Can. J. Math., Vol. XXXVIII, No. 3, 1986, pp. 619-632

ANTICHAINS AND FINITE SETS THAT MEET ALL MAXIMAL CHAINS

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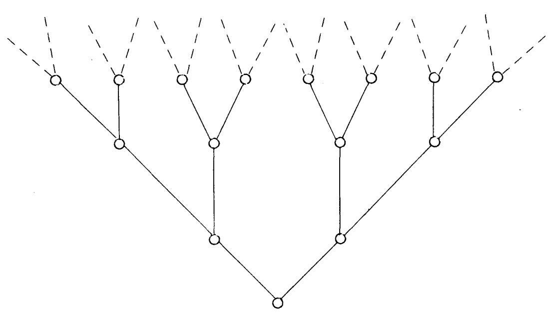
This paper is inspired by two apparently different ideas. Let P be an ordered set and let $\mathbf{M}(P)$ stand for the set of all of its maximal chains. The collection of all sets of the form

$$A(x) = \{C \in \mathbf{M}(P) | x \notin C\}$$

and

$$B(x) = \{C \in \mathbf{M}P) | x \in C\},$$

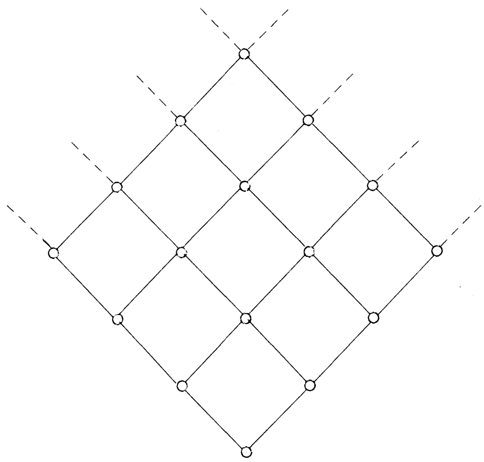
where $x \in P$, is a subbase for the open sets of a topology on $\mathbf{M}(P)$. (Actually, it is easy to check that the $B(x)$ sets themselves form a subbase.) In other words, as $\mathbf{M}(P)$ is a subset of the power set $\mathbf{2}^{|P|}$ of P, we can regard $\mathbf{M}(P)$ as a subspace of $\mathbf{2}^{|P|}$ with the usual product topology. M. Bell and J. Ginsburg [**1**] have shown that the topological space $\mathbf{M}(P)$ is compact if and only if, for each $x \in P$, there is a finite subset $C(x)$ of P all of whose elements are noncomparable to x and such that $\{x\} \cup C(x)$ meets each maximal chain.



$\mathbf{2}^{\omega}$

Received November 28, 1983 and in revised form June 25, 1985.

https://doi.org/10.4153/CJM-1986-031-3 Published online by Cambridge University Press



$\underset{\sim}{\omega} \times \underset{\sim}{\omega}$

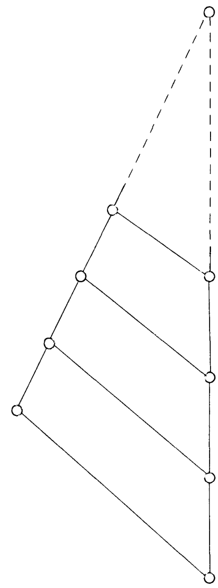


Figure 1

The primary concern in [**1**] is to describe those compact spaces which can be represented as $\mathbf{M}(P)$ for some ordered set P. That is the first idea behind this paper.

In contrast, our interest here is with the order theoretical counterpart of this compactness condition. What are the properties of the ordered sets P for which $\mathbf{M}(P)$ is compact? Some examples are illustrated in Figure 1. What kinds of subsets of an ordered set meet each maximal chain? That is the second motivation.

A maximum sized antichain of an ordered set need not meet each maximal chain. For instance, in Figure 2, $\{b, c, d, e\}$ does not meet the chain $\{a, f\}$. In fact, it may even be that no antichain at all meets every maximal chain (cf. Figure 3). Moreover, in the ordered set of Figure 2 the smallest subset that

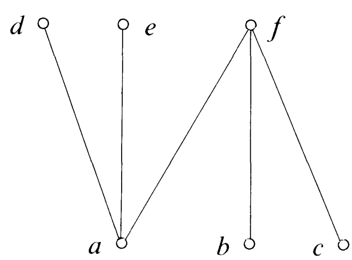


Figure 2

meets each maximal chain is a two-element chain, while every maximal antichain has at least three elements. This contrasts with the "chain decomposition theorem" of Dilworth **[2]** which in a sense suggests a "crossroad" between chains and antichains in an ordered set.

The main idea that we treat here is this. A *cutset* for an element x of an ordered set P is a subset $C(x)$ of P all of whose elements are noncomparable to x and such that each maximal chain of P includes x or some element of $C(x)$. For instance, in the ordered set of Figure 2, we may take $C(a) = \{b, c\}$, $C(b) = \{a, c\}$, and so on.

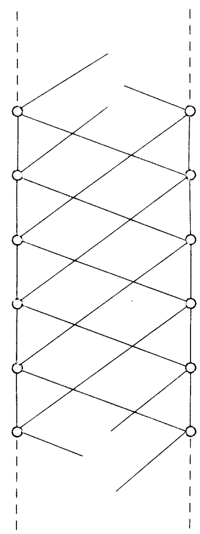


Figure 3

Let m be a nonnegative integer. Say that an ordered set P has the *m-cutset property* if there is a cutset $C(x)$ with at most m elements for each $x \in P$. The degenerate case $m = 0$ corresponds to the case that P itself is a chain. It is not hard to prove that if $m = 1$ then P cannot contain a three-element antichain. For, suppose that $\{a, b, c\}$ is a three-element antichain in an ordered set P with the 1-cutset property and suppose that $C(a) = \{a_1\}$, $C(b) = \{b_1\}$ and $C(c) = \{c_1\}$. Then a_1 is noncomparable to a but must be comparable to b and to c, both. Say $a_1 > b$ and $a_1 > c$. Now b_1 is comparable to a and to c. If $b_1 > c$ then $b_1 > a$ and so b_1 is noncomparable to a_1, and then neither a nor a_1 can be on any maximal chain including c and b_1. Therefore, $b_1 < a$ and $b_1 < c$. Now, c_1 is comparable to a and to b both. If $c_1 > a$ and $c_1 > b$ then both a and a_1 miss every maximal chain through b and c_1. If $c_1 < a$ and $c_1 < b$ then both b and b_1 miss every maximal chain through c_1 and a. Therefore, P cannot, after all, have a three-element antichain.

In contrast the ordered set illustrated in Figure 4 has the 3-cutset property yet it contains an infinite antichain; in fact, every maximal antichain containing x is infinite. Note, for instance, that we may take

$$C(a_i) = \{b_{i-1}, d\}, \ C(b_i) = \{a_{i+1}, d\},$$
$$C(u_i) = \{a_{i+1}, b_{i-1}, d\}, \ C(x) = \{a_1, c_1\}$$

and so on, while there is no finite maximal antichain including x. What if P satisfies the 2-cutset property? N. Sauer and R. E. Woodrow [**5**] have shown that in an ordered set with the 2-cutset property every element is contained in a 4-element maximal antichain.

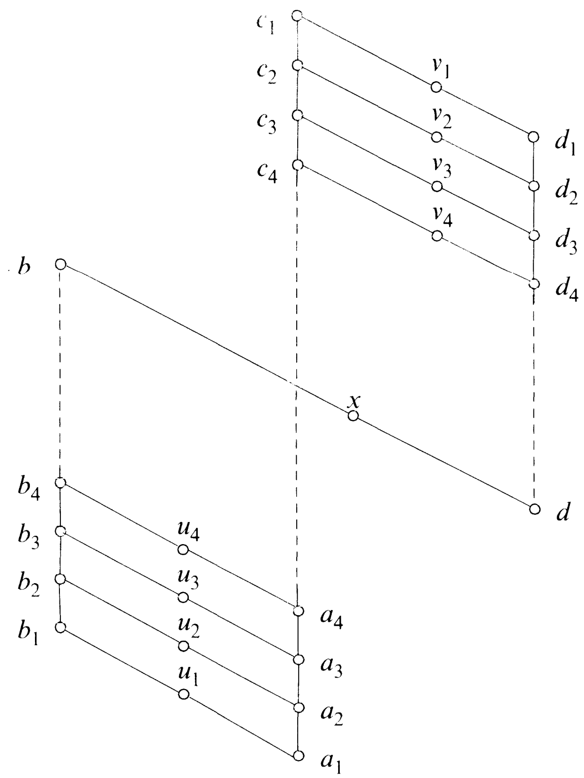


Figure 4

Say that an ordered set has the *finite cutset property* if there is a finite cutset $C(x)$ for each $x \in P$. We shall show that an ordered set with the finite cutset property in which each chain is finite must itself be finite (see Corollary 3).

An ordered set may have "big" cutsets but only small antichains. An example can be constructed from $\underset{\sim}{\kappa}$-chains as suggested in Figure 5.

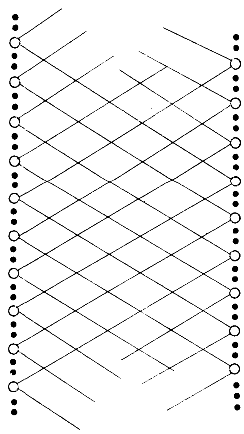


Figure 5

Also, an ordered set may have "big" antichains even if the cutsets are quite small. The ordered set of Figure 6 has the 2-cutset property but it contains a $\underset{\sim}{\kappa}$-element antichain.

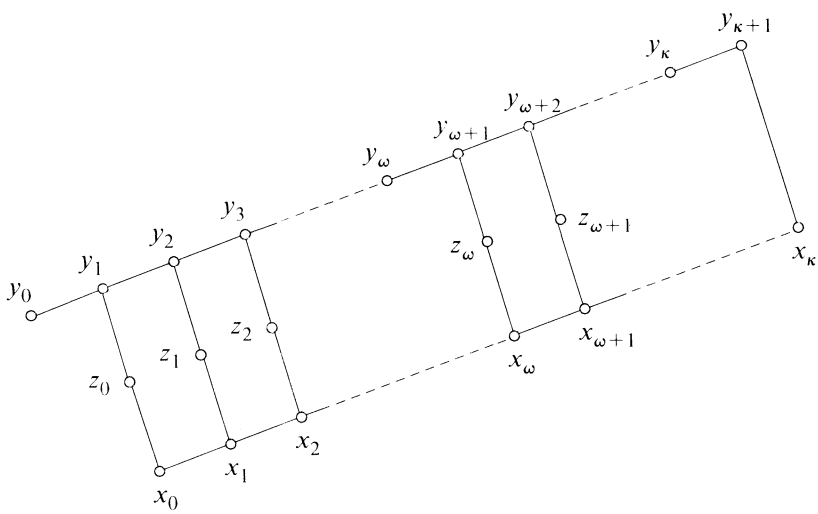


Figure 6

This ordered set is not chain complete; for instance, the ω-chain $\{x_0 < x_1 < x_2 < \dots\}$ has no supremum, although x_ω and y_ω are upper bounds.

Our principal result is this.

THEOREM 1. *A countably chain complete ordered set with the finite cutset property contains no uncountable antichain.*

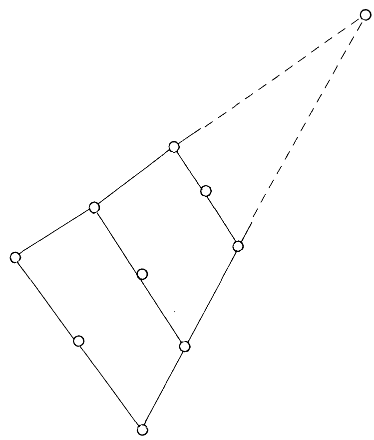


Figure 7

We have conjectured that in a chain complete ordered set with the finite cutset property each element is contained in a finite maximal antichain. And this is settled in the affirmative by Sauer and Woodrow in **[5]**. Still, an ordered set may be chain complete, satisfy the finite cutset property (even the 2-cutset property) and yet contain infinite antichains (see Figure 7).

We also establish a result with a topological bent. Actually, it uses Theorem 1.

THEOREM 2. *Let X be a compact topological space. If there is a chain complete ordered set P whose space* $\mathbf{M}(P)$ *of maximal chains is homeomorphic to X then the cellularity* $\mathbf{c}(X)$ *of X satisfies* $\mathbf{c}(X) \leqq 2^{\omega}$.

The *cellularity* $\mathbf{c}(X)$ of a topological space X is defined by

$$\mathbf{c}(X) = \sup\{\kappa | X \text{ contains a family of } \kappa \text{ disjoint nonempty open sets}\}.$$

Finite cutsets and uncountable antichains. For the proof of Theorem 1 we shall use a lemma about "special" infima (or suprema).

LEMMA 1. *Let P be an ordered set with the finite cutset property. Let C be a chain in P and let $c_0 = \inf_P C$. Then there is $c_1 \in C$ such that for each $x \in P$ if $x < c_1$ then $x \leqq c_0$ or $x \geqq c_0$.*

Proof. Let $C(c_0)$ be a finite cutset for c_0. Then, of course, each element $a \in C(c_0)$ is noncomparable to c_0. Actually, for each $a \in C(c_0)$ there is $c_a \in C$ noncomparable to a (otherwise, if $a \leqq c$ for all $c \in C$ then $a \leqq \inf_P C = c_0$). As $C(c_0)$ is finite and the c_a's lie in a chain we may construct

$$c_1 = \inf\{c_a | a \in C(c_0)\} \in C.$$

Now, let $x < c_1$ and let D be any maximal chain including x and c_1. Suppose that $c_0 \notin D$. Then some $a \in C(c_0)$ must belong to D. If $a \leqq c_1$ then $a \leqq c_a$ which is impossible. Otherwise, if $a > c_1$ then $a > c_0$ and that too is impossible.

Here is a typical application of this lemma.

PROPOSITION 2. *Let P be a chain complete ordered set with the finite cutset property. If P contains an infinite antichain then P contains a subset isomorphic to the ordered set* $\mathbf{T}$ *or its dual* $\mathbf{T}^d$ *(see Figure* 8).

The ordered set $\mathbf{T}$.

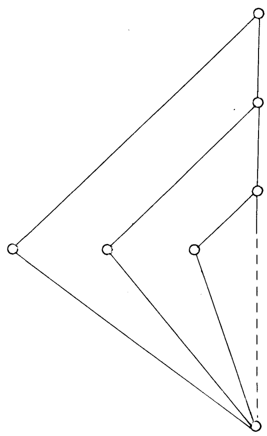


Figure 8

Proof. Let $A = \{a_1, a_2, \dots\}$ be an infinite antichain in P. There is $b_1 \in C(a_1)$ which meets infinitely many of the maximal chains through a_1, $a_2, \dots$. Then either b_1 is above infinitely many a_i's or below infinitely many of the a_i's. Suppose that $b_1 > a_2, a_3, \dots$, (after possibly relabelling the indices). Now, choose $b_2 \in C(a_2)$ which meets infinitely many of the maximal chains through $a_3 < b_1$, $a_4 < b_1$, $a_5 < b_1, \dots$. Either b_2 is above or below infinitely many of $a_3, a_4, \dots$, and it is in any case below b_1 (since $b_2 \geqq b_1$ implies $b_2 \geqq a_2$). For simplicity let us suppose that b_2 is above a_3, $a_4, \dots$ (after possibly relabelling the indices). In this way we construct a descending chain $b_1 > b_2 > \dots$ of elements, each $b_i \in C(a_i)$, and $b_i > a_j$ if and only if $j > i$ (see Figure 9). (If the successive b_i's are not always on the same side of the a_i's then at least a subsequence of the b_i's can be chosen all on the same side. If the b_i's are below then we construct an ascending chain $b_1 < b_2 < \dots$, each $b_i \in C(a_i)$, and $b_i < a_j$ if and only if $j > i$.) Now, set $C = \{b_1 > b_2 > \dots\}$ and $c_0 = \inf_P C$. Note that $c_0 \notin C$. Choose $c_1 \in C$ as in Lemma 1. Then $c_1 = b_i$ for some i, that is, $c_1 > a_{i+1}, a_{i+2}, \dots$ and that, according to Lemma 1 means that $c_0 < a_{i+1}, a_{i+2}, \dots$.

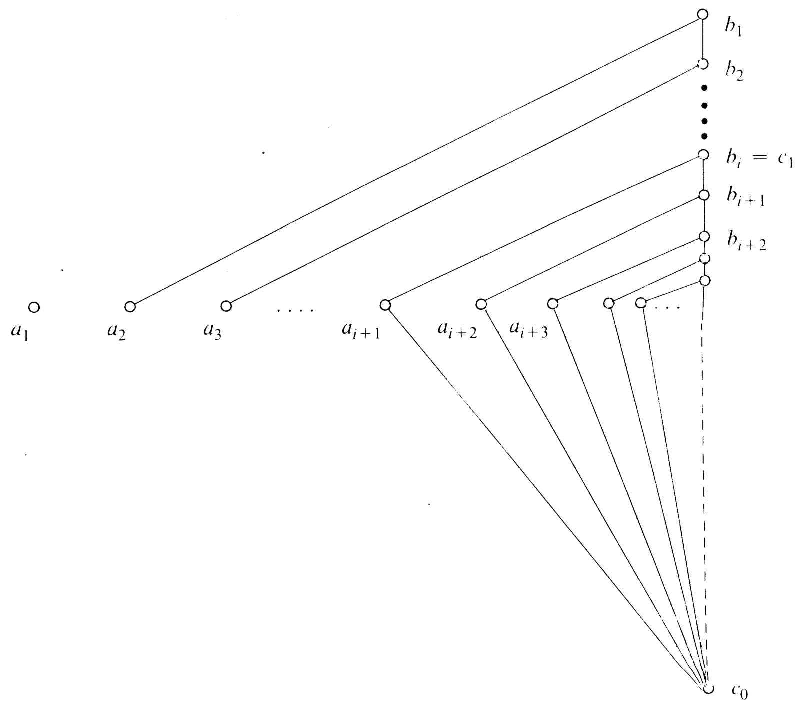


Figure 9

COROLLARY 3. *Let P be an ordered set with the finite cutset property. If every chain of P is finite then P itself is finite too.*

Here follow two simple proofs of this fact. The first is an order theoretical proof. If P contains an infinite antichain then according to Proposition 2 P will contain either an infinite descending chain or an infinite ascending chain. Finally, if each chain and each antichain is finite then, of course, P is finite.

The second proof is quite different in character and applies the result from [**1**] cited at the outset. Let C be a maximal chain of P. Then

$$\{C\} = \bigcap_{x \in C} B(x)$$

is an open set in $\mathbf{M}(P)$ (this is a finite intersection). As P has the finite cutset property $\mathbf{M}(P)$ is compact. That means that P can be covered by finitely many chains, and so P is finite.

We turn now to the proof of Theorem 1. Let A be an uncountable antichain in P. For each $a \in A$ there is a finite cutset $C(a)$. We may choose a positive integer r such that $|C(a)| = r$ for an uncountable subset of A. After possibly relabelling call this subset A, itself. Let $r^{\underline{\omega}}$ stand for the tree with countably many levels starting from a root and branching into r vertices at each vertex. Let us write

$$r^{\underline{\omega}} = \bigcup_{n \in \omega} r^n,$$

where r^n is the set of all functions f of $n = \{0, 1, \dots, n - 1\}$ to $r = \{0, 1, \dots, r - 1\}$. Now, for each $\sigma \in r^{\underline{\omega}}$ we shall construct (i) sets U_σ and (ii) elements a_σ, b_σ. To this end call $A = A_0$ and let a_0 be any element of A. Recall $C(a_0)$ is an r-element cutset for a_0. Put $U_0 = C(a_0) - A_0$ and $b_0 = 1_P$, the maximum element of P; we may suppose without loss of any generality that it exists (see Figure 10).

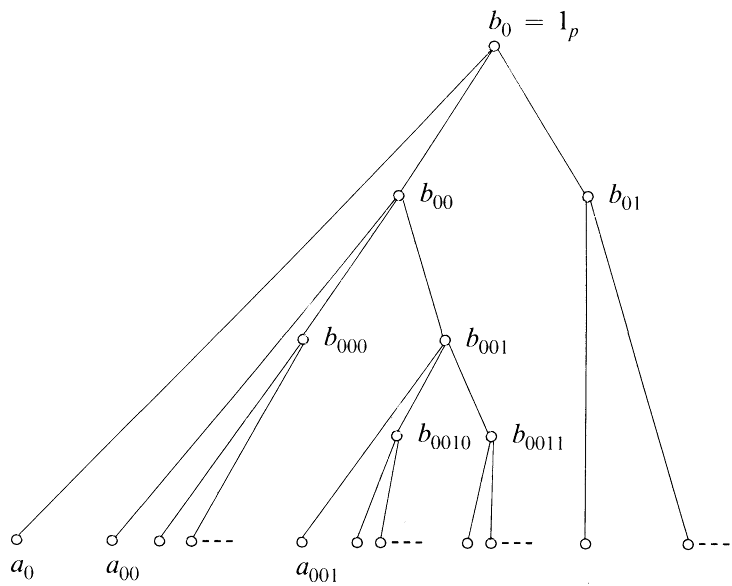


Figure 10

Suppose that for each $k \leqq n$, A_k and each

$$\sigma \in \bigcup_{k \leqq n} r^k$$

has been constructed. Let $\sigma \in r^{n+1}$ and write $\sigma = \tau \cup \{ (n, i) \}$, where $\tau = \sigma|n$ and $i = \sigma(n)$. Now, set

$$A_{n+1} = A_n - (\bigcup_{\tau \in r^n} (C(a_\sigma) \cap A) \cup \{a_\tau | \tau \in r^n\}).$$

Suppose

$$U_\tau = \{u_{\tau,0}, u_{\tau,1}, \dots, u_{\tau,r-1}\}.$$

Note that the cutsets have at most r elements and that

$$U_\tau = C(a_\tau) - A.$$

Choose $b_\sigma = u_{\tau,i}$ and

$$a_\sigma \in A_{n+1} \cap \bigcap_{k \leqq n+1} X(b_{\sigma|k})$$

where

$$X(y) = \{z \in P | z \leqq y \text{ or } z \geqq y\}$$

(provided this is nonempty), and otherwise let $a_\sigma = 1$. Also, put

$$U_\sigma = (C(a_\sigma) \cap \bigcap_{k \leqq n+1} X(b_{\sigma|k})) - A$$

as long as it is nonempty and otherwise $U_\sigma = \{1\}$. This completes the inductive construction.

Now, each A_n is a cofinite subset of A so there is

$$a \in \bigcap_{n \in \omega} A_n.$$

We now define $f \in r^{\underline{\omega}}$ such that $a \in X(b_{f|n})$ for all $n \in \omega$. Note that $a \in A_1$ but $a \neq a_0$. Take any maximal chain C, $a \in C$. Then C meets $C(a_0)$ and so C meets U_0, say in $u_{0,i}$. Then define $f(0) = i$. Now proceed inductively. Suppose all $f(0), f(1), \ldots, f(n)$ such that $a \in X(b_{f|k})$ for all $k \leqq n + 1$ are defined. To define $f(n + 1)$ consider $a \in A_{n+1}$. Let C be any maximal chain containing

$$\{a\} \cup \{b_{f|k} | k \leqq n + 1\}.$$

(From the construction, the b's form a chain.) Then

$$C \cap C(a_{f|n+1}) \neq \emptyset.$$

Actually $C \cap U_{f|n+1} \neq \emptyset$, say

$$u_{f|n+1,i} \in C \cap U_{f|n+1}.$$

Define $f(n + 1) = i$.

Now consider $\{b_{f|n} | n \in \omega\}$. (For these b's the corresponding U_σ's are nonempty so the b's themselves must all be distinct.) Say

$$S = \{n | b_{f|n} > a\}$$

is infinite (or else $\{n | b_{f|n} < a\}$ is infinite). Then by construction $\{b_{f|n} | n \in S\}$ forms a decreasing sequence. Let

$$c_0 = \inf_P \{b_{f|n} | n \in S\}.$$

Note that $a \leqq c_0$. According to Lemma 1,

$$c_0 \leqq a_{f|n-1} \quad \text{for many } n \in S,$$

so $a \leqq a_{f|n-1}$ and that is a contradiction. This completes the proof of Theorem 1.

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We conjecture that in a chain complete ordered set with the finite cutset property every uncountable subset contains an uncountable chain.

Cellularity. In [**1**] Bell and Ginsburg consider compact topological spaces of the form $\mathbf{M}(P)$ for some ordered set P. They show, for instance, that the one-point compactification of an uncountable discrete space is not of the form $\mathbf{M}(P)$.

Which compact spaces arise from an $\mathbf{M}(P)$ for a chain complete ordered set P? While this is not in general known Theorem 2 provides an interesting property of such topological spaces. Before we turn to its proof we record this useful fact.

LEMMA 4. *Let P be a chain complete ordered set with the finite cutset property. Then P contains no subset isomorphic to* $(\underset{\sim}{\omega} \oplus \mathbf{1}) \times \mathbf{2}$ *or to its dual.*

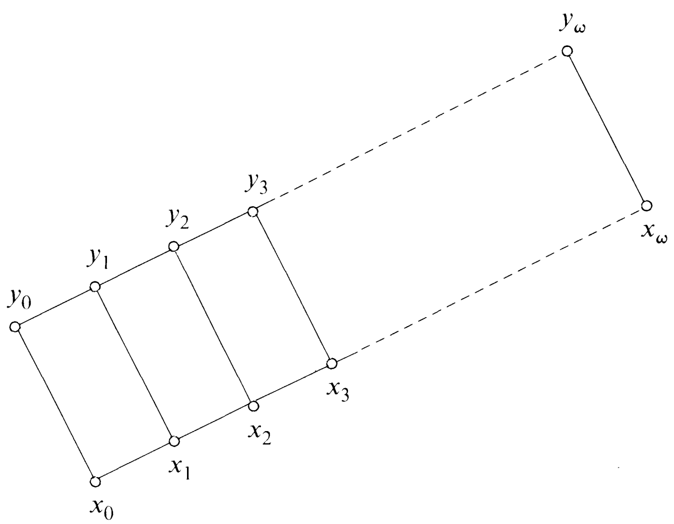

Figure 11

Proof. Suppose to the contrary that there are chains $x_0 < x_1 < x_2 < \dots < x_\omega$ and $y_0 < y_1 < y_2 < \dots < y_\omega$ in P whose union is isomorphic to $(\underset{\sim}{\omega} \oplus \mathbf{1}) \times \mathbf{2}$ (cf. Figure 11). Let $C = \{x_0 < x_1 < x_2 < \dots\}$ and set

$$c_0 = \sup_P\{x_i | i = 0, 1, 2, \dots\} \leqq x_\omega.$$

According to Lemma 1 there is $c_1 \in C$ (note that $c_0 \notin C$) such that $z \geqq c_1$ implies z is comparable to c_0. But $c_1 = x_i$ for some i and y_i is noncomparable to x_ω and x_{i+1} whence it must be noncomparable to c_0, too.

And now to the proof of Theorem 2.

Suppose that $\mathbf{M}(P)$ is compact, that P is a chain complete ordered set and suppose that $\mathbf{M}(P)$ contains a family of $(2^{\omega})^{+}$ disjoint open sets, say

$$F = \{G_{\alpha} | \alpha < (2^{\omega})^{+}\}.$$

Each G_{α} contains a nonempty finite intersection of subbasic open sets, say

$$G_{\alpha} \supseteq \bigcap_{x \in F_{\alpha}} B(x)$$

for some finite $F_{\alpha} \subseteq P$. Without loss of generality there is an integer n such that $|F_{\alpha}| = n$ for each $\alpha < (2^{\omega})^{+}$. Let $B(F_{\alpha})$ stand for

$$\bigcap_{x \in F_{\alpha}} B(x).$$

Note that

$$B(F_{\alpha}) = \{C \in \mathbf{M}(P) \,|\, F_{\alpha} \subseteq C\}$$

and that F_{α} is a chain in P. If $\alpha \neq \beta$ then

$$B(F_{\alpha}) \cap B(F_{\beta}) = \emptyset$$

and then $F_{\alpha} \cup F_{\beta}$ is not a chain. Let

$$F_{\alpha} = \{x_{\alpha,1}, x_{\alpha,2}, \ldots, x_{\alpha,n}\}$$

where $x_{\alpha,1} < x_{\alpha,2} < \ldots < x_{\alpha,n}$. For $\alpha \neq \beta$ there is $i \leqq n$ and $j \leqq n$ such that $x_{\alpha,i}$ is noncomparable to $x_{\beta,j}$. Let $\{\alpha, \beta\}^{<}$ denote the pair $\{\alpha, \beta\}$ where $\alpha < \beta$. Partition the pairs $\{\alpha, \beta\}$ of ordinals less than $(2^{\omega})^{+}$ into sets D_i, $E_{i,j}$ as follows: for $i = 1, 2, \ldots, n$, $\{\alpha, \beta\}^{<} \in D_i$ if $x_{\alpha,i}$ is noncomparable to $x_{\beta,i}$; for $i \neq j$, $\{\alpha, \beta\}^{<} \in E_{(i,j)}$ if $x_{\alpha,k}$ is comparable to $x_{\beta,k}$ for all k and $x_{\alpha,i}$ is noncomparable to $x_{\beta,j}$. Every pair $\{\alpha, \beta\}$ is in one of the sets D_i, $E_{(i,j)}$. As $(2^{\omega})^{+} \rightarrow (\omega^{+})^{2}_{\omega}$ (cf. [3]) there is a subset S of $(2^{\omega})^{+}$ such that $|S| = \omega_1$ and the pairs $\{\alpha, \beta\}$ from S are either all in one D_i or all in one $E_{(i,j)}$. If they are all in D_i then $\{x_{\alpha,i} | \alpha \in S\}$ would be an uncountable antichain and that is impossible according to Theorem 1. So they must all be in $E_{(i,j)}$. Say $i < j$. Therefore, for $\alpha, \beta \in S$ and $\alpha < \beta$, $x_{\alpha,i}$ is noncomparable to $x_{\beta,j}$ but, for each i, $\{x_{\alpha,i} | \alpha \in S\}$ is a chain. If $x_{\beta,i} > x_{\alpha,i}$ then $x_{\beta,j} > x_{\alpha,i}$; hence $x_{\beta,i} < x_{\alpha,i}$. Also, $x_{\beta,j} < x_{\alpha,j}$. It follows that

$$\{x_{\alpha,i} | \alpha \in S\} \cup \{x_{\alpha,j} | \alpha \in S\}$$

is a subset of P isomorphic to $\omega_1^{d} \times 2$, which according to Lemma 4 is impossible. And that completes the proof of Theorem 2.

One further remark is in order. Let P be a chain complete ordered set with the finite cutset property. If, in addition, each uncountable subset of P contains an uncountable chain (as conjectured above) then $\mathbf{c}(\mathbf{M}(P)) \leqq \omega$. To see this suppose that the $B(F_\alpha)$, for $\alpha < \omega_1$, are disjoint open sets and

$$F_\alpha = \{x_{\alpha,1}, x_{\alpha,2}, \dots, x_{\alpha,n}\}.$$

We may suppose that, for each i, $\{x_{\alpha,i}|\alpha < \omega_1\}$ is a chain. Construct a partition of the pairs α,β, $\alpha,\beta < \omega_1$ as above. Then using the partition relation $\omega_1 \rightarrow (\omega + 1)^2_k$ (cf. [**3**]) we find, as above, a subset of P isomorphic to $(\underset{\sim}{\omega} \oplus \mathbf{1}) \times \mathbf{2}$, which is a contradiction.

However, let T be a Souslin tree and set $P = T \oplus T$ (cf. [**4**]). Then P has no uncountable antichain, P contains no subset isomorphic to $\underset{\sim}{\alpha} \times \mathbf{2}$ (for any $\underset{\sim}{\alpha} > \mathbf{2}$) and yet $\mathbf{M}(P)$ does contain an uncountable family of disjoint open sets.

Two final remarks. Some of the results reported in this article can be viewed in graph-theoretical terms. This is because a maximal chain in an ordered set is a (maximal) complete subgraph of its comparability graph. However, the transitivity property of an ordered set cannot usually be dropped in our results. For instance, let G be the graph with vertex set consisting of two disjoint sets $A = \{a_1, a_2, \dots\}$, $B = \{b_1, b_2, \dots\}$ each of size κ such that A is complete, B is independent and each a_i is joined to b_j if and only if $j \neq i$. Then for each i, the pair $\{a_i, b_i\}$ intersects every maximal complete subgraph (and so, in a sense, G has the "1-cutset" property). Still, unlike ordered sets, G is not the union of two (in fact, less than κ) complete subgraphs.

What if for each element x in an ordered set P there is a cutset $C(x)$ which is restricted by some order theoretical constraint rather than by its size? Say $C(x)$ is required to be a chain, for each $x \in P$. Then P is not necessarily the union of two chains (see Figure 12). Can P contain an infinite antichain? Can P even contain a five-element antichain?

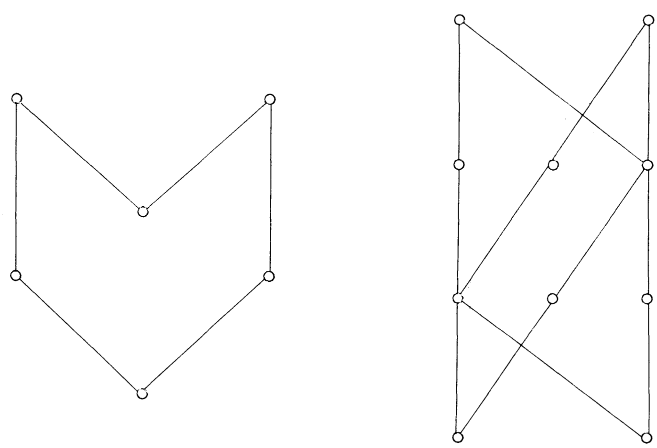

Figure 12

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https://doi.org/10.4153/CJM-1986-031-3 Published online by Cambridge University Press