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WIGNER'S THEOREM IN $\mathcal{L}^\infty(\Gamma)$-TYPE SPACES

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Abstract

We investigate surjective solutions of the functional equation

$$\{\|f(x)+f(y)\|, \|f(x)-f(y)\|\} = \{\|x+y\|, \|x-y\|\} \quad (x, y \in X),$$

where $f : X \to Y$ is a map between two real $\mathcal{L}^\infty(\Gamma)$-type spaces. We show that all such solutions are phase equivalent to real linear isometries. This can be considered as an extension of Wigner's theorem on symmetry for real $\mathcal{L}^\infty(\Gamma)$-type spaces.

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1. Introduction

Let X and Y be real normed spaces. We say that a mapping $f : X \to Y$ is an isometry if it satisfies the equality

$$\|f(x) - f(y)\| = \|x - y\| \quad (x, y \in X).$$

This equality implies strong structural properties for the mapping f. The classical Mazur–Ulam theorem [5] states that every surjective isometry between X and Y is affine. We say that a mapping $f : X \to Y$ is phase equivalent to a linear isometry if there exists a function $\varepsilon : X \to \{-1, 1\}$ such that εf is a linear isometry. The fundamental theorem of Wigner on symmetry characterises the mappings that are phase equivalent to linear isometries in real Hilbert spaces. That is, when X and Y are real Hilbert spaces, all mappings $f : X \to Y$ that are phase equivalent to linear isometries are precisely the solutions of the functional equation

$$|\langle f(x), f(y)\rangle| = |\langle x, y\rangle| \quad (x, y \in X).$$

Wigner's theorem plays a fundamental role in quantum mechanics and has several equivalent formulations and extensions (see, for example, [1, 2, 4, 6–10, 12]). In [4], a real version of Wigner's theorem was given by using the functional equation

$$\{\|f(x)+f(y)\|, \|f(x)-f(y)\|\} = \{\|x+y\|, \|x-y\|\} \quad (x, y \in X). \tag{1.1}$$

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It is easy to see that, when X and Y are real normed spaces, all mappings $f : X \to Y$ that are phase equivalent to real linear isometries are also the solutions of the functional equation (1.1). In [4], Maksa and Páles proved that the converse also holds provided that X and Y are real inner product spaces, and they posed the question: what are the solutions $f : X \to Y$ of (1.1) when X and Y are normed but not necessarily inner product spaces? Huang and Tan [3] gave a partial answer to the above question for real atomic L_p spaces with $p > 0$.

The aim of this note is to answer the above question for real $\mathcal{L}^\infty(\Gamma)$-type spaces. We will show that the surjective solutions of (1.1) are phase equivalent to linear isometries provided that X and Y are real $\mathcal{L}^\infty(\Gamma)$-type spaces. Indeed, we give a representation theorem of surjective mappings which are phase equivalent to linear isometries in $\mathcal{L}^\infty(\Gamma)$-type spaces.

2. Main results

Throughout this section, all spaces are over the real field $\mathbb{R}$. Let X and Y be normed spaces. We use S_X and S_Y to denote their respective unit spheres. The space of all bounded real-valued functions on an index set Γ equipped with the supremum norm is denoted by $\ell^\infty(\Gamma)$ and any of its subspaces containing all e_γ's ($\gamma \in \Gamma$) are called $\mathcal{L}^\infty(\Gamma)$-type spaces. For example, the spaces $c_0(\Gamma), c(\Gamma), \ell^\infty(\Gamma)$, particularly, c_0, c, ℓ^∞, are $\mathcal{L}^\infty(\Gamma)$-type spaces. The $\ell^\infty(\Gamma)$-space is

$$\ell^\infty(\Gamma) = \Big\{x = \{\xi_\gamma\}_{\gamma\in\Gamma} : \|x\| = \sup_{\gamma\in\Gamma} |\xi_\gamma| < \infty,\ \xi_\gamma \in \mathbb{R},\ \gamma \in \Gamma\Big\}.$$

For every $x = \{\xi_\gamma\}_{\gamma\in\Gamma} \in \mathcal{L}^\infty(\Gamma)$, we write $x = \{\xi_\gamma\}$ and omit the subscripts $\gamma \in \Gamma$ for simplicity of notation. We denote the support of x by Γ_x, that is,

$$\Gamma_x = \{\gamma \in \Gamma : x(\gamma) \neq 0\}.$$

The star of x with respect to $S_{\mathcal{L}^\infty(\Gamma)}$ is defined by

$$St(x) = \{y : y \in S_{\mathcal{L}^\infty(\Gamma)}, \|y + x\| = 2\}.$$

We first cite a basic result for star sets in $\mathcal{L}^\infty(\Gamma)$-type spaces.

Lemma 2.1 [11, Lemma 2]. *Let x be in $S_{\mathcal{L}^\infty(\Gamma)}$. If there exists an $x_0 \in St(x)$ satisfying $\|y - x_0\| \leq 1$ for all $y \in St(x)$, then Γ_{x_0} is a singleton.*

In order to prove the first main result, we need the following lemma.

Lemma 2.2. *Let $X = \mathcal{L}^\infty(\Gamma)$ and $Y = \mathcal{L}^\infty(\Delta)$. Suppose that $f : X \to Y$ is a surjective mapping satisfying (1.1). Let $\gamma \in \Gamma$ and denote by $\Delta_{f(e_\gamma)}$ the support of $f(e_\gamma)$. Then $\Delta_{f(e_\gamma)}$ is a singleton.*

Proof. Suppose that $\Delta_{f(e_\gamma)}$ contains more than one point. Since f is surjective, by Lemma 2.1 there is a vector $x \in X$ with $f(x) \in St(f(e_\gamma))$ such that $\|f(x) - f(e_\gamma)\| > 1$. This implies that

$$\|f(x) + f(e_\gamma)\| + \|f(x) - f(e_\gamma)\| > 3.$$

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By (1.1), f is norm preserving and thus $x \in S_X$. Hence, for every $\gamma \in \Gamma$,

$$\|f(x) + f(e_\gamma)\| + \|f(x) - f(e_\gamma)\| = \|x + e_\gamma\| + \|x - e_\gamma\| \leq 3,$$

which is a contradiction. The proof is complete. □

The following theorem is a representation theorem for surjective mappings between two real $\mathcal{L}^\infty(\Gamma)$-type spaces satisfying (1.1). For any $a, b \in \mathbb{R}$, we shall write $a \vee b = \max\{a, b\}$.

Theorem 2.3. *Let $X = \mathcal{L}^\infty(\Gamma)$ and $Y = \mathcal{L}^\infty(\Delta)$. Suppose that $f : X \to Y$ is a surjective mapping satisfying (1.1). Then there exists a bijection $\pi : \Gamma \to \Delta$ such that for every $x = \{\xi_\gamma\} \in X$, we have $f(x) = \{\eta_{\pi(\gamma)}\} \in Y$ with $|\eta_{\pi(\gamma)}| = |\xi_\gamma|$ for every $\gamma \in \Gamma$.*

Proof. From Lemma 2.2, we can define a map $\pi : \Gamma \to \Delta$ by $\{\pi(\gamma)\} = \Delta_{f(e_\gamma)}$ for each $\gamma \in \Gamma$. We now prove that π is bijective. If $\pi(\gamma_1) = \pi(\gamma_2)$, by (1.1) and Lemma 2.2,

$$\begin{aligned} 2 &= \|f(e_{\gamma_1}) - f(e_{\gamma_2})\| \vee \|f(e_{\gamma_1}) + f(e_{\gamma_2})\| \\ &= \|e_{\gamma_1} - e_{\gamma_2}\| \vee \|e_{\gamma_1} + e_{\gamma_2}\| \leq 2. \end{aligned}$$

So, $\|e_{\gamma_1} - e_{\gamma_2}\| \vee \|e_{\gamma_1} + e_{\gamma_2}\| = 2$, which implies that $\gamma_1 = \gamma_2$. To see that π is surjective, suppose on the contrary that there is a $\delta_0 \in \Delta/\pi(\Gamma)$. As f is surjective, there exists $x \in S_X$ such that $f(x) = e_{\delta_0}$. For every $\gamma \in \Gamma$,

$$\begin{aligned} \|x + e_\gamma\| + \|x - e_\gamma\| &= \|f(x) + f(e_\gamma)\| + \|f(x) - f(e_\gamma)\| \\ &= \|e_{\delta_0} + f(e_\gamma)\| + \|e_{\delta_0} - f(e_\gamma)\| \\ &= 2. \end{aligned}$$

The equation $\|x + e_\gamma\| + \|x - e_\gamma\| = 2$ for all $\gamma \in \Gamma$ implies that $x = \pm e_{\gamma_1}$ for some $\gamma_1 \in \Gamma$ or $x = 0$. Since $\delta_0 \in \Delta/\pi(\Gamma)$, we must have $x = 0$, which is a contradiction.

We shall prove that f has the desired property. Since f is norm preserving, we need only consider the vectors in the unit sphere of X. For every $x = \{\xi_\gamma\} \in S_X$, we can write $f(x) = \{\eta_{\pi(\gamma)}\} \in Y$. For every $\gamma \in \Gamma$, we have $f(e_\gamma) = \pm e_{\pi(\gamma)}$ and so

$$\begin{aligned} 1 + |\xi_\gamma| &= \|x + e_\gamma\| \vee \|x - e_\gamma\| \\ &= \|f(x) + f(e_\gamma)\| \vee \|f(x) - f(e_\gamma)\| \\ &= 1 + |\eta_{\pi(\gamma)}|. \end{aligned}$$

Thus, $|\xi_\gamma| = |\eta_{\pi(\gamma)}|$ for every $\gamma \in \Gamma$. The proof is complete. □

For our second main result, we need one more lemma. For $x = \{\xi_\gamma\} \in \mathcal{L}^\infty(\Gamma)$, we shall use the notation $e_x = \{\theta_\gamma\}$, where $\theta_\gamma = \text{sign}(\xi_\gamma)$ for every $\gamma \in \Gamma$ (if $\xi_\gamma = 0$, we put $\theta_\gamma = 0$ throughout what follows). Obviously, $e_{x+y} = e_x + e_y$, $e_{-x} = -e_x$ and $e_{\lambda x} = e_x$ for all $x, y \in \mathcal{L}^\infty(\Gamma)$ with $\Gamma_x \cap \Gamma_y = \emptyset$ and $\lambda > 0$.

Lemma 2.4. *Let $X = \mathcal{L}^\infty(\Gamma)$ and $Y = \mathcal{L}^\infty(\Delta)$. If $f : X \to Y$ is a surjective mapping satisfying (1.1), then $e_{f(x)} = \pm f(e_x)$ for every $x \in \mathcal{L}^\infty(\Gamma)$.*

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Proof. By (1.1),

$$\{\|f(e_x)+f(\|x\|e_x)\|,\|f(e_x)-f(\|x\|e_x)\|\}=\{\|e_x+\|x\|e_x\|,\|e_x-\|x\|e_x\|\}$$
$$=\{1+\|x\|,|1-\|x\||\}. \tag{2.1}$$

From Theorem 2.3, for every $\gamma\in\Gamma_x$, $|f(e_x)(\pi(\gamma))|=1$ and $|f(\|x\|e_x)(\pi(\gamma))|=\|x\|$. This together with (2.1) implies that

$$e_{f(\|x\|e_x)}=\pm f(e_x).$$

On the other hand,

$$\{\|f(\|x\|e_x)+f(x)\|,\|f(\|x\|e_x)-f(x)\|\}=\{\|\,\|x\|e_x+x\|,\|\,\|x\|e_x-x\|\}$$
$$=\Big\{2\|x\|,\|x\|-\inf_{\gamma\in\Gamma_x}|x(\gamma)|\Big\}.$$

Note that $|f(x)(\pi(\gamma))|=|x(\gamma)|$ for any $\gamma\in\Gamma_x$. This implies that

$$e_{f(x)}=\pm e_{f(\|x\|e_x)}=\pm f(e_x).$$ □

The next result shows that a surjective mapping satisfying (1.1) is close to linear.

Lemma 2.5. *Let $X=\mathcal{L}^\infty(\Gamma)$ and $Y=\mathcal{L}^\infty(\Delta)$. Suppose that $f:X\to Y$ is a surjective mapping satisfying (1.1). Then:*

(a) $f(\lambda x)=\pm\lambda f(x)$ *for every $x\in X$, $\lambda\in\mathbb{R}$;*

(b) *there exist two real numbers α and β with $|\alpha|=|\beta|=1$ such that*

$$f(x+y)=\alpha f(x)+\beta f(y)$$

for all nonzero vectors x and y in X with $\Gamma_x\cap\Gamma_y=\emptyset$.

Proof. (a) It suffices to show that the conclusion holds for every x in the unit sphere of X. From (1.1), $f(-e_x)=\pm f(e_x)$. Applying Lemma 2.4,

$$e_{f(\lambda x)}=\pm f(e_{\lambda x})=\pm f(e_x)=\pm e_{f(x)}.$$

This and Theorem 2.3 imply that

$$f(\lambda x)=\pm\lambda f(x).$$

(b) By Theorem 2.3, we only need to check that

$$e_{f(x+y)}=\alpha e_{f(x)}+\beta e_{f(y)}$$

for some real numbers α and β with $|\alpha|=|\beta|=1$. This is equivalent to showing that

$$f(e_x+e_y)=f(e_{x+y})=\alpha f(e_x)+\beta f(e_y)$$

for some real numbers α and β with $|\alpha|=|\beta|=1$. Write

$$f(e_x)=\{\xi'_{\pi(\gamma)}\},\quad f(e_y)=\{\eta'_{\pi(\gamma)}\},\quad f(e_x+e_y)=\{\xi''_{\pi(\gamma)}+\eta''_{\pi(\gamma)}\},$$

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where $|\xi''_{\pi(\gamma)}| = |\xi'_{\pi(\gamma)}| = 1$ for every $\gamma \in \Gamma_x$ and $|\eta''_{\pi(\gamma)}| = |\eta'_{\pi(\gamma)}| = 1$ for every $\gamma \in \Gamma_y$. By Lemma 2.4 and (1.1),

$$\Big\{\sup_{\gamma\in\Gamma_x} |\xi''_{\pi(\gamma)} + \xi'_{\pi(\gamma)}| \vee 1, \sup_{\gamma\in\Gamma_x} |\xi''_{\pi(\gamma)} - \xi'_{\pi(\gamma)}| \vee 1\Big\}$$
$$= \{\|f(e_x + e_y) + f(e_x)\|, \|f(e_x + e_y) - f(e_x)\|\} = \{2, 1\}.$$

It follows that $\{\xi''_{\pi(\gamma)}\} = \pm f(e_x)$ and, similarly, $\{\eta''_{\pi(\gamma)}\} = \pm f(e_y)$. This completes the proof. □

Theorem 2.6. *Let $X = \mathcal{L}^\infty(\Gamma)$ and $Y = \mathcal{L}^\infty(\Delta)$. Suppose that $f : X \to Y$ is a surjective mapping satisfying (1.1). Then f is phase equivalent to a linear isometry.*

Proof. We first show that f is phase equivalent to a homogeneous map. It follows from the axiom of choice that there is a set $L \subset X$ such that for any $x \in X$ with $x \neq 0$, there exists exactly one element $y \in L$ such that $x = \lambda y$ for some $\lambda \in \mathbb{R}$. The desired map $f' : X \to Y$ can be defined by

$$f'(x) = f'(\lambda y) = \lambda f(y) \quad \text{for all } x = \lambda y \in X.$$

Therefore, we may assume that f is homogeneous. Fix $\gamma_0 \in \Gamma$ and let

$$Z = \{x \in X : \Gamma_x \cap \{\gamma_0\} = \emptyset\}.$$

By Lemma 2.5, for every $z \in Z$, we can write

$$f(z + e_{\gamma_0}) = \alpha(z) f(z) + \beta(z) f(e_{\gamma_0}), \quad |\alpha(z)| = |\beta(z)| = 1.$$

We shall show that for all $z \in Z$ with $z \neq 0$ and $\lambda \in \mathbb{R}$ with $\lambda \neq 0$,

$$\alpha(z)\beta(z) = \alpha(\lambda z)\beta(\lambda z). \tag{2.2}$$

It suffices to show that (2.2) holds for every z in the unit sphere of Z. Then, by (1.1), if $|\lambda| > 1$,

$$\{|\lambda(\alpha(z) + \alpha(\lambda z))| \vee |\lambda\beta(z) + \beta(\lambda z)|, |\lambda(\alpha(z) - \alpha(\lambda z))| \vee |\lambda\beta(z) - \beta(\lambda z)|\}$$
$$= \{\|f(\lambda z + \lambda e_{\gamma_0}) + f(\lambda z + e_{\gamma_0})\|, \|f(\lambda z + \lambda e_{\gamma_0}) - f(\lambda z + e_{\gamma_0})\|\}$$
$$= \{|\lambda - 1|, 2|\lambda|\}.$$

This proves the equation (2.2) in the case of $|\lambda| > 1$. If $|\lambda| < 1$, by considering $f(\lambda z + e_{\gamma_0})$ and $f(z + e_{\gamma_0})$ instead of $f(\lambda z + \lambda e_{\gamma_0})$ and $f(\lambda z + e_{\gamma_0})$, respectively, we can also derive (2.2). The case $|\lambda| = 1$ follows from these two cases.

Define a mapping $g : X \to Y$ as follows:

$$g(z) = \alpha(z)\beta(z) f(z), \quad g(z + \lambda e_{\gamma_0}) = g(z) + \lambda f(e_{\gamma_0})$$

for all $z \in Z$ and $\lambda \in \mathbb{R}$ with $\lambda \neq 0$. By (2.2), g is phase equivalent to f and, for all z_1, z_2 in Z with $\|z_1\| \le 1, \|z_2\| \le 1$,

$$\{2, \|g(z_1) - g(z_2)\|\} = \{\|g(z_1 + e_{\gamma_0}) + g(z_2 + e_{\gamma_0})\|, \|g(z_1 + e_{\gamma_0}) - g(z_2 + e_{\gamma_0})\|\}$$
$$= \{\|z_1 + z_2 + 2e_{\gamma_0}\|, \|z_1 - z_2\|\}$$
$$= \{2, \|z_1 - z_2\|\}.$$

Since g is homogeneous, we conclude from this that $\|g(z_1) - g(z_2)\| = \|z_1 - z_2\|$ for all $z_1, z_2 \in Z$. This and its definition are enough to show that g is an isometry from X onto Y. The Mazur–Ulam theorem implies that g is a linear isometry. □

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