

HAMILTONIAN CYCLES IN CERTAIN GRAPHS

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Abstract

It is observed that arrays which arise in the scheduling of tournaments exist if and only if there are Hamiltonian cycles in certain graphs. The graphs are generalizations of those which arise in the "Footballers of Croam" problem. It is proven that such Hamiltonian cycles exist in infinite classes of the graphs.

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1. Introduction

We consider the following problem. Given positive integers m , n and g , is it possible to arrange all the subsets of size m of an n -set S in an array with g columns such that

- (i) each subset appears exactly once in the array;
- (ii) there are exactly g subsets in each row, except possibly the last;
- (iii) no element of S appears more than once in any row;
- (iv) no element of S appears in two successive rows;
- (v) no element of S appears in both the first and the last row.

Grant (1975) pointed out that arrays with these properties (and similar arrays with property (v) relaxed) arise in the scheduling of tournaments; the interested reader should see his paper for details. Grant studied the case $m = 2$, for arbitrary n and g .

The arrays are also related to a well-known problem of graph theory. Let $G_{n,m}$ denote the graph whose vertices are labelled with the m -subsets of an n -set, two vertices being adjacent if and only if their inducing sets are disjoint. Then $G_{n,m}$ has $\binom{n}{m}$ vertices and $\frac{1}{2} \binom{n}{m} \binom{n-m}{m}$ edges. An array satisfying the five conditions

of the problem in the case $g = 1$ exists if and only if $G_{n,m}$ has a Hamiltonian cycle.

The case $n = 2k - 1$, $m = k - 1$, $g = 1$ has been considered by several authors. Meredith and Lloyd (1972) called $G_{2k-1,k-1}$ “the graph 0_k ” and conjectured that, for $k \geq 4$, 0_k is an edge-disjoint union of Hamiltonian cycles plus, perhaps, a 1-factor. It is easily verified that 0_2 has a Hamiltonian cycle and that 0_3 does not (0_3 is in fact the Petersen graph). The graphs 0_4 , 0_5 , 0_6 , 0_7 and 0_8 have all been shown to have Hamiltonian cycles (see Balaban, 1972; Meredith and Lloyd, 1973; Mather, 1976). The problem appears to be very difficult.

For the rest of the paper we restrict our attention to the “graphical” case $g = 1$, but consider general m and n with $n \geq 2m + 1$. (If $n \leq 2m$ the array is impossible and, in fact, the graph is disconnected, unless $m = 1$.) Without loss of generality we may choose the n -set S to be $\{1, 2, \dots, n\}$.

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2. Graph-theoretic preliminaries

We shall use two standard results of graph theory—Phillip Hall’s Theorem on matchings and Ore’s Theorem on the existence of Hamiltonian cycles.

If G is a bipartite graph on two sets X and Y , written as $G = G(X, Y)$, then a *complete matching* of X into Y is a set of edges such that no two edges are incident and every vertex of X appears in some edge. The following theorem is due to P. Hall (1935).

THEOREM 2.1. *Let $G = G(X, Y)$ be a bipartite graph. There exists a complete matching of X into Y if and only if for all $A \subseteq X$, $|A| \leq |R(A)|$ where $R(A)$ is the set of vertices in Y adjacent to vertices of A .*

COROLLARY 2.1.1. *In a bipartite graph $G(X, Y)$ there exists a complete matching of X into Y if every vertex in X is adjacent to k or more vertices in Y and every vertex in Y is adjacent to k or fewer vertices in X .*

The proof of the corollary follows easily from the theorem.

THEOREM 2.2 (Ore, 1960). *A graph with $p \geq 3$ vertices has a Hamiltonian cycle if, for every pair u and v of non-adjacent vertices,*

$$\text{degree } u + \text{degree } v \geq p.$$

COROLLARY 2.2.1. *$G_{n,m}$ has a Hamiltonian cycle whenever $2\binom{n-m}{m} \geq \binom{n}{m}$.*

3. The cases $m = 1, 2, 3$

The existence of a Hamiltonian cycle in $G_{n,m}$ is easy to discuss when $m = 1$ or 2 . The case $m = 3$ is more difficult and takes up most of this section.

It is interesting to notice that $G_{n,1}$ is the complete graph K_n , and $G_{n,2}$ is the complement of the line graph of K_n .

THEOREM 3.1. *The graph $G_{n,1}$, $n \geq 3$, has a Hamiltonian cycle.*

The proof is trivial.

THEOREM 3.2. *The graph $G_{n,2}$, $n \geq 6$, has a Hamiltonian cycle.*

PROOF. The graph $G_{n,2}$ has $\binom{n}{2}$ vertices and is regular of degree $\binom{n-2}{2}$. As $2\binom{n-2}{2} \geq \binom{n}{2}$ for $n \geq 8$, then by Corollary 2.2.1, $G_{n,2}$, $n \geq 8$, always has a Hamiltonian cycle. The following sequences of vertices illustrate Hamiltonian cycles in $G_{6,2}$ and $G_{7,2}$ respectively:

25, 16, 23, 45, 12, 34, 56, 13, 24, 35, 46, 15, 26, 14, 36;

25, 37, 12, 34, 56, 17, 23, 45, 67, 13, 24, 35, 46, 57, 16, 27, 36, 47, 15, 26, 14.

THEOREM 3.3. *The graph $G_{n,3}$, $n \geq 7$, has a Hamiltonian cycle.*

PROOF. Theorem 2.2 implies that $G_{n,3}$ has a Hamiltonian cycle for $n \geq 16$ and Theorem 4.2 will prove that $G_{n,3}$ has a Hamiltonian cycle when $n = 12, 13$ or 15 leaving only the cases $n = 7, 8, 9, 10, 11$ or 14 ; however, it is shorter to give a general proof than to exhibit the six Hamiltonian cycles.

The proof is by induction. We begin by exhibiting sequences of vertices which are Hamiltonian cycles in $G_{7,3}$ (due to Balaban, 1972) and in $G_{8,3}$, respectively:

236, 157, 246, 135, 467, 125, 346, 257, 134, 256, 137, 456, 123, 567, 234,
167, 235, 147, 356, 127, 345, 267, 145, 237, 146, 357, 124, 367, 245, 136,
247, 156, 347, 126, 457;

247, 138, 567, 348, 156, 234, 678, 124, 356, 178, 235, 146, 578, 134, 256,
347, 568, 137, 248, 357, 468, 157, 236, 145, 268, 147, 368, 257, 168, 237,
458, 127, 346, 125, 378, 245, 136, 278, 456, 123, 478, 126, 345, 128, 457,
238, 167, 258, 367, 148, 267, 135, 246, 158, 467, 358.

Now assume that the graph $G_{n-2,3}$ has a Hamiltonian cycle. Also assume that $n \geq 9$.

Writing V for the vertex set of $G_{n,3}$, partition V into four subsets, V_1, V_2, V_3 and V_4 as follows: V_1 consists of all vertices whose indexing sets contain neither of the elements 1 and 2; V_2 consists of all vertices whose indexing sets contain 1 but not 2; V_3 has all vertices whose indexing sets contain 2 but not 1; and V_4 has all vertices whose indexing sets contain both 1 and 2. Obviously $V = V_1 \cup V_2 \cup V_3 \cup V_4$ and $|V_1| = \binom{n-2}{3}, |V_2| = |V_3| = \binom{n-2}{2}$ and $|V_4| = \binom{n-2}{1}$. By the induction hypothesis the subgraph induced by V_1 has a Hamiltonian cycle; we denote it by C .

We now consider inserting the vertices of V_4 into C in the following way. Say $v = v_1 v_2 v_3$ and $u = u_1 u_2 u_3$ are two adjacent vertices in C and $w = 12w_1$ is a vertex of V_4 where $w_1 \notin \{v_1, v_2, v_3, u_1, u_2, u_3\}$. Then we can extend C to include w by deleting the edge (u, v) and adding the edges (u, w) and (w, v) . For any edge of C there are $n-8$ possible vertices in V_4 that could be inserted into this edge. So with each of the $\binom{n-2}{3}$ edges of C we can associate a set of $n-8$ vertices of V_4 .

We next count the number of these sets which contain a given vertex of V_4 . Consider the vertex $u = 12u_1$ of V_4 . As u_1 appears in the indexing set of $\binom{n-3}{2}$ vertices of V_1 , none of which are adjacent, then u is contained in

$$\binom{n-2}{3} - 2 \binom{n-3}{2} = (n-8)(n-4)(n-3)/6$$

of the sets associated with the edges of C ; and as $n \geq 9$ each vertex of V_4 is contained in at least $5(n-8)$ of these sets.

Construct a bipartite graph $G = G(X, Y)$ where X is the set of vertices of V_4 and Y is the set of edges in C . If $x \in X$ and $y \in Y$ then x is adjacent to y if and only if x is contained in the set associated with the edge y . Thus each vertex in X has degree $(n-8)(n-4)(n-3)/6 \geq 5(n-8)$, as $n \geq 9$, and each vertex of Y has degree $n-8$. Putting $k = n-8$ we can, using Corollary 2.1, associate each vertex of V_4 with a distinct edge of C . Therefore the cycle C can be extended to a cycle C' which passes through all vertices of $V_1 \cup V_4$.

We consider separately the cases when $\binom{n-2}{2}$ is odd and when $\binom{n-2}{2}$ is even.

Case 1. When $\binom{n-2}{2}$ is odd. Consider the vertex sets V_2 and V_3 . In V_2 delete the element 1 from all indexing sets and draw the graph $G_{n-2,2}$ on this new set of vertices, V'_2 . Similarly delete 2 from V_3 and draw the graph $G_{n-2,2}$ on V' . Since

by Theorem 3.2, $G_{n-2,2}$ has a Hamiltonian cycle for $n-2 \geq 6$ we can draw Hamiltonian cycles through the vertices of V'_2 and V'_3 . Permute the elements of the indexing sets in these cycles so that we have $u = 34$ adjacent to $v = 56$ and then delete the edges (u, v) from both cycles to obtain two Hamiltonian paths of odd length; their lengths are $\binom{n-2}{2}$. In V'_2 relabel $u_1 = 34$ with $u'_1 = 134$. Then if $w = w_1 w_2$ is adjacent to u relabel it with $w' = 2w_1 w_2$. Continue in this manner until $v_1 = 56$ is relabelled with $v'_1 = 156$ (as $\binom{n-2}{2}$ is odd). For V'_3 relabel $u_2 = 34$ with $u'_2 = 234$ and continue as for V'_2 until $v_2 = 56$ is relabelled $v'_2 = 256$. Then draw an edge joining the vertices $u'_1 = 134$ and $v'_2 = 256$. We now have a Hamiltonian path through the vertices of $V_2 \cup V_3$.

We return now to the Hamiltonian cycle C' through the vertices of $V_1 \cup V_4$. Here there must be two adjacent vertices neither of which has the elements 1 and 2 in its indexing set as otherwise we have $n-2 \geq \binom{n-2}{3}$ which is never true for $n \geq 9$. Permute the elements of the indexing sets in $V_1 \cup V_4$ so that vertex $u_3 = 347$ is adjacent to $v_3 = 568$. We can then draw edges from $u'_2 = 234$ to $v_3 = 568$ and from $v'_2 = 256$ to $u_3 = 347$ and we now have a Hamiltonian cycle in $G_{n,3}$, $n \geq 9$, $\binom{n-2}{2}$ odd.

Case 2. When $\binom{n-2}{2}$ is even. Note that we now only require $n \geq 10$ as $\binom{9-2}{2}$ is odd. Construct the two paths through the vertices of $V_2 \cup V_3$ as was done in Case 1. However, this time the endpoints of one path, P_1 , are $u_1 = 234$, $v_1 = 156$ and of the other, P_2 , are $u_2 = 134$, $v_2 = 256$. Again, relabel the Hamiltonian cycle C' through $V_1 \cup V_4$ so that $u_3 = 347$ and $v_3 = 568$ are adjacent. We now have a cycle and two paths which we wish to join together to form a Hamiltonian cycle in $G_{n,3}$, $n \geq 10$, $\binom{n-2}{2}$ even. First add the edge (u_2, v_2) . The number of vertices in P_2 with 1 in their indexing sets is $\frac{1}{2} \binom{n-2}{2}$ whilst the number with either 3 or 4 in their indexing sets is $2 \binom{n-4}{1} + 1$. As $\frac{1}{2} \binom{n-2}{2} > 2 \binom{n-4}{1} + 1$ for $n \geq 10$ then clearly we have in P_2 a vertex $v = 1x_1y_1$ where neither x_1 nor y_1 is 3 or 4. Let $u = 2x_2y_2$, $u \neq 256$, be the vertex adjacent to v in P_2 . Delete the edge (u, v) and add the edges (v_1, u_3) and (v, u_1) . The indexing set of u can contain at most two of 5, 6 and 8. If it contains any of these make the relevant exchange of the elements x_2 and 9 or y_2 and 10 in C' ; then add the edge (u, v_3) . On deleting (u_3, v_3) we have a Hamiltonian cycle in $G_{n,3}$, $n \geq 10$, $\binom{n-2}{2}$ even.

Therefore, combining the two cases, we have that the graph $G_{n,3}$, $n \geq 7$, always contains a Hamiltonian cycle.

4. Cases where n is large relative to m

In this section we derive two theorems which state that $G_{n,m}$ has a Hamiltonian cycle in cases where n is sufficiently large relative to m . Theorem 4.2 uses a result on block designs. A balanced incomplete block design is said to be *resolvable* if its blocks can be partitioned into sets in such a way that every set contains each treatment precisely once. (The reader is referred to Chapters 7, 8 and 10 of Street and Wallis, 1977.)

THEOREM 4.1 (Baranyai, 1975). *The balanced incomplete block design with parameters $(km, \binom{km}{m}, \binom{km-1}{m-1}, m, \binom{km-2}{m-2})$ in which the blocks are all the m -sets of a set of km treatments is resolvable.*

We can now proceed to the lemmas needed for the proof of the main result.

LEMMA 4.2.1. *Given an n -set S let X be the set of all r -sets of S and Y the set of all l -sets of S , $n \geq l+r$ and $l > r$. We can associate with each r -set an l -set which contains the r -set, that is, if the r -set is $x_1 x_2 \dots x_r$, then the associated l -set is $x_1 x_2 \dots x_r x_{r+1} \dots x_l$; so that different l -sets are associated with different r -sets.*

PROOF. Construct a bipartite graph $G = G(X, Y)$ in the following way. Let the vertices of X be the r -sets of S and the vertices of Y be the l -sets of S , and let vertices be adjacent if one contains the other. Then each vertex in X has degree $\binom{n-r}{l-r}$ and each vertex in Y has degree $\binom{l}{r}$. So if $\binom{n-r}{l-r} \geq \binom{l}{r}$ then by Corollary 2.1.1 there is a complete matching of X into Y and hence the lemma is proven.

If $n \geq l+r$, $l > r$, then $r < \frac{1}{2}n$. Now either $n-l$ or l is $\leq \frac{1}{2}n$ and both are bigger than or equal to r so either $\binom{n}{n-l} \geq \binom{n}{r}$ or $\binom{n}{l} \geq \binom{n}{r}$. But both these statements are the same. Hence $(n-r)!r! \geq (n-l)!l!$ and so $\binom{n-r}{l-r} \geq \binom{l}{r}$.

LEMMA 4.2.2. *The graph $G_{n,m}$, $n = km$, always has a Hamiltonian cycle when $k \geq m+1$.*

PROOF. The vertices of $G_{n,m}$ are labelled with the blocks of the balanced incomplete block design of Theorem 4.1. Therefore, we can partition the vertex set

of $G_{n,m}$ into $p = \binom{km-1}{m-1}$ classes R_1, R_2, \dots, R_p each of size k . It is easy to see that the subgraph induced by any one of these p classes will be the complete graph on k vertices, K_k . In R_i we have a vertex v_i labelled with the indexing set $1x_1^i x_2^i \dots x_{m-1}^i$, for all $i \in \{1, 2, \dots, p\}$. As at most m of the vertices of R_{i+1} can contain elements of the indexing set of v_i and as there are $k \geq m+1$ vertices in R_{i+1} then there is at least one vertex u_{i+1} whose indexing set is disjoint from that of v_i . Therefore, we can draw an edge from v_i to u_{i+1} . Continue in this manner joining R_i to R_{i+1} finally joining R_p to R_1 . Obviously, as u_i and v_i are distinct and both are vertices in a complete graph K_k , there is a Hamiltonian path in K_k of which they are the endpoints. The Hamiltonian cycle C in $G_{n,m}$, $n = km$, $k \geq m+1$, can now clearly be seen.

THEOREM 4.2. *The graph $G_{n,m}$ where $n = km+l$, $0 \leq l < m$, will always have a Hamiltonian cycle if $k \geq \max\{2^{l+1}, m+1\}$.*

PROOF. Assume that $k \geq \max\{2^{l+1}, m+1\}$ where $n = km+l$, $0 \leq l < m$. In the case $l = 0$ we have $k \geq \max\{2, m+1\}$ and as $m \geq 1$ then always $k \geq m+1$ and by Lemma 4.2.2, $G_{n,m}$, $n = km$, always has a Hamiltonian cycle. Thus we can assume $0 < l < m$.

Let S be the n -set from which the vertices of $G_{n,m}$, $n = km+l$, are labelled. Select a subset of l elements from S , say $Y = \{y_1, y_2, \dots, y_l\}$, and use the remaining km elements to construct $G' = G_{km,m}$. Draw the Hamiltonian cycle C in G' as constructed in Lemma 4.2.2.

We want to insert the remaining vertices of $G_{n,m}$ into the edges of C , as was done in the first part of Theorem 3.3.

Before we begin some notation is needed. Write $\begin{bmatrix} l \\ r \end{bmatrix} \begin{bmatrix} km \\ m-r \end{bmatrix}$ to indicate an m -set of S of the form $y_1 y_2 \dots y_r x_1 x_2 \dots x_{m-r}$ where $y_i \in Y$, $x_i \in S \setminus \{Y \cup \{1\}\}$, and $1 \begin{bmatrix} l \\ r \end{bmatrix} \begin{bmatrix} km \\ m-r-1 \end{bmatrix}$ to indicate an m -set $1y_1 y_2 \dots y_r x_1 x_2 \dots x_{m-r-1}$ where $y_i \in Y$, $x_i \in S \setminus \{Y \cup \{1\}\}$. When we speak of u_i , v_i and R_i they are the u_i , v_i and R_i of Lemma 4.2.2.

Thus the vertices of $G_{n,m}$, not in G' , are labelled

$$\begin{aligned} & \begin{bmatrix} l \\ 1 \end{bmatrix} \begin{bmatrix} km \\ m-1 \end{bmatrix}, 1 \begin{bmatrix} l \\ 1 \end{bmatrix} \begin{bmatrix} km \\ m-2 \end{bmatrix}, \begin{bmatrix} l \\ 2 \end{bmatrix} \begin{bmatrix} km \\ m-2 \end{bmatrix}, 1 \begin{bmatrix} l \\ 2 \end{bmatrix} \begin{bmatrix} km \\ m-3 \end{bmatrix}, \dots, \\ & \begin{bmatrix} l \\ l-1 \end{bmatrix} \begin{bmatrix} km \\ m-l+1 \end{bmatrix}, 1 \begin{bmatrix} l \\ l-1 \end{bmatrix} \begin{bmatrix} km \\ m-l \end{bmatrix}, \begin{bmatrix} l \\ l \end{bmatrix} \begin{bmatrix} km \\ m-l \end{bmatrix}, 1 \begin{bmatrix} l \\ l \end{bmatrix} \begin{bmatrix} km \\ m-l-1 \end{bmatrix}. \end{aligned}$$

Insert these vertices into the Hamiltonian cycle C of G' in the following way.

With each v_i in C we can associate $\binom{l}{1}$ vertices of the form $\left[\binom{l}{1} \right] \left[\binom{km}{m-1} \right]$, where if $v_i = 1x_1x_2 \dots x_{m-1}$ then $\left[\binom{km}{m-1} \right] = x_1x_2 \dots x_{m-1}$. Consider the path of C in R_i from v_i to u_i . Since x_j does not occur in the indexing set of any vertex in this path except for v_i then we can extend each edge of the path, except for that edge incident with v_i , to include one of the $\binom{l}{1}$ vertices. This can be done provided the path has at least $\binom{l}{0} + \binom{l}{1}$ edges, i.e. R_i has $1 + \binom{l}{0} + \binom{l}{1}$ vertices. And as $k \geq \max\{2^{l+1}, m+1\}$ this is always true.

Take the vertices of the form $1 \left[\binom{l}{1} \right] \left[\binom{km}{m-2} \right]$ and $\left[\binom{l}{2} \right] \left[\binom{km}{m-2} \right]$. For each $\left[\binom{km}{m-2} \right] = x_1x_2 \dots x_{m-2}$ there is a different $v_i, i \in \{1, 2, \dots, p = \binom{km-1}{m-1}\}$, such that $v_i = 1x_1x_2 \dots x_{m-2}z_1$. (This follows from Lemma 4.2.1.) So to extend the path in R_i from v_i to u_i to include these $\binom{l}{1} + \binom{l}{2}$ vertices R_i must have a further $\binom{l}{1} + \binom{l}{2}$ edges, i.e. R_i must have at least $1 + \left[\binom{l}{0} + \binom{l}{1} \right] + \left[\binom{l}{1} + \binom{l}{2} \right]$ vertices; and there will be this many vertices as $k \geq \max\{2^{l+1}, m+1\}$.

Continue in this manner until finally we have only to add $1 \left[\binom{l}{l} \right] \left[\binom{km}{m-l-1} \right]$ where for each $\left[\binom{km}{m-l-1} \right] = x_1x_2 \dots x_{m-l-1}$ there is a different $v_i, i \in \{1, 2, \dots, p\}$ such that $v_i = 1x_1x_2 \dots x_{m-l-1}z_1z_2 \dots z_l$ (again by Lemma 4.2.1). Then to extend the path in R_i from v_i to u_i it follows that R_i must have a further $\binom{l}{l}$ edges and so have at least

$$1 + \left[\binom{l}{0} + \binom{l}{1} \right] + \left[\binom{l}{1} + \binom{l}{2} \right] + \left[\binom{l}{2} + \binom{l}{3} \right] + \dots + \left[\binom{l}{l-1} + \binom{l}{l} \right] + \binom{l}{l}$$

$$= 2 \left[\binom{l}{0} + \binom{l}{1} + \binom{l}{2} + \dots + \binom{l}{l} \right] = 2^{l+1} \text{ vertices.}$$

As $k \geq \max\{2^{l+1}, m+1\}$ this is always possible.

Therefore $G_{n,m}, n = km+l$, has a Hamiltonian cycle when $k \geq \max\{2^{l+1}, m+1\}$.

Given a positive integer m , we shall define $N(m)$ to be the smallest positive integer such that, for all $n \geq N(m), 2 \binom{n-m}{m} \geq \binom{n}{m}$.

Lemma 4.3.1.

$$\frac{m \cdot m\sqrt{2}}{m\sqrt{2}-1} - 1 \leq N(m) \leq \frac{m \cdot m\sqrt{2}}{m\sqrt{2}-1} + m$$

PROOF. It is easiest to consider the ratio

$$M = \binom{n}{m} / \binom{n-m}{m};$$

$$M = \left(1 - \frac{m}{n}\right) \left(1 - \frac{m}{n-1}\right) \cdots \left(1 - \frac{m}{n-m+1}\right).$$

Clearly

$$\left(1 - \frac{m}{n-m+1}\right)^m \leq M \leq \left(1 - \frac{m}{n}\right)^m.$$

Now

$$\left(1 - \frac{m}{n-m+1}\right)^m \geq \frac{1}{2}$$

implies $M \geq \frac{1}{2}$, and the left-hand inequality is equivalent to

$$n \geq \frac{m \cdot m\sqrt{2}}{m\sqrt{2}-1} + (m-1),$$

so the smallest integer not less than this is an upper bound for $N(m)$, and

$$N(m) \leq \frac{m \cdot m\sqrt{2}}{m\sqrt{2}-1} + m.$$

If

$$\left(1 - \frac{m}{n}\right)^m < \frac{1}{2}$$

then $M < \frac{1}{2}$ and $n < N(m)$. So

$$\frac{m \cdot m\sqrt{2}}{m\sqrt{2}-1} \leq N(m).$$

From the lemma and Corollary 2.2.1 we have

THEOREM 4.3. *If*

$$n \geq \frac{m\sqrt{2} \cdot m}{m\sqrt{2}-1} + m$$

then $G_{n,m}$ has a Hamiltonian cycle.

We note that $N(m)$ is asymptotically equal to $m^2 \log_2 e$, and approaches this value from above. The relative error in using this approximation is quite small (about 5% at $m = 20$).

References

- A. T. Balaban (1972), "Chemical graphs, Part XIII: Combinatorial patterns", *Rev. Roumaine Math. Pures Appl.* **17**, 3–16.
- Zs. Baranyai (1975), "On the factorization of the complete uniform hypergraph", *Infinite and Finite Sets*, Vol. 1, pp. 91–108 (North-Holland Publishing Co.).
- Douglas D. Grant (1975), "How to run a hockey tournament—a combinatorial scheduling problem", *Proc. 5th British Combinatorial Conf.*, pp. 239–254 (Utilitas Math. Pub. Co., Winnipeg).
- P. Hall (1935), "On representatives of subsets", *J. London Math. Soc.* **10**, 26–30.
- Michael Mather (1976), "The Rugby footballers of Croam", *J. Combinatorial Theory (Series B)*, **20**, 62–63.
- Guy H. J. Meredith and E. Keith Lloyd (1972), "The Hamiltonian graphs 0_4 to 0_7 ", *Combinatorics* (Institute of Mathematics and its Applications, Southend-on-Sea).
- Guy H. J. Meredith and E. Keith Lloyd (1973), "The footballers of Croam", *J. Combinatorial Theory (Series B)*, **15**, 161–166.
- Oystein Ore (1960), "Note on Hamilton circuits", *American Math. Monthly*, **67**, 55.
- Anne Penfold Street and W. D. Wallis (1977), *Combinatorial Theory; An Introduction* (C.B.R.C., Winnipeg).

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