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ON THE MARTINGALES UPCROSSINGS INEQUALITY

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In this note we present a very simple proof of the upcrossings inequality (see [6], and the note at the end) for martingale sequences—one of the basic results in the theory of martingales—which does not make use of the notion of optional random variable, as is done in the usual proofs of the inequality.

Let X_1, X_2 be random variables forming a sub-martingale sequence relative to the increasing σ-fields $\mathscr{F}_1$ and $\mathscr{F}_2$. Let $a<b$.

Then $(X_1-a)^+$, $(X_2-a)^+$ is also a sub-martingale. Hence letting Y denote $[(X_2-a)^+-(X_1-a)^+/b-a]$, we have

$$\int_A Y \geq 0, \qquad \text{for each } A \in \mathscr{F}_1. \tag{1}$$

(Here, and in the sequel, all random variables under consideration are assumed to be defined on $(\Omega, \mathscr{F}, P)$ and integration is understood to be with respect to P.)

Define

$$\begin{aligned} U &= 1, \quad \text{if } (X_1-a)^+ \leq 0, \quad (X_2-a)^+ \geq b-a \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Note that $\{(X_1-a)^+\leq 0, (X_2-a)^+\geq b-a\}=\{X_1\leq a, X_2\geq b\}$. Thus U represents the number of upcrossings of $[a, b]$ by X_1, X_2.

Now, since $U=0$ on $\{X_1>a\}$,

$$0 = \int_{A\cap\{X_1>a\}} U \leq \int_{A\cap\{X_1>a\}} Y, \qquad A \in \mathscr{F}_1. \tag{2}$$

Next, $Y\geq 0$ on $\{X_1\leq a\}$. Therefore, since $U=0$ on $\{(X_2-a)^+<b-a\}$, we have, in particular,

$$U \leq Y \quad \text{on} \quad \{X_1 \leq a\} \cap \{(X_2-a)^+ < b-a\}. \tag{3}$$

Lastly, $U=1$ on $\{X_1\leq a\} \cap \{(X_2-a)^+\geq b-a\}$, whereas $Y\geq 1$ on it. Hence

$$U \leq Y \quad \text{on} \quad \{X_1 \leq a\} \cap \{(X_2-a)^+ \geq b-a\}. \tag{4}$$

Thus from (2), (3), and (4), we obtain $\int_A U\leq\int_A Y$, $A \in \mathscr{F}_1$, which is equivalent to

$$E(U \mid \mathscr{F}_1) \leq E(Y \mid \mathscr{F}_1) = E\left[\frac{(X_2-a)^+-(X_1-a)^+}{b-a} \,\middle|\, \mathscr{F}_1\right] \tag{5}$$

Next, let us say that a sub-martingale $X_1, \ldots, X_n$, relative to an increasing family of σ-fields $\mathscr{F}_1, \ldots, \mathscr{F}_n$, is of length n—'length' referring to the number of random variables in the sequence. Let U be the number of upcrossings of $[a, b]$ by $X_1, \ldots, X_n$. We prove the sub-martingale inequality

$$E(U \mid \mathscr{F}_1) \leq E\left[\frac{(X_n-a)^+-(X_1-a)^+}{b-a} \,\middle|\, \mathscr{F}_1\right] \tag{6}$$

by induction on the length.

(6) has already been proved to hold for a sub-martingale sequence of length 2. Suppose it holds when the length is not greater than $n-1$.

On $\{X_1>a\}$, U equals the number of upcrossings by the sequence $X_2, \ldots, X_n$, which is of length $n-1$. Hence, by the induction assumption, and relation (1), we have that (6) holds on $\{X_1>a\}$.

Next, define X_{n+1} to be identical with X_n. Let N be the first value of i for which $X_i \geq b$ and $i<n+1$. Should no such i exist, define N to be $n+1$. Let V and W be the respective upcrossings by the sequences X_1, X_N and $X_N, \ldots, X_{n+1}$. Let k be a positive integer such that $2 \leq k \leq n+1$. Clearly, we have on $\{X_1 \leq a\}$, and $\{N=k\}$ that

$$V \leq [(X_k-a)^+-(X_1-a)^+]/(b-a)$$

and, by the induction assumption,

$$E(W \mid \mathscr{F}_k) \leq E[((X_n-a)^+-(X_k-a)^+)/(b-a) \mid \mathscr{F}_k].$$

Hence, on $\{X_1 \leq a\}$, since $U=V+W$, we have

$$E(UI_{\{N=k\}} \mid \mathscr{F}_1) \leq E\{I_{\{N=k\}}((X_n-a)^+-(X_1-a)^+)/(b-a) \mid \mathscr{F}_1\}.$$

Summing the last relation over k from 2 to $n+1$ yields (6) on $\{X_1 \leq a\}$, since (6) has already been shown to hold on $\{X_1>a\}$, the induction argument is complete.

REMARK. The most used form of the martingale inequality [1] follows, of course from (6) by taking expectations.

REFERENCE

1. K. L. Chung, *A course in probability theory*, Harcourt Brace, & World Inc., (1968), p. 304.

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