

FINITE NORMAL 2-GEODESIC TRANSITIVE CAYLEY GRAPHS

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Abstract

For an odd prime p , a p -transposition group is a group generated by a set of involutions such that the product of any two has order 2 or p . We first classify a family of $(G, 2)$ -geodesic transitive Cayley graphs $\Gamma := \text{Cay}(T, S)$ where S is a set of involutions and $T : \text{Inn}(T) \leq G \leq T : \text{Aut}(T, S)$. In this case, T is either an elementary abelian 2-group or a p -transposition group. Then under the further assumption that G acts quasiprimitively on the vertex set of Γ , we prove that: (1) if Γ is not $(G, 2)$ -arc transitive, then this quasiprimitive action is the holomorph affine type; (2) if T is a p -transposition group and S is a conjugacy class, then $p = 3$ and Γ is $(G, 2)$ -arc transitive.

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1. Introduction

In this paper, graphs are finite, simple, connected and undirected. In a noncomplete graph Γ , a vertex triple (u, v, w) with v adjacent to both u and w is called a 2-arc if $u \neq w$, and a 2-geodesic if in addition u, w are not adjacent. An arc is an ordered pair of adjacent vertices. The graph Γ is said to be $(G, 2)$ -arc transitive or $(G, 2)$ -geodesic transitive if its automorphism subgroup G is transitive on arcs, and also on 2-arcs or 2-geodesics, respectively. Clearly, every 2-geodesic is a 2-arc, but some 2-arcs may not be 2-geodesics. If Γ has girth 3 (length of the shortest cycle is 3), then the 2-arcs contained in 3-cycles are not 2-geodesics. The graph in Figure 1 is the octahedron which is $(G, 2)$ -geodesic transitive but not $(G, 2)$ -arc transitive with valency 4. Thus the family of noncomplete $(G, 2)$ -arc transitive graphs is properly contained in the family of $(G, 2)$ -geodesic transitive graphs.

The first remarkable result about $(G, 2)$ -arc transitive graphs comes from Tutte [20, 21], and since then, this family of graphs has been studied extensively; see [1, 12, 14, 16, 18, 22]. The study of finite $(G, 2)$ -geodesic transitive graphs was initiated

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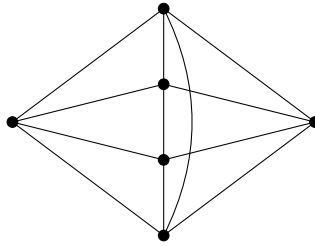


FIGURE 1. Octahedron.

in [4], where Devillers *et al.* classified tetravalent connected $(G, 2)$ -geodesic transitive graphs. Later, in [7], they determined the family of prime valency $(G, 2)$ -geodesic transitive graphs. The possible local structures of $(G, 2)$ -geodesic transitive graphs are characterized in [5]. In [6], a general study of normal $(G, 2)$ -geodesic transitive Cayley graphs was given.

For a finite group T , and a subset S of T such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\Gamma := \text{Cay}(T, S)$ of T with respect to S is the graph with vertex set T and edge set $\{\{g, sg\} \mid g \in T, s \in S\}$. In particular, Γ is connected if and only if $T = \langle S \rangle$. The group $R(T) = \{\sigma_t \mid t \in T\}$ of right translations $\sigma_t : x \mapsto xt$ is a subgroup of the automorphism group $\text{Aut}(\Gamma)$ and acts regularly on the vertex set. We may identify T with $R(T)$. Godsil [10, Lemma 2.1] observed that $N_{\text{Aut}(\Gamma)}(T) = T : \text{Aut}(T, S)$ where $\text{Aut}(T, S) = \{\sigma \in \text{Aut}(T) \mid S^\sigma = S\}$. The family of Cayley graphs Γ such that $N_{\text{Aut}(\Gamma)}(T) = \text{Aut}(\Gamma)$ has been studied under various additional conditions; see [8, 13, 15, 19, 23].

DEFINITION 1.1. Let $\Gamma = \text{Cay}(T, S)$ for a finite group T and a subset $S \subseteq T \setminus \{1\}$, $S = S^{-1}$. Then Γ is said to be *normal 2-geodesic transitive* if it is $(G, 2)$ -geodesic transitive for a group G satisfying $T : \text{Inn}(T) \leq G \leq T : \text{Aut}(T, S)$. We also say that Γ is *normal $(G, 2)$ -geodesic transitive* if we wish to specify the group G .

There are many normal $(G, 2)$ -geodesic transitive Cayley graphs. One simple example is $\text{Cay}(T, S)$ where $T = \langle a \rangle \cong \mathbb{Z}_r$ for some $r \geq 4$, $S = \{a, a^{-1}\}$, and $G = T : H$ where $H = \langle \alpha \rangle$ with $\alpha : a \mapsto a^{-1}$.

In [6], a reduction theorem for the family of normal $(G, 2)$ -geodesic transitive Cayley graphs was produced and those which are complete multipartite graphs were also classified. It was pointed out that the study of normal $(G, 2)$ -geodesic transitive Cayley graphs $\text{Cay}(T, S)$ reduces to the following three problems: investigating the case where T is a minimal normal subgroup of G , studying the 2-geodesic transitive covers of these graphs, and investigating the 2-geodesic transitive covers of complete graphs. This paper is a contribution to the first problem. In the case where T is a minimal normal subgroup of G , many such groups satisfy $T : \text{Inn}(T) \leq G$. We study normal $(G, 2)$ -geodesic transitive Cayley graphs $\text{Cay}(T, S)$ with S being a set of involutions.

Let T be a finite group. Then T' denotes the derived subgroup of T . For an odd prime p , a *p -transposition group* is a group generated by a set of involutions such that

the product of any two has order 2 or p . Such groups have been studied by Fisher [9] and Aschbacher [2, 3]. For a vertex u of a graph Γ , $\Gamma_2(u)$ denotes the set of vertices which are at distance 2 from u in Γ .

THEOREM 1.2. *Let $\Gamma = \text{Cay}(T, S)$ be a connected normal $(G, 2)$ -geodesic transitive graph with S being a set of involutions. Then all elements of $\Gamma_2(1)$ have the same prime order, say p , and either $p = 2$ or T is a center-free p -transposition group. Further, one of the following holds.*

- (1) *If $p = 2$, then $T \cong \mathbb{Z}_2^n$ for some $n \geq 2$.*
- (2) *If S is a conjugacy class and $p = 3$, then Γ is $(G, 2)$ -arc transitive and $|T : T'| \leq 2$.*
- (3) *If S is a conjugacy class and $p > 3$, then T is solvable but not characteristically simple.*
- (4) *If $S = a_1^T \cup a_2^T \cup \dots \cup a_r^T$ and $p \geq 3$ where $a_i \in S$ and $r \geq 2$, then $|a_i^T| \geq 2$, each $\langle a_i^T \rangle$ is a proper nonabelian normal subgroup of T and the girth of Γ is 3.*

REMARK 1.3. (1) This theorem partially classifies the case where S is a set of involutions. It is also interesting to investigate the case where S contains no involutions.

(2) In Theorem 1.2(1), examples exist for each n . Let $T = \langle b_1 \rangle \times \dots \times \langle b_n \rangle \cong \mathbb{Z}_2^n$, $S = T \setminus \{1, b_1 \dots b_n\}$ and let $G = T : GL(n, 2)_{b_1 \dots b_n}$. Then $\text{Cay}(T, S) \cong K_{2^{n-1}[2]}$ is normal $(G, 2)$ -geodesic transitive.

(3) In Theorem 1.2(2), Γ is $(G, 2)$ -arc transitive, and this family of graphs has been studied extensively.

(4) Let Γ be a graph in Theorem 1.2 which is not $(G, 2)$ -arc transitive. Then Γ is noncomplete, arc transitive of girth 3. Thus Γ has valency at least 4. Since Γ has girth 3, it follows that a, b are adjacent for some $a, b \in S$. Since S is a set of involutions, it follows that (a, b, c, a) is a triangle of $[S]$ (the subgraph induced by the set S) for some $c \in S$. If Γ has valency 4, then by [5, Theorem 1.1], $[S]$ is isomorphic to C_4 or $2K_2$ which has no triangles, a contradiction. If Γ has valency 5, then by [5, Theorem 1.1], $[S]$ is isomorphic to C_5 which has no triangles, again a contradiction. Thus Γ has valency at least 6. One example graph of valency 6 is the graph in (2) with $n = 3$.

A transitive permutation group is said to be *quasiprimitive* if its every nontrivial normal subgroup is transitive. The family of quasiprimitive permutation groups has been classified into eight types by Praeger [17] analogous to the O’Nan–Scott theorem in primitive permutation groups. These eight types are holomorph affine (HA), holomorph simple (HS), holomorph compound (HC), almost simple (AS), simple diagonal (SD), compound diagonal (CD), product action (PA) and twisted wreath product (TW). Let $\Gamma = \text{Cay}(T, S)$ be a normal $(G, 2)$ -geodesic transitive graph. Suppose that G is quasiprimitive on $V(\Gamma)$. If Γ is $(G, 2)$ -arc transitive, then [17] showed that the possible quasiprimitive types are HA, AS, PA and TW. If Γ is not $(G, 2)$ -arc transitive, that is, it has girth 3, then as an application of Theorem 1.2, we have the following observation.

COROLLARY 1.4. *Let $\Gamma = \text{Cay}(T, S)$ be a connected normal $(G, 2)$ -geodesic transitive graph with S being a set of involutions. Suppose that G acts quasiprimively on $V(\Gamma)$.*

- (1) *If Γ is not $(G, 2)$ -arc transitive, then the quasiprimitive action is of HA type.*
- (2) *If T is a p -transposition group and S is a conjugacy class, then $p = 3$ and Γ is $(G, 2)$ -arc transitive.*

2. Proof of Theorem 1.2

We prove Theorem 1.2 by a series of lemmas. In all lemmas of this section, we assume that $\Gamma = \text{Cay}(T, S)$ is connected normal $(G, 2)$ -geodesic transitive.

REMARK 2.1. Suppose that $\Gamma = \text{Cay}(T, S)$ is a connected normal $(G, 2)$ -geodesic transitive graph. Then the stabilizer of the identity $G_1 \leq \text{Aut}(T)$ acts transitively on both S and $\Gamma_2(1)$. Thus all elements of S have the same order and all elements of $\Gamma_2(1)$ have the same order.

For two vertices u, v of Γ , we denote by $d_\Gamma(u, v)$ the distance between u and v in Γ .

LEMMA 2.2. *Let N be a normal subgroup of G such that $N < T$. Then $d_\Gamma(x, y) \geq 3$ for distinct $x, y \in N$ if and only if $\Gamma_2(1) \cap N = \emptyset$.*

Further, if $d_\Gamma(x, y) \geq 3$ for some distinct $x, y \in N$, then $ns \in S$ where $s \in S$ and $n \in N$ implies that $n = 1$.

PROOF. Assume first that $d_\Gamma(x, y) \geq 3$ for distinct $x, y \in N$. Then $d_\Gamma(1, z) \geq 3$ for all $z \in N \setminus \{1\}$, and hence $\Gamma_2(1) \cap N = \emptyset$. Conversely, assume that $\Gamma_2(1) \cap N = \emptyset$. Since N is a normal subgroup of G , by [6, Lemma 3.1(1)], $d_\Gamma(u, v) \geq 2$ for distinct $u, v \in N$. If $d_\Gamma(u, v) = 2$ for some $u, v \in N$, then by [6, Lemma 3.1(2)], either $\Gamma_N \cong K_2$ or $\Gamma \cong K_{n[b]}$ and $\Gamma_N \cong K_n$ for some $n \geq 3, b \geq 2$. In either case, $\Gamma_2(1) \cap N \neq \emptyset$, a contradiction. Thus $d_\Gamma(x, y) \geq 3$ for distinct $x, y \in N$.

Now suppose that $d_\Gamma(x, y) \geq 3$ for all distinct $x, y \in N$. Then $N \cap S = \emptyset$ and $\Gamma_2(1) \cap N = \emptyset$, so $N \cap (S \cup \Gamma_2(1)) = \emptyset$. Assume that $ns \in S$ where $s \in S$ and $n \in N$. If $n \neq 1$, then $ns = s'$ for some $s' (\neq s) \in S$. Thus $n = s's^{-1} \in S$ or $\Gamma_2(1)$, which contradicts the fact that $N \cap (S \cup \Gamma_2(1)) = \emptyset$. Hence $n = 1$. □

Let a be an element of a group T . Then $o(a)$ denotes the order of a .

LEMMA 2.3. *If $a, b \in S$ such that $o(a) = 2$ and $ab \in \Gamma_2(1)$, then $o(ab) \in \{2, 4, i\}$ where i is an odd integer.*

PROOF. Suppose that $a, b \in S$ such that $o(a) = 2$ and $ab \in \Gamma_2(1)$. Then by Remark 2.1, all elements of S are involutions and all elements of $\Gamma_2(1)$ have the same order $o(ab)$. If $(ab)^2 = 1$, then $o(ab) = 2$. Suppose that $(ab)^2 \neq 1$. Then $(ab)^2 = abab = aa^b \in S$ or $\Gamma_2(1)$, and hence $o(ab) = 4$ or an odd integer i , respectively. □

LEMMA 2.4. *Suppose that $a, b \in S$ such that $o(a) = p$ is a prime and $ab \in \Gamma_2(1)$ of order i . Let N be a normal subgroup of G such that $N < T$, and let $\bar{\Gamma} = \text{Cay}(\bar{T}, \bar{S})$ where $\bar{T} = T/N, \bar{S} = SN/N$. Then*

- (1) $o(Ns_1) = p$ and $o(Ns_1Ns_2)$ divides i whenever $Ns_1Ns_2 \in \bar{\Gamma}_2(1_{\bar{T}})$ and $s_1, s_2 \in S$.
- (2) Suppose that $p = 2$ and $i \in \{4, q\}$ where q is an odd prime. If $\bar{\Gamma}$ is noncomplete and $Ns_1Ns_2 \in \bar{\Gamma}_2(1_{\bar{T}})$, then $o(Ns_1Ns_2) = i$ where $Ns_1, Ns_2 \in \bar{S}$.

PROOF. (1) Since $\Gamma = \text{Cay}(T, S)$ is normal $(G, 2)$ -geodesic transitive and N is a normal subgroup of G such that $N < T$, it follows from [6, Lemma 3.1(1)] that $S \cap N = \emptyset$ and $Ns_1 = s_1N$. Thus $Ns_1 \neq N$, and $(Ns_1)^p = Ns_1^p = N$. As p is a prime, $o(Ns_1) = p$.

Now suppose that $Ns_1Ns_2 \in \bar{\Gamma}_2(1_{\bar{T}})$. Then $s_1s_2 \notin S \cup \{1\}$, and so $s_1s_2 \in \Gamma_2(1)$. Since all elements of $\Gamma_2(1)$ have the same order, $o(s_1s_2) = i$, hence $(Ns_1s_2)^i = N(s_1s_2)^i = N$. Thus $o(Ns_1Ns_2)$ divides i .

(2) Suppose that $\bar{\Gamma}$ is noncomplete and $Ns_1Ns_2 \in \bar{\Gamma}_2(1_{\bar{T}})$ where $Ns_1, Ns_2 \in \bar{S}$. Then by (1), $o(Ns_1Ns_2) = t (\neq 1)$ divides i . If $i = q$, then $t = q$ as q is a prime. Suppose that $t < i = 4$. Then $t = 2$. Hence $N(s_1s_2)^2 = N$. Since $Ns_1Ns_2 \in \bar{\Gamma}_2(1_{\bar{T}})$, it follows that $s_1s_2 \in \Gamma_2(1)$, and so $o(s_1s_2) = 4$. Thus $o((s_1s_2)^2) = 2$, and so $(s_1s_2)^2 \notin \{1\} \cup \Gamma_2(1)$. Since $p = 2$, all elements of S are involutions, and it follows that $(s_1s_2)^2 = s_1s_1^2 \in N \cap S$, contradicting [6, Lemma 3.1(1)]. Thus $t = 4 = i$. □

LEMMA 2.5. Suppose that $a, b \in S$ such that $o(a) = 2$, $ab \in \Gamma_2(1)$ with $o(ab) \neq 2$. Let N be a normal subgroup of G such that $N < T$ and $\bar{\Gamma} = \text{Cay}(\bar{T}, \bar{S})$ where $\bar{T} = T/N, \bar{S} = SN/N$. Then $\bar{\Gamma}$ is complete if and only if $d_{\bar{\Gamma}}(n_1, n_2) = 2$ for some $n_1, n_2 \in N$.

PROOF. Since $o(a) = 2$ and all elements of S have the same order, it follows that S is a set of involutions. Suppose first that $d_{\bar{\Gamma}}(n_1, n_2) = 2$ for some $n_1, n_2 \in N$. Then by [6, Lemma 3.1(2)], $\bar{\Gamma} \cong K_r$ is complete for some $r \geq 2$.

Conversely, suppose that $\bar{\Gamma}$ is complete. Since N is an orbit of G , it follows that $d_{\bar{\Gamma}}(u, v) \geq 2$ for distinct $u, v \in N$. Assume that $d_{\bar{\Gamma}}(u, v) \geq 3$ for distinct $u, v \in N$. Then by Lemma 2.2, $\Gamma_2(1) \cap N = \emptyset$. Since Γ is noncomplete, there exist $x, y \in S$ such that $yx \in \Gamma_2(1)$. Then $yx \notin N$, and hence $Nx \neq Ny$. Since $\bar{\Gamma}$ is complete, it follows that Nx and Ny are adjacent in $\bar{\Gamma}$, so $Ny = NsNx$ for some $Ns \in \bar{S}$. Thus $Ny = Nsx$, and hence $y = nsx$ for some $n \in N$. Since $s, x \in S$, it follows that either $sx = 1$ or $sx \in S \cup \Gamma_2(1)$. If $sx = 1$, then $y = n$, contradicting [6, Lemma 3.1(1)] that $S \cap N = \emptyset$. If $sx \in S$, then by Lemma 2.2, $n = 1$ and $y = sx$, and so $yx = s \in S$, contradicting the fact that $yx \in \Gamma_2(1)$. Thus $sx \in \Gamma_2(1)$.

Since N is a normal subgroup of T , it follows that $N^s = N$, and so $nsns = nn'$ for some $n' \in N$. Since $ab \in \Gamma_2(1)$, it follows from Lemma 2.3 that $o(ab) \in \{4, i\}$ where i is an odd integer. Since all elements of $\Gamma_2(1)$ have the same order, it follows that $o(yx) = o(ab) \in \{4, i\}$. First, suppose that $o(yx) = 4$. Recall that $y = nsx$, so $yx = ns$. Hence $nn' = (ns)^2 = (yx)^2 = yy^x \in S$, and so $nn' \in S \cap N$, contradicting [6, Lemma 3.1(1)]. Next suppose that $o(yx) = i$ is an odd integer. Then $nn' = (ns)^2 = (yx)^2 = yy^x \in \Gamma_2(1)$, which contradicts the fact that $\Gamma_2(1) \cap N = \emptyset$. Thus, Nx and Ny are not adjacent, and hence $\bar{\Gamma}$ is noncomplete, which contradicts our assumption. Thus $d_{\bar{\Gamma}}(n_1, n_2) = 2$ for some $n_1, n_2 \in N$. □

The center of a group T is denoted by $Z(T)$.

LEMMA 2.6. *Let N be a normal subgroup of G such that $N < T$. Let $\bar{\Gamma} = \text{Cay}(\bar{T}, \bar{S})$ where $\bar{T} = T/N, \bar{S} = SN/N$. Let \bar{G} be the group induced by G on $V(\bar{\Gamma})$. Then $\bar{T} : \text{Inn}(\bar{T}) \leq \bar{G}$.*

PROOF. Since Γ is a normal $(G, 2)$ -geodesic transitive graph, it follows that $T : \text{Inn}(T) \leq G$. Hence $\text{Inn}(T) \leq G_1$, the stabilizer of the identity in G . Let $\phi(g)$ be the induced action on \bar{T} by $g \in G_1$. Then $(Nt)^{\phi(g)} = Nt^g$ for any $Nt \in \bar{T}$. Let $Nx_1, Nx_2 \in \bar{T}$. Then $(Nx_1Nx_2)^{\phi(g)} = N(x_1x_2)^g = Nx_1^g x_2^g = Nx_1^g Nx_2^g = (Nx_1)^{\phi(g)}(Nx_2)^{\phi(g)}$. Thus $\phi(g)$ is a homomorphism. Assume that $(Nx_1)^{\phi(g)} = (Nx_2)^{\phi(g)}$. Then $Nx_1^g = Nx_2^g$, so $N(x_1x_2^{-1})^g = N$, and hence $(x_1x_2^{-1})^g \in N$, that is, $x_1x_2^{-1} \in N^{g^{-1}} = N$. Thus $Nx_1 = Nx_2$, and hence $\phi(g)$ is injective. Since \bar{T} is finite, it follows that $\phi(g)$ is surjective. Thus $\phi(g) \in \text{Aut}(\bar{T})$.

Define a map $\sigma : \phi(\text{Inn}(T)) \mapsto \text{Inn}(\bar{T})$ by $\phi(\hat{t}) \mapsto \widehat{Nt}$. Suppose that $\widehat{Nx} = \widehat{Ny}$. Then for any $Na \in \bar{T}$, we have $(Na)^{\phi(\hat{x})} = Na^x = Nx^{-1}ax$ and $(Na)^{\phi(\hat{y})} = Na^y = Ny^{-1}ay$. Since $\widehat{Nx} = \widehat{Ny}$, it follows that $xy^{-1} \in N$, and so $axy^{-1}a^{-1} \in N, x^{-1}axy^{-1}a^{-1}x \in N$. Hence $Nx^{-1}ax = Nx^{-1}ay = Ny^{-1}ay$. Thus $(Na)^{\phi(\hat{x})} = (Na)^{\phi(\hat{y})}$, and so $\phi(\hat{x}) = \phi(\hat{y})$. Thus σ is well defined and also injective. Let $\widehat{Nx} (\neq 1) \in \text{Inn}(\bar{T})$. Then $x \notin Z(T)$, else $Nx \in Z(\bar{T})$, a contradiction. Thus $\sigma(\phi(\hat{x})) = \widehat{Nx}$, and hence σ is surjective. Thus σ is a bijection. Finally, for any vertex $Nt \in \bar{T}$, $(Nt)^{\phi(\hat{x})} = Nt^x = (Nt)^{\widehat{Nx}}$, it follows that $\phi(\text{Inn}(T))$ and $\text{Inn}(\bar{T})$ are permutationally isomorphic. Thus $\text{Inn}(\bar{T}) \cong \phi(\text{Inn}(T)) \leq \phi(G_1) \cong \bar{G}_1$, and so $\bar{T} : \text{Inn}(\bar{T}) \leq \bar{T} : \bar{G}_1$. Finally, by [6, Lemma 3.2(1)], $\bar{G} = \bar{T} : \bar{G}_1$, so $\bar{T} : \text{Inn}(\bar{T}) \leq \bar{G}$. □

LEMMA 2.7. *Let $a, b \in S$ such that $o(a) = 2$ and $ab \in \Gamma_2(1)$. Then T is abelian if and only if $o(ab) = 2$, which holds in turn if and only if $\Gamma_2(1) \cap Z(T) \neq \emptyset$.*

PROOF. Since $a \in S$ and $o(a) = 2$, it follows that all elements of S are involutions and all elements of $\Gamma_2(1)$ have the same order $o(ab)$. Since $T = \langle S \rangle$, it follows that T is abelian if and only if $o(ab) = 2$.

Now suppose that $\Gamma_2(1) \cap Z(T) \neq \emptyset$ and $t \in \Gamma_2(1) \cap Z(T)$. Then $t = xy$ for some $x, y \in S$. Assume that $o(ab) = i \neq 2$. Then $o(xy) = i$. Since $xy \in Z(T)$, it follows that $(xy)x = x(xy) = y$. Thus $xy = yx$, and hence $o(xy) = 2$, contradicting that $o(xy) = i \neq 2$. Thus $o(ab) = 2$. Conversely, if $o(ab) = 2$, then T is abelian, and so $\Gamma_2(1) \cap Z(T) \neq \emptyset$. □

LEMMA 2.8. *Let $a, b \in S$ such that $o(a) = 2, ab \in \Gamma_2(1)$ and $o(ab) \in \{4, p\}$ where p is odd prime. Suppose that $Z := Z(T) \neq 1$. Let $\bar{T} = T/Z, \bar{S} = S/Z, \bar{\Gamma} = \text{Cay}(\bar{T}, \bar{S})$ and \bar{G} be the group induced by G on \bar{T} . Then $\bar{\Gamma}$ is normal $(\bar{G}, 2)$ -geodesic transitive with $\bar{T} : \text{Inn}(\bar{T}) \leq \bar{G}, o(Zs_1) = 2$ and $o(Zs_1Zs_2) = o(ab)$ where $Zs_1, Zs_2 \in \bar{S}$ and $Zs_1Zs_2 \in \bar{\Gamma}_2(1)$.*

PROOF. Since Z is a characteristic subgroup of T , it follows that Z is a normal subgroup of G . Let \bar{G}_1 be the group induced by G_1 on $V(\bar{\Gamma})$. Since $ab \in \Gamma_2(1)$ is not an involution, it follows from Lemma 2.7 that $\Gamma_2(1) \cap Z = \emptyset$. Hence by Lemma 2.2, $d_\Gamma(u, v) \geq 3$ for all distinct $u, v \in Z$. By Lemma 2.5 and [6, Lemma 3.1(1)], Γ is a cover of $\bar{\Gamma}$

and $\bar{\Gamma}$ is noncomplete connected $(\bar{G}, 2)$ -geodesic transitive where $\bar{G} = \bar{T} \cdot \bar{G}_1$. Hence $|\bar{S}| = |S|$ and $\bar{T} = \langle \bar{S} \rangle$. By Lemma 2.4(1), $o(Zs) = 2$ for every $Zs \in \bar{S}$, and so $\bar{S} = \bar{S}^{-1}$. Therefore, $\bar{\Gamma}$ is a normal $(\bar{G}, 2)$ -geodesic transitive graph.

Let $Zs_1, Zs_2 \in \bar{S}$ such that $Zs_1Zs_2 \in \bar{\Gamma}_2(1)$. Then by Lemma 2.4(2), $o(Zs_1Zs_2) = o(ab)$. Further, by Lemma 2.6, $\bar{T} : \text{Inn}(\bar{T}) \leq \bar{G}$. □

For a prime p , $O_p(G)$ denotes the maximal normal p -subgroup of G .

LEMMA 2.9. *Let $a, b \in S$ such that $o(a) = 2$ and $ab \in \Gamma_2(1)$. Then the following two statements hold.*

- (1) $o(ab) \neq 4$.
- (2) *Suppose that $o(ab)$ is an odd prime. Then $Z(T) = 1$.*

PROOF. (1) Assume that $o(ab) = 4$. Then all elements of $\Gamma_2(1)$ have order 4. Let T be a minimal group with these properties.

Since $T : \text{Inn}(T) \leq G$, it follows that $\text{Inn}(T) \leq G_1$, and so $S = a_1^T \cup a_2^T \cup \dots \cup a_r^T$ where $a_i \in S$. Let $T_i = \langle a_i^T \rangle$. Then $T_i \trianglelefteq T$. If $|a_i^T| = 1$, then since G_1 is transitive on S , it follows that $|x^T| = 1$ for every $x \in S$, that is, $S \subset Z(T)$. Thus $o(xy) = 2$ for any $x \neq y \in S$ contradicting that all elements of $\Gamma_2(1)$ have order 4. Thus $|a_i^T| \geq 2$. Let $x, y \in a_i^T$ be two distinct elements. Then either $\langle x, y \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\langle x, y \rangle \cong D_8$. Hence by [11, page 117], $a_i^T \subseteq O_2(T_i)$. Since $O_2(T_i)$ is a characteristic subgroup of T_i , it follows that $O_2(T_i) \trianglelefteq T$. Since $O_2(T)$ is the maximal normal 2-subgroup of T , it follows that $O_2(T_i) \leq O_2(T)$, and hence $S \subseteq O_2(T)$. Thus T is a nonabelian 2-group, as $T = \langle S \rangle$, $ab \in \Gamma_2(1)$ and $o(ab) = 4$. Hence $1 < Z(T) < T$.

Let $Z = Z(T)$, $\bar{T} = T/Z$, $\bar{S} = SZ/Z$ and $\bar{\Gamma} = \text{Cay}(\bar{T}, \bar{S})$. Let \bar{G} be the group induced by G on $V(\bar{\Gamma})$. By Lemma 2.8, $\bar{\Gamma}$ is normal $(\bar{G}, 2)$ -geodesic transitive with $\bar{T} : \text{Inn}(\bar{T}) \leq \bar{G}$, and $o(Zs_1) = 2$, $o(Zs_1Zs_2) = o(ab) = 4$ where $Zs_1, Zs_2 \in \bar{S}$ and $Zs_1Zs_2 \in \bar{\Gamma}_2(1)$. However, $\bar{T} < T$, contradicting that T is minimal. Thus, $o(ab) \neq 4$.

(2) Suppose that $Z := Z(T) \neq 1$. Let T be a minimal such group. Let $\bar{T} = T/Z$, $\bar{S} = SZ/Z$ and $\bar{\Gamma} = \text{Cay}(\bar{T}, \bar{S})$. Let \bar{G} be the group induced by G on $V(\bar{\Gamma})$. Since $ab \in \Gamma_2(1)$ with odd prime order, say p , T is nonabelian. Hence by Lemma 2.8, $\bar{\Gamma}$ is normal $(\bar{G}, 2)$ -geodesic transitive with $\bar{T} : \text{Inn}(\bar{T}) \leq \bar{G}$, $o(Zs) = 2$ for every $Zs \in \bar{S}$, $o(Zs_1Zs_2) = p$ where $Zs_1Zs_2 \in \bar{\Gamma}_2(1)$. However, $\bar{T} < T$, contradicting that T is a minimal such group. Thus, $Z(T) = 1$. □

The following lemma proves the key part of our main theorem.

LEMMA 2.10. *Let $a, b \in S$ such that $o(a) = 2$ and $ab \in \Gamma_2(1)$ of odd prime order. Then T is not nilpotent and one of the following holds.*

- (1) $S = a^T$. If $o(ab) = 3$, then Γ is $(G, 2)$ -arc transitive and $|T : T'| \leq 2$. If $o(ab) > 3$, then T is solvable but not characteristically simple.
- (2) $S = a_1^T \cup a_2^T \cup \dots \cup a_r^T$ where $a_i \in S$ and $r > 1$. Then $xy = yx$ for any $x \in a_i^T$ and $y \in a_j^T$ whenever $i \neq j$, $|a_i^T| \geq 2$, each $\langle a_i^T \rangle$ is a proper normal subgroup of T and Γ has girth 3.

PROOF. Suppose that T is nilpotent. Then its Sylow 2-subgroup H is a characteristic subgroup of T . Since all elements of S are involutions, it follows that $H = \langle S \rangle = T$, and so T is a 2-group, and this contradicts that T has odd prime order elements. Thus T is not nilpotent.

(1) Let $S = a^T$. Suppose first that $o(ab) = 3$. Then all elements of $\Gamma_2(1)$ have order 3. Let $K = \{ab \mid ab = ba \neq 1, a, b \in S\}$. If Γ has girth 3, then there exists $c \in S$ such that a, c are adjacent, that is, $c = da$ for some $d \in S$. Since $o(da) = 2$, $da = ad \neq 1$, and hence $da \in K$. By [9, Lemma 4.1.1], $S \cap K = \emptyset$, and hence $da \in \Gamma_2(1)$, contradicting that all elements of $\Gamma_2(1)$ have order 3. Thus the girth of Γ is at least 4 and Γ is $(G, 2)$ -arc transitive. Further, by [9, page 235], $|T : T'| \leq 2$.

Now suppose that $o(ab) = p > 3$. Then for any two distinct elements $x, y \in S$, $o(xy) = 2$ or p . Thus S is a conjugacy class of p -transpositions. If T is nonsolvable, then by [2, Lemma 5.2], $p = 3$, contradicts our assumption. Thus T is solvable. Suppose that T is characteristically simple. Then $T = M_1 \times M_2 \times \dots \times M_r$ where $M_i \cong M_j$ is simple. Since T is solvable, it follows that each M_i is solvable, and hence $M_i \cong \mathbb{Z}_{p'}$, where p' is a prime. If $p' = 2$, then T has no elements of order p ; if $p' = p$, then T has no involutions, a contradiction. Thus T is solvable but not characteristically simple.

(2) Suppose that $S = a_1^T \cup a_2^T \cup \dots \cup a_r^T$ where $a_i \in S$ and $r > 1$. Suppose that $o(ab) = p$. Let $S_i = a_i^T$, $S_j = a_j^T$ where $i \neq j$, and let $x \in S_i$, $y \in S_j$. Assume that $xy \neq yx$. Then $o(xy) \neq 2$, that is, $xy \notin S$. So $xy \in \Gamma_2(1)$, and hence $o(xy) = p$, $(xy)^p = 1$. Thus $(xy)^{p-1}x = y \in S_j$, that is, $x(y(xy)^{p-2})x \in S_j$, and hence $y(xy)^{p-2} \in S_j^x = S_j$. Since $y(xy)^{p-2} = y(xy)^{p-3}xy$, it follows that $y(xy)^{p-3}xy \in S_j$, $(xy)^{p-3}x \in S_j^y = S_j$. Repeating the above process, we finally obtain $yxy \in S_j$, $x \in S_j^y = S_j$, a contradiction. Thus $xy = yx$.

Let $T_i = \langle S_i \rangle$. Then $T_i \trianglelefteq T$. If $|S_i| = 1$ for some i , then since G_1 is transitive on S , it follows that $|x^T| = 1$ for every $x \in S$, that is, $S \subset Z(T)$. By Lemma 2.9(2), $Z(T) = 1$, and so $T = \langle S \rangle = Z(T) = 1$, a contradiction. Thus $|S_i| \geq 2$, and so each T_i has at least three elements.

Suppose that $T = \langle S_k \rangle$ for some $k \in \{1, \dots, r\}$. Since $G_1 \leq \text{Aut}(T, S)$ is transitive on S , it follows that $T = \langle S_i \rangle$ for each i . Since $xy = yx$ for any $x \in S_i$ and $y \in S_j$ whenever $i \neq j$, it follows that $T = \langle S_2 \rangle \leq C_T(\langle S_1 \rangle) = C_T(T) = Z(T)$. By Lemma 2.9(2), $T = Z(T) = 1$, a contradiction. Thus $\langle S_k \rangle < T$ for each k , and so each $\langle S_k \rangle$ is a nontrivial proper normal subgroup of T .

Finally, since $a_2a_1 = a_1a_2$, it follows that $o(a_1a_2) = 2$, that is, $a_1a_2 \in S$. Thus Γ has girth 3. □

We are now ready to prove the main theorem.

PROOF OF THEOREM 1.2. Suppose that $a, b \in S$ such that $o(a) = 2$ and $ab \in \Gamma_2(1)$. Then it follows from Lemma 2.3 that $o(ab) = j \in \{2, 4, i\}$ where i is an odd integer. Further, as $T : \text{Inn}(T) \leq G$, by Lemma 2.9(1), $j \neq 4$, and so $j \in \{2, i\}$.

Suppose that $j = 2$. Then by Lemma 2.7, T is abelian. Thus $T \cong \mathbb{Z}_2^n$ for some $n \geq 2$, and (1) holds.

Now let $j = i$. Suppose that $i = mn$ with $m > 1, n > 1$. Then both m, n are odd integers, as i is odd. Let $x = (ab)^m$. Then $o(x) = n$. Moreover, $x = (ab)^m = a(bab \dots ab) = a(ba)^k b(ab)^k = ab^{(ab)^k}$ where $k = (m - 1)/2$. Since $\text{Inn}(T) \leq G_1$, it follows that $b^{(ab)^k} \in S$, and hence x lies in $S \cup \Gamma_2(1)$, contradicting the fact that elements of $S \cup \Gamma_2(1)$ have order 2 or mn . Thus i is an odd prime and T is a p -transposition group. By Lemma 2.9(2), $Z(T) = 1$. Since Γ is $(G, 2)$ -geodesic transitive and $T : \text{Inn}(T) \leq G$, it follows that $S = a_1^T \cup a_2^T \cup \dots \cup a_r^T$ where $a_k \in S$. Thus (2)–(4) follow from Lemma 2.10. □

PROOF OF COROLLARY 1.4. (1) Suppose that Γ is not $(G, 2)$ -arc transitive. Since Γ is a normal $(G, 2)$ -geodesic transitive graph, it follows that T is a normal subgroup of G . Assume that G is quasiprimitive of type X on $V(\Gamma)$. Then T is a minimal normal subgroup of G , $X \in \{\text{HA}, \text{HS}, \text{AS}, \text{HC}, \text{TW}\}$ and $T : \text{Inn}(T) \leq G \leq T : \text{Aut}(T, S)$. Since Γ is not $(G, 2)$ -arc transitive, the girth of Γ is 3, and it follows from Theorem 1.2 that T is not simple. Hence X is not any of the types AS, HS, HC and TW, that is, X is of type HA.

(2) Suppose that T is a p -transposition group and S is a conjugacy class. Assume $p > 3$. Then by Theorem 1.2 (3), T is solvable but not characteristically simple. Hence T has a proper normal subgroup N which is also a normal subgroup of G . However, N is not transitive on $V(\Gamma)$, contradicting that G is quasiprimitive on $V(\Gamma)$. Thus $p = 3$. By Theorem 1.2(2), T is a 3-transposition group and Γ is $(G, 2)$ -arc transitive. □

We give another corollary of Theorem 1.2. Let Γ be a G -vertex transitive graph. If N is an intransitive normal subgroup of G , then the set of N -orbits $\mathcal{B} = \{B_1, B_2, \dots, B_n\}$ forms a G -invariant partition of $V(\Gamma)$. The *quotient graph* Γ_N of Γ is the graph with vertex set \mathcal{B} such that B_i, B_j are adjacent in Γ_N if and only if there exist $x \in B_i, y \in B_j$ such that x, y are adjacent in Γ . Further, Γ is said to be a *cover* of Γ_N if for each edge $\{B_i, B_j\}$ of Γ_N and $v \in B_i$, we have $|\Gamma(v) \cap B_j| = 1$. ($\Gamma(v)$ denotes the set of vertices of Γ at distance 1 from v .)

COROLLARY 2.11. *Let T, S, Γ, G be as in Theorem 1.2. Let T be a p -transposition group with $p > 3$. If S is a conjugacy class and Γ is not $(G, 2)$ -arc transitive, then one of the following holds.*

- (1) $\Gamma \cong K_{n[b]}$ for some $n \geq 3, b \geq 2$.
- (2) There exists a normal subgroup N of G such that $N < T$ and Γ_N is complete and either $\Gamma_N \cong K_2$ or Γ is a cover of Γ_N .

PROOF. Suppose that S is a conjugacy class and Γ is not $(G, 2)$ -arc transitive. Since $p > 3$, it follows from Theorem 1.2(3) that T is not characteristically simple, so T is not a minimal normal subgroup of G . If $\Gamma \cong K_{n[b]}$ for some $n \geq 3, b \geq 2$, then (1) holds. Suppose not this case. Note that Γ is not $(G, 2)$ -arc transitive. Then it follows from [6, Theorem 1.3] that there exists a normal subgroup N of G such that $N < T$, and either $\Gamma_N \cong K_2$ or Γ is a cover of Γ_N , and $\Gamma_N \cong \text{Cay}(\overline{T}, \overline{S})$ where $\overline{T} = T/N$ and $\overline{S} = SN/N$. Further, by Lemma 2.4(1), $o(Ns) = 2$ for every $Ns \in \overline{S}$. Suppose that Γ is a cover

of Γ_N . Assume that Γ_N is not complete. Let T be a minimal such group. Then by [6, Lemma 3.3], $\text{Cay}(\overline{T}, \overline{S})$ is a normal $(\overline{G}, 2)$ -geodesic transitive Cayley graph where $\overline{G} \cong G/N$ is the group induced by G on $V(\overline{\Gamma})$, and $\overline{S} = (Ns)^{\overline{T}}$, contradicting that T is minimal. Thus Γ_N is a complete graph. \square

3. Examples

EXAMPLE 3.1. Let $T = \langle a_1, \dots, a_i, b_1, \dots, b_i \rangle \cong \mathbb{Z}_2^n$ where $n = 2i, i \geq 1$. Let $S = S_a \cup S_b$ where $S_a = \langle a_1, \dots, a_i \rangle \setminus \{1\}$ and $S_b = \langle b_1, \dots, b_i \rangle \setminus \{1\}$. Then $\Gamma = \text{Cay}(T, S)$ is a normal $(G, 2)$ -geodesic transitive Cayley graph where $G = T : \text{Aut}(T, S)$. In particular, Γ is a graph in Theorem 1.2(1) of girth 3 and diameter 2.

PROOF. Since $T \cong \mathbb{Z}_2^n$, it follows that for any $x, y \in S, xy \in S$ if and only if x, y are adjacent in Γ . Since $S_a = \langle a_1, \dots, a_i \rangle \setminus \{1\}, S_b = \langle b_1, \dots, b_i \rangle \setminus \{1\}$, and $S = S_a \cup S_b$, it follows that $a_j a_k, b_j b_k \in S$ and $a_j b_k \in \Gamma_2(1)$.

We identify T with the n -dimensional vector space over the finite field F_2 , and identify the identity with the zero vector. Then $\{a_1, \dots, a_i, b_1, \dots, b_i\}$ is a basis of T . Let M be the subspace generated by $\{a_1, \dots, a_i\}$ and N be the subspace generated by $\{b_1, \dots, b_i\}$. Let $A = \text{Aut}(T)$, and let A_M, A_N be the stabilizers of the subspaces M, N , respectively. Then $A_M \cong A_N \cong GL(i, 2) \times GL(i, 2) \leq G_1 = \text{Aut}(T, S)$ and $GL(i, 2) \wr S_2 \leq G_1$. Further, $GL(i, 2) \wr S_2$ is transitive on both S and $\Gamma_2(1)$.

Note that $\Gamma_2(1) \cap \Gamma(a_1) = \{x + a_1 \mid x \in S_b\}$. Since $\text{Aut}(N)$ is transitive on $\Gamma_2(1) \cap \Gamma(a_1)$ and $\text{Aut}(N) \leq G_{1,a_1}$, it follows that G_{1,a_1} is transitive on $\Gamma_2(1) \cap \Gamma(a_1)$. Hence Γ is $(G, 2)$ -geodesic transitive. In particular, if $i \geq 2$, then $a_1 a_2 \in S$, so Γ has girth 3.

Let $x \in S$ and $b \in S_b$. If $x \in S_a$, then $xa_1 \in S$, and so $xa_1 b \in \Gamma_2(1)$. If $x \in S_b$, then $xb \in S$, and so $xa_1 b = xba_1 \in \Gamma_2(1)$. Thus $\Gamma_3(1) \cap \Gamma(a_1 b) = \emptyset$. Since Γ is $(G, 2)$ -geodesic transitive, it follows that Γ has diameter 2. \square

EXAMPLE 3.2. (1) Let $T = S_3$ and S be the set of all transpositions of T . Then $\text{Cay}(T, S) \cong K_{3,3}$. In particular, Γ is a graph in Theorem 1.2(2).

(2) Let $T = \mathbb{Z}_3^n : \mathbb{Z}_2$ where \mathbb{Z}_2 is the involution exchanging the two nonidentity elements of \mathbb{Z}_3 . Then T is a Frobenius group. Let S be the set of all involutions of T . Then $\text{Cay}(T, S)$ is a graph in Theorem 1.2(2).

(3) Let $T = \mathbb{Z}_p^n : \mathbb{Z}_2$ where $p > 3$ is odd prime and \mathbb{Z}_2 is the involution exchanging the generator of \mathbb{Z}_p and its inverse. Then T is a Frobenius group. Let S be the set of all involutions of T . Then $\text{Cay}(T, S)$ is a graph in Theorem 1.2(3).

We do not have examples for Theorem 1.2(4) at the time of writing.

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