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A NOTE ON A SECOND-ORDER NONLINEAR DIFFERENTIAL SYSTEM

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Abstract. We investigate the boundedness of solutions of a second order nonlinear differential system.

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We consider the nonlinear system

$$\begin{cases} x' = \frac{1}{a(x)}[c(y) - b(x)], \\ y' = -a(x)[h(x) - e(t)], \end{cases} \tag{1}$$

where $a : \mathbb{R} \to (0, \infty)$, $b, c, h : \mathbb{R} \to \mathbb{R}$ and $e : \mathbb{R}_+ \to \mathbb{R}$ are continuous.

As particular cases of the system (1) we have the unforced Liénard equation

$$x'' + f(x)x' + h(x) = 0 \tag{2}$$

for $a(x) = 1$, $b(x) = \int_0^x f(s)ds$, $c(x) = x$, $x \in \mathbb{R}$; $e(t) = 0$, $t \in \mathbb{R}_+$, and the second order nonlinear differential equation

$$x'' + \Big(f(x) + g(x)x'\Big)x' + h(x) = e(t) \tag{3}$$

for $a(x) = \exp(\int_0^x g(s)ds)$, $b(x) = \int_0^x a(s)f(s)ds$, $c(x) = x$, $x \in \mathbb{R}$.

Arising from problems in applied sciences (theory of feedback electronic circuits, motion of a mass-spring system, cf. [**2**]), equations (2) and (3) have been investigated extensively. See [**1**], [**3**], [**10**], and the citations therein. Generalizations of these two equations were considered in recent years: in [**6**] and [**11**] vector-valued functions are considered whereas in [**7**] the system (1) is analysed in an attempt to unify the methods known for particular cases in a general result on an important qualitative aspect—the boundedness of solutions.

We are concerned with the problem of boundedness of solutions for the system (1). Our results encompass earlier works if restricted to the equations (2) and (3), viewed as particular cases of system (1). We extend the results from [**7**] for the system (1) so that we can deal with cases where the methods from [**7**] fail to work.

Let us set

$$C(y) = \int_0^y c(s)ds, \quad E(t) = \int_0^t |e(s)|ds, \quad H(x) = \int_0^x a^2(s)h(s)ds,$$

and denote by $\Re$ the class of nondecreasing functions $w \in C(\mathbb{R}_+, (0, \infty))$ satisfying $\int_1^\infty \frac{1}{w(s)}ds = \infty$.

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THEOREM 1. *Assume that*
(i) *there exists* $w \in \Re$ *such that* $a(x)|c(y)| \leq C(y) + w(H(x)), \quad (x, y \in \mathbb{R})$;
(ii) $b(x)h(x) \geq 0$ *for all* $x \in \mathbb{R}$;
(iii) $C(y) > 0$ *for* $y \neq 0$ *and* $\limsup_{|y|\to\infty} C(y) = \infty$;
(iv) $E(\infty) < \infty$;
(v) $H(x) > 0$ *for* $x \neq 0$.
Then every solution of (1) *is bounded if*
(vi) $\limsup_{|x|\to\infty} (H(x) + \text{sign}(x)b(x)) = \infty$.

Proof. Set $V(t, x, y) = C(y) + H(x), \; t \in \mathbb{R}_+, \; x, y \in \mathbb{R}$. We have (along solutions)

$$\frac{dV}{dt} = a(x)c(y)e(t) - a(x)b(x)h(x) \leq a(x)|c(y)||e(t)| \leq$$

$$\leq |e(t)|\Big(V(t, x, y) + w(V(t, x, y))\Big).$$

Let $(x(t), y(t))$ be a solution of (1). By Conti's comparison method **[5]**, in view of (iv) and since

$$\int_1^\infty \frac{1}{w(s) + s} ds = \infty$$

for $w \in \Re$, (cf. **[4]**), we have that there exists an $M > 0$ such that

$$V(t, x(t), y(t)) = C(y(t)) + H(x(t)) \leq M, \quad t \in \mathbb{R}_+.$$

By (iii) we obtain the existence of an $K > 0$ such that $|y(t)| \leq K, \; t \in \mathbb{R}_+$.

If $\limsup_{r\to\infty} H(r) = \infty$ or $\limsup_{r\to-\infty} H(r) = \infty$ we conclude that $x(t)$ is bounded above (respectively below) since

$$0 \leq H(x(t)) \leq M, \quad t \in \mathbb{R}_+.$$

If $\limsup_{r\to\infty} H(r) < \infty$ we have that $\limsup_{r\to\infty} b(r) = \infty$. Set

$$W(t, x, y) = x, \quad t \in \mathbb{R}_+, \; x, y \in \mathbb{R},$$

so that (along solutions) we have

$$\frac{dW}{dt} = \frac{1}{a(x)}[c(y) - b(x)].$$

If $\limsup_{t\to\infty} x(t) = \infty$ we would deduce that there exist x_1 and x_2 such that $x(0) = x_0 < x_1 < x_2$ and

$$\frac{dW}{dt} < 0 \text{ for } x_1 \leq x \leq x_2, \; -K \leq y \leq K.$$

If $0 < t_1 < t_2$ are such that $x(t_1) = x_1$ and $x(t_2) = x_2$, we have

$$W(t_1, x(t_1), y(t_1)) = x_1 < x_2 = W(t_2, x(t_2), y(t_2)),$$

which is in contradiction with the previous relation. Hence $x(t)$ is bounded above.

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The case $\limsup_{r\to-\infty} H(r) < \infty$ can be handled similarly by choosing $W(t,x,y) = -x$, $t \in \mathbb{R}_+$, $x, y \in \mathbb{R}$, proving that $x(t)$ is bounded below. $\square$

Corollary 1 [7]. *Assume that*
(i) *there exists a positive constant m such that* $a(x)|c(y)| \le C(y) + m$, $(x, y \in \mathbb{R})$;
(ii) $b(x)h(x) \ge 0$ *for all* $x \in \mathbb{R}$;
(iii) $C(y) > 0$ *for* $y \ne 0$ *and* $\limsup_{|y|\to\infty} C(y) = \infty$;
(iv) $E(\infty) < \infty$;
(v) $H(x) > 0$ *for* $x \ne 0$.
Then every solution of (1) *is bounded if*
(vi) $\limsup_{|x|\to\infty} (H(x) + \mathrm{sign}(x)b(x)) = \infty$.

As a particular case of Corollary 1 we have a boundedness result of Antosiewicz [1]. The relation of Theorem 1 with the boundedness theorem of Qian [7] is given in the following example.

Example. Consider the second order nonlinear system

$$\begin{cases} x' = \frac{1}{a(x)}[y - b(x)], \\ y' = -a(x)[b(x) - e(t)], \end{cases} \tag{4}$$

where $a(x) = \max\{1, x\}$, $x \in \mathbb{R}$, and $b(x) = 2$, $x \le 1$, $b(x) = \frac{2}{x}$, $x > 1$. If $\int_0^\infty |e(s)|ds < \infty$ we have that every solution of (4) is bounded since

$$a(x)|y| \le \frac{y^2}{2} + H(x) + 1, \quad x, y \in \mathbb{R}.$$

Since $a(x)$ is not bounded we cannot apply the result from [7]. $\square$

As a particular case of the nonlinear system (1) we have

$$\begin{cases} x' = \frac{1}{a(x)}[y - b(x)], \\ y' = -a(x)[h(x) - e(t)]. \end{cases} \tag{5}$$

Theorem 2. *Assume that*
(i) $a(x)$ *is bounded on* $\mathbb{R}$ *and* $\liminf_{|x|\to\infty} a(x) > 0$;
(ii) $b(x)h(x) \ge 0$ *for all* $x \in \mathbb{R}$;
(iii) $E(\infty) < \infty$;
(iv) $xh(x) > 0$ *for* $x \ne 0$.
Then every solution of (5) *is bounded if and only if*
(v) $\limsup_{|x|\to\infty} (H(x) + \mathrm{sign}(x)b(x)) = \infty$.

Proof. Since $a(x)$ is bounded and $C(y) = \frac{y^2}{2}$ we have that (v) implies that every solution of (5) is bounded in view of Corollary 1.

In order to prove the necessity of condition (v) we will exhibit an unbounded solution of (5) if (v) does not hold.

Suppose for instance that (the other case being similar)

$$\limsup_{x\to\infty} (H(x) + b(x)) < \infty.$$

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Let $k, K, M > 0$ be such that

$$a(x) + b(x) \le K, \quad x \in \mathbb{R}_+,$$

$$0 < k \le a(x), \quad x \in \mathbb{R},$$

$$H(x) + E(\infty) < M, \quad x \in \mathbb{R}.$$

Let $(x(t), y(t))$ be a solution of (5) with initial conditions $x(0) = 1$, $y(0) = K + KM + \frac{M}{k}$. We have

$$y(t) = K + KM + \frac{M}{k} - \int_0^t a(x(s))h(x(s))ds + \int_0^t a(x(s))e(s)ds.$$

As long as $y(t) \ge K$ we have that $\frac{dx(t)}{dt} \ge 1$ (from the first equation of the system (5)) and

$$\int_0^t a(x(s))h(x(s))ds \le \frac{1}{k}\int_0^t a^2(x(s))h(x(s))ds \le$$

$$\le \frac{1}{k}\int_0^t a^2(x(s))h(x(s))x'(s)ds = \frac{1}{k}\int_{x(0)}^{x(t)} a^2(s)h(s)ds.$$

With $x(0) = 1$, $y(0) = K + KM + \frac{M}{k}$, we have

$$y(t) \ge K + KM + \frac{M}{k} - \frac{1}{k}\int_{x(0)}^{x(t)} a^2(s)h(s)ds - KM \ge K, \quad t \in \mathbb{R}_+,$$

thus $x(t) \ge t + 1$, $t \in \mathbb{R}_+$; i.e. $(x(t), y(t))$ is an unbounded solution of (5). □

Corollary 2 [**3**]. *Suppose that $f, h : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $f(x) \ge 0$, $xh(x) \ge 0$ for $x \in \mathbb{R}$. Then every solution of the unforced Liénard equation*

$$x'' + f(x)x' + h(x) = 0 \tag{2}$$

is bounded if and only if

$$\limsup_{|x|\to\infty}\left(\left|\int_0^x f(s)ds\right| + \int_0^x h(s)ds\right) = \infty.$$

We consider now equation (3) which is equivalent to the system

$$\begin{cases} x' = \frac{1}{a(x)}[y - b(x)], \\ y' = -a(x)[h(x) - e(t) \end{cases} \tag{3}$$

with $a(x) = \exp(\int_0^x g(s)ds)$, $b(x) = \int_0^x a(s)f(s)ds$, $x \in \mathbb{R}$.

Corollary 3. *Assume that*
(i) $g(x)$ *is bounded on* $\mathbb{R}$*;*
(ii) $b(x)h(x) \geq 0$ *for all* $x \in \mathbb{R}$*;*
(iii) $E(\infty) < \infty$*;*
(iv) $xh(x) > 0$ *for* $x \neq 0$.
Then for every solution $x(t)$ *of* (3) *there exists* $K > 0$ *such that*

$$|x(t)| + |x'(t)| \leq K, \quad t \in \mathbb{R}_+,$$

if and only if
(v) $\limsup_{|x| \to \infty} (H(x) + \operatorname{sign}(x)b(x)) = \infty$.

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