) LEMMA. *The set of elements in $B \cup Br_iB$ $(i = 1, 2)$ forms a subgroup of G.*

PROOF. Let $g = b\omega(r_i)b' \in Br_iB$ where b, $b' \in B$. Then the element $g' = (b')^{-1}\omega(r_i)(\omega(r_i)^{-2}b^{-1}) \in Br_iB$ and is an inverse of g.

Let $G_1 = B \cup Br_1B = B \cup Ba_2B$. Clearly to show that G_1 is closed with respect to multiplication, we need only to show that $a_2\sigma_2^\delta a_2 \in G_1$ $(\delta = 0, 1, -1)$; since B has the form $\langle \sigma_2 \rangle (\langle v \rangle \langle \sigma_1, \rho_1, \rho_2, \rho_3, \rho_4 \rangle)$ and $\langle v \rangle \langle \sigma_1, \rho_1, \rho_2, \rho_3, \rho_4 \rangle$ is normalized by a_2. If $\delta = 0$, then $a_2\sigma_2^\delta a_2 = t \in B$. If δ is 1, then by (4.3), $a_2\sigma_2 a_2 = \sigma_2^{-1}a_2(t\sigma_2^{-1}) \in Ba_2B$. Similarly of $\delta = -1$, we get $a_2\sigma_2^{-1}a_2 = t\sigma_2 a_2\sigma_2 t \in Ba_2B$. Hence we have shown that G_1 is a subgroup of G.

Next to show that $G_2 = B \cup Br_2B$ is a subgroup of G, we need to show that $u\rho_3^i u\rho_4^j \in G_2$ $(i, j = 0, 1, -1)$. By using (4.3), and similar reasoning as in the last case, this is in fact true.

(4.5) LEMMA. *For any i and $w \in W$, if $l(r_iw) \geqq l(w)$, then $r_iBw \subseteqq Br_iwB$.*

PROOF. First of all, we construct table I showing the action of a_2 and u on P by conjugation.

TABLE I

	σ_1	σ_2	ρ_1	ρ_2	ρ_3	ρ_4
a_2	σ_1	—	ρ_3	ρ_4	ρ_1^{-1}	ρ_2^{-1}
u	σ_2	σ_1	ρ_1	ρ_2^{-1}	—	—

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To prove this lemma, we construct table II, showing $l(r_i w)$ and $l(w)$ for all i and $w \in W$. Clearly we need only to see that $r_1 \sigma_2 w \subseteqq B r_1 w B$ and $r_2 \rho_3^i \rho_4^j w \subseteqq B r_2 w B$ $(i, j = 0, 1, -1)$. It is easily verified that for those $w \in W$ such that $l(r_2 w) \geqq l(w)$, we can always get $r_1 \sigma_2 w \in B r_1 w y_1$ and $r_2 \rho_3^i \rho_4^j w \in B r_2 \rho_3^i \rho_4^j y_2$, using the informations in table I. Hence the lemma is completely proved.

TABLE II

w	$l(w)$	$l(r_1 w)$	y_1	$l(r_2 w)$	y_2
1	0	1	1	1	1
r_1	1	0		2	$\rho_1^{-i} \rho_2^{-j}$
r_2	1	2	σ_1	0	
$r_1 r_2$	2	1		3	$\rho_1^{-i} \rho_2^j$
$r_2 r_1$	2	3	σ_1	1	
$r_1 r_2 r_1$	3	2		4	$\rho_3^{-i} \rho_4^j$
$r_2 r_1 r_2$	3	4	σ_2	2	
$r_1 r_2 r_1 r_2$	4	3		3	

(4.6) LEMMA. *The set of elements* $G_0 = BNB$ *is a subgroup of* G *and if we have* $B w_1 B = B w_2 B$, *then* $w_1 = w_2$.

PROOF. It follows from (4.4), (4.5) and Tits [8].

We shall next compute the order of G_0. Define for any $w \in W$, the group B_w generated by elements $x \in P$ such that $\omega(w) \times \omega(w)^{-1} \in t_1 P t_1$. The groups B_w for all $w \in W$ are shown in the next table.

TABLE III

w	1	r_1	r_2	$r_1 r_2$
B_w	1	$\langle \sigma_2 \rangle$	$\langle \rho_3, \rho_4 \rangle$	$\langle \sigma_1, \rho_3, \rho_4 \rangle$
$(B_w)'$	P	$\langle \sigma_1, \rho_1, \rho_2, \rho_3, \rho_4 \rangle$	M	$\langle \sigma_2, \rho_1, \rho_2 \rangle$

$r_2 r_1$	$r_1 r_2 r_1$	$r_2 r_1 r_2$	$r_1 r_2 r_1 r_2$
$\langle \sigma_2, \rho_1, \rho_2 \rangle$	M	$\langle \sigma_1, \rho_1, \rho_2, \rho_3, \rho_4 \rangle$	P
$\langle \sigma_1, \rho_3, \rho_4 \rangle$	$\langle \rho_3, \rho_4 \rangle$	$\langle \sigma_2 \rangle$	1

We observe that for every B_w, there exists the subgroup $(B_w)'$ such that $B_w (B_w)' = P$ and $B_w \cap (B_w)' = 1$ (see (4.1)).

(4.7) LEMMA. *The order of* G_0 *is* $2^7 \cdot 3^6 \cdot 5 \cdot 7$.

PROOF. We show first that every element of G_0 can be written in the 'normal' form $hp\omega(w)p_w$ where $h \in \langle v \rangle$, $p \in P$ and $p_w \in B_w$. By (4.6), every element x in G_0 has the form $x = b_1\omega(w)b_2$ where $b_1, b_2 \in B$. Since we have $P = B_w(B_w)'$ we may write $b_2 = hp_2'p_2$ where $h \in \langle v \rangle$, $p_2 \in B_w$ and $p_2' \in (B_w)'$. From the facts $\omega(w)h\omega(w)^{-1} \in \langle v \rangle$ and $\omega(w)p_2'\omega(w)^{-1} \in P$, we get $x = b\omega(w)p_2$ showing the existence of the 'normal' form.

To show the uniqueness of the 'normal' form, suppose that

$$b\omega(w)b_w = b'\omega(w')b'_{w'}.$$

By (4.6), we have $w = w'$. Therefore we get

$$(b')^{-1}b = \omega(w)b_w(b'_w)^{-1}_{t_1}\omega(w)^{-1}.$$

Since $(b')^{-1}b \in B$ and

$$\omega(w)b_w(bw')^{-1}\omega(w)^{-1} \in P^{t_1},$$

we obtain

$$(b')^{-1}b \in B \cap P^{t_1} \subseteqq P.$$

The uniqueness follows by (4.1).

By (4.1), the 8 double cosets in BNB are distinct, therefore we have

$$|G_0| = |B| \sum_w |B_w| = 2^7 \cdot 3^6 \cdot 5 \cdot 7.$$

To conclude the proof of the theorem, we require the following result of Thompson.

LEMMA (*Thompson*). *Let $\mathscr{M}$ be a subgroup of $\mathfrak{G}$ such that*

(a') $|\mathscr{M}|$ *is even.*

(b') $\mathscr{M}$ *contains the centralizer of each of its involutions.*

(c') $\bigcap_{s \in \mathfrak{G}} \mathscr{M}^s$ *is of odd order.*

Let $\mathscr{S}$ be a S_2-subgroup of $\mathscr{M}$ and let I be an involution in $Z(\mathscr{S})$. We have

(d') $N(\mathscr{S}) \subseteqq \mathscr{M}$.

Then

(i) $i(\mathscr{M}) = 1$ (*the number of conjugate classes of involution in* $\mathscr{M}$)

(ii) $\mathscr{M}$ *contains a subgroup* $\mathscr{M}_0$ *of odd order such that* $\mathscr{M} = \mathscr{M}_0 C_{\mathscr{M}}(I)$.

Using the informations of our tables (I, II, III), (4.2) and the structures of P and $\langle v \rangle$, we can multiply any two elements of G_0 in the 'normal' form to get the product *uniquely* in the 'normal' form. Now if X is any finite group satisfying properties (a) and (b) of the theorem, then X contains a subgroup X_0 of order $|U_4(3)|$ with uniquely determined multiplication table. Hence taking X to be $U_4(3)$, we see that $X_0 = U_4(3)$ and so $G_0 \cong U_4(3)$.

Consequently G_0 satisfies conditions (a′), (b′) and (d′) of Thompson lemma. Suppose the (c′) is also fulfilled, then we obtain that G_0 contains a subgroup M_0 of odd order such that $G_0 = M_0 C_G(t) = M_0 H$.

Suppose that $|M_0 \cap H| = 3^2$, then we have $|M_0| = 3^6 \cdot 5 \cdot 7$. Let S_3 be a Sylow 3-subgroup of M_0. By (3.4) we get $N_{M_0}(S_3) = S_3$. This is a contradiction since $|M_0 : N_{M_0}(S_3)| = 5 \cdot 7 \not\equiv 1 \pmod 3$. Hence we must have $|M_0| = 3^4 \cdot 5 \cdot 7$ or $3^5 \cdot 5 \cdot 7$. Now M_0 is soluble and so by P. Hall (4], there exists a subgroup of order $5 \cdot 7$ in M_0. Clearly K is abelian. Let S_7 be the Sylow 7-subgroup of K. By Sylow's Theorem, we get that S_7 is normal in M_0. Applying Sylow's theorem again, we obtain that $N_{G_0}(S_7)$ is $2^4 \cdot 3^6 \cdot 5 \cdot 7$, $2^5 \cdot 3^4 \cdot 5 \cdot 7$, $2^2 \cdot 3^4 \cdot 5 \cdot 7$ or $2 \cdot 3^6 \cdot 5 \cdot 7$. The first 3 cases are not possible, since this would then imply that an involution of G_0 is centralized by elements of order 7, a contradiction of structure of H. Thus we have $|N_{G_0}(S_7)| = 2 \cdot 3^6 \cdot 5 \cdot 7$. Now a Sylow 2-subgroup of $N_{G_0}(S_7)$ is cyclic of order 2. Therefore, by Burnside [4], there is a subgroup of order $3^6 \cdot 5 \cdot 7$ in $N_{G_0}(S_7)$ and this gives a contradiction as before.

Thus we must get $\bigcap_{g \in G} G_0^g$ is even. By (2.6), the group G is simple. Hence $G = G_0 \cong U_4(3)$, proving our theorem.

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