Relations for quadratic Hodge integrals via stable maps

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Abstract. Following Faber-Pandharipande, we use the virtual localization formula for the moduli space of stable maps to compute relations between Hodge integrals. We prove that certain generating series of these integrals are polynomials.

Let $\overline{M}_{g,n}$ be the moduli space of $n$-pointed genus $g$ stable curves. It is a proper smooth Deligne Mumford (DM) stack of dimension $3g - 3 + n$. We denote by $\pi: \overline{C}_{g,n} \to \overline{M}_{g,n}$ the universal curve and by $\sigma_i: \overline{M}_{g,n} \to \overline{C}_{g,n}$ the sections associated to the marking $i$ for all $1 \leq i \leq n$. We denote by $\omega_{\overline{C}_{g,n}/\overline{M}_{g,n}}$ the relative dualizing sheaf of $\pi$.

We will consider the following classes in $A^*(\overline{M}_{g,n})$:

- For all $0 \leq i \leq g$, $\lambda_i$ stands for the $i$-th Chern class of the Hodge bundle, i.e. the vector bundle $E = \pi^*\omega_{\overline{C}_{g,n}/\overline{M}_{g,n}}$. For all $\alpha \in \mathbb{C}$, we denote $\Lambda_g(\alpha) = \sum_{j=0}^g \alpha^{g-j} \lambda_j$, and $\Lambda_g^\vee(\alpha) = (-1)^g \Lambda_g(-\alpha)$.

- For all $1 \leq i \leq n$, we denote $\psi_i$ the Chern class of the cotangent line at the $i$th marking $L_i = \sigma_i^*(\omega_{\overline{C}_{g,n}/\overline{M}_{g,n}})$.

A Hodge integral is an intersection number of the form:

$$\int_{\overline{M}_{g,n}} \psi_1^{k_1} ... \psi_n^{k_n} \Lambda_g(t_1) ... \Lambda_g(t_m),$$

where $k_1, ..., k_n$ are non-negative integers and $t_1, ..., t_m$ are complex numbers. If $m = 1, 2, \text{ or } 3$, then the above integral is called a linear, double, or triple Hodge integrals respectively. Relations between linear Hodge integrals where proved in [FP00a] using the Gromov-Witten theory of $\mathbb{P}^1$ and the localization formula of [GP99]. This approach was also used in [FP00b] and [TZ03] to prove certain properties of triple Hodge integrals. Linear and triple Hodge integrals naturally appeared in the GW-theory of Calabi-Yau 3-folds, thus explaining a more abundant literature on the topic. However, double Hodge integrals have appeared recently in the Quantization of Witten-Kontsevich generating series (see [Blo20]), in the theory of spin Hurwitz numbers (see [GKL21]), and in the GW theory of blow-ups of smooth surfaces (see [GKLS22]).

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In the present note, we consider the following power series in $\mathbb{C}[\alpha][[t]]$ defined using double Hodge integrals

$$
P_a(\alpha, t) = \sum_{g \geq 0} t^g \left( \int_{\overline{M}_{g,n}} \frac{\Lambda^g_1(1) \Lambda^g(\alpha)}{1 - \psi_0} \prod_{i=1}^n (2a_i + 1)!! (-4\psi_i)^{a_i} \right) \exp \left( \frac{t}{24} \right)
$$

where $a = (a_1, \ldots, a_n)$ is a vector of non-negative integers. If $n = 1$, we use the convention: $\int_{\overline{M}_{0,2}} \psi_1^a \frac{\Lambda_1^1(1) \Lambda_1^1(\alpha)}{1 - \psi_1} = (-1)^a$.

**Theorem 0.1** $P_a(\alpha, t)$ is a monic polynomial in $\mathbb{C}[\alpha][[t]]$ of degree $|a|$ in $t$.

Here we provide the first values of $P_a(-\alpha - 1, t)$. In the list below we omit the variables $-\alpha - 1$ and $t$ in the notation:

- $P(1) = 1$
- $P(1) = t + 12$
- $P(2) = t^2 - 10\alpha t + 240$
- $P(1,1) = t^2 - 12t$
- $P(3) = t^3 + (-77/3\alpha - 28)t^2 + 280t + 6720$
- $P(2,1) = t^3 + (-10\alpha - 48)t^2 + (240\alpha + 240)t$
- $P(1,1,1) = t^3 - 72t^2 + 432t$
- $P(4) = t^4 + (-43\alpha - 72)t^3 + (126\alpha^2 + 756\alpha + 840)t^2 + 10080t + 241920$
- $P(3,1) = t^4 + (-77/3\alpha - 100)t^3 + (1232\alpha + 1624)t^2$
- $P(2,2) = t^4 + (20\alpha + 100)t^3 + (-100\alpha^2 - 1360\alpha - 1680)t^2$
- $P(2,1,1) = t^4 + (-10\alpha - 132)t^3 + (840\alpha + 3120)t^2 + (-8640\alpha - 8640)t$
- $P(1,1,1,1) = t^4 - 168t^3 + 5616t^2 - 20736t$.

Considering these first values, we conjecture that $P_a$ is a polynomial of total degree $|a|$ in both variables $t$ and $\alpha$.

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1 Preliminaries

We denote by \( \overline{M}_{g,n}(\mathbb{P}^1, 1) \) the moduli space of stable maps of degree 1 to \( \mathbb{P}^1 \). It is a proper DM stack of virtual dimension \( 2g + n \). Here we can define in an analogous way the Hodge bundle \( E \), the cotangent line bundles \( L_i \) and we denote again \( \lambda_i \) and \( \psi_i \) the respective Chern classes. We also have the forgetful and evaluation maps

\[
\pi: \overline{M}_{g,n+1}(\mathbb{P}^1, 1) \to \overline{M}_{g,n}(\mathbb{P}^1, 1), \quad \text{and} \quad ev_i: \overline{M}_{g,n+1}(\mathbb{P}^1, 1) \to \mathbb{P}^1.
\]

Throughout this note the enumeration of markings starts from 0. Furthermore, \( \pi \) is the morphism that forgets the marking \( p_0 \) and \( ev_i \) is the evaluation of a stable map to the \( i \)-th marked point. The vector bundle \( T := R^1\pi_*(ev_0^*O_{\mathbb{P}^1}(-1)) \) is of rank \( g \) and we denote by \( y \) its top Chern class. We will denote:

\[
\langle \prod_{i=0}^{n-1} \tau_{a_i}(\omega)|y\rangle_{g,1} := \int_{[\overline{M}_{g,n}(\mathbb{P}^1, 1)]^{vir}} \prod_{i=0}^{n-1} \psi_i^{a_i} ev_i(\omega) y
\]

where \( \omega \) denotes the class of a point in \( \mathbb{P}^1 \).

**Theorem 1.1 (Localization Formula, [GP99], [FP00a])** Let \( g \in \mathbb{Z}_{\geq 0} \), let \( a \in \mathbb{Z}_{\geq 0}^n \) such that \( |a| \leq g \). Then, for all complex numbers \( \alpha \), and \( t \in \mathbb{C}^* \), we have

\[
\langle \prod_{i=1}^{n} \tau_{a_i}(\omega)|y\rangle_{g,1} = \sum_{g_1+g_2=g} \int_{\overline{M}_{g_1,n+1}} t^n \prod_{i=1}^{n} \psi_i^{a_i} \frac{\Lambda^\vee_{g_1}(t)\Lambda^\vee_{g_2}(\alpha t)}{t(t-\psi_0)} \times \int_{\overline{M}_{g_2,1}} \frac{\Lambda^\vee_{g_2}(-t)\Lambda^\vee_{g_2}((\alpha+1)t)}{-t(-t-\psi_0)}.
\]

Here we use the convention \( \int_{\overline{M}_{0,1}} \psi_0^a = 1 \).

**Proposition 1.2 (4.1 of [TZ03])** For all complex numbers \( \alpha \) we have

\[
F(\alpha, t) = 1 + \sum_{g \geq 0} t^{2g} \int_{\overline{M}_{g,1}} \frac{\Lambda^\vee_g(1)\Lambda^\vee_g(\alpha)}{1-\psi_0} = \exp \left( -\frac{t^2}{24} \right).
\]

Besides, we have the String and Dilaton equation for Hodge integrals.

**Proposition 1.3** Let \( g, n \in \mathbb{Z}_{\geq 0} \) such that \( 2g - 2 + n > 0 \).

(i) [Dilaton equation for Hodge integrals] Let \( (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n \) and assume that there exist \( i_0 \) such that \( a_{i_0} = 1 \). Then

\[
\int_{\overline{M}_{g,n+1}} \frac{\psi_{i_0} \prod_{i \neq i_0} \psi_i^{a_i} \prod_{j=1}^{n-1} \psi_j^{b_k}}{1-\psi_0} = (2g - 2 + n) \int_{\overline{M}_{g,n}} \frac{\prod_{j=1}^{n-1} \psi_j^{a_i} \prod_{j=1}^{n} \psi_j^{b_k}}{1-\psi_0}.
\]
(ii) [String equation for Hodge integrals] Let \((a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n\) and assume that there exist \(i_0\) such that \(a_{i_0} = 0\). Then we have

\[
\int_\mathcal{M}_{g,n+1} \frac{\prod_{i=1}^n \psi_i a_i \prod_{j=1}^g \lambda_k^{b_k}}{1 - \psi_0} = \int_\mathcal{M}_{g,n} \frac{\prod_{i=1}^{n-1} \psi_i a_i \prod_{j=1}^g \lambda_k^{b_k}}{1 - \psi_0} + \sum_{j=1}^n \int_\mathcal{M}_{g,n} \psi_j a_j \prod_{i \neq j} \psi_i a_i \prod_{k=1}^g \lambda_k^{b_k} \over (1 - \psi_0).
\]

2 The calculation

Note that the GW-invariant \(\langle \prod_{i=1}^n \tau_{a_i}(\omega) | y \rangle_{g,1}^{\mathbb{Z}}\) is 0 unless \(|a| = g\) for dimensional reasons. Indeed, \(\dim_C [\mathcal{M}_{g,n}(\mathbb{P}^1, 1)]^{\text{et}} = 2g + n\) and the cycle we are integrating is in codimension \(g + |a| + n\). Using the above localization formula, and Lemma 2.1 of [TZ03] the intersection number \(\langle \prod_{i=1}^n \tau_{a_i}(\omega) | y \rangle_{g,1}^{\mathbb{Z}}\) is expressed as:

\[
\sum_{g_1 + g_2 = g} t^{n} \int_{\mathcal{M}_{g_1,n+1}} \psi_i a_i \Lambda_{g_1}(1) \Lambda_{g_1}(\alpha) \frac{\Lambda_{g_2}(t) \Lambda_{g_2}(\alpha t)}{t(t - \psi_0)} \cdot \int_{\mathcal{M}_{g_2,1}} \frac{\Lambda_{g_2}(t) \Lambda_{g_2}(\alpha t)}{t(t - \psi_0)}
\]

\[
= \sum_{g_1 + g_2 = g} t^{n} \int_{\mathcal{M}_{g_1,n+1}} \psi_i a_i \Lambda_{g_1}(1) \Lambda_{g_1}(\alpha) \frac{\Lambda_{g_2}(1) \Lambda_{g_2}(\alpha)}{1 - \psi_0} \cdot \int_{\mathcal{M}_{g_2,1}} \psi_{3g_2 - 2} - t^{n} \int_{\mathcal{M}_{g_2,1}} \psi_{3g_2 - 2}
\]

In the last equation we used Proposition 1.2 in order to replace \(\int_{\mathcal{M}_{g_2,1}} \Lambda_{g_2}(1) \Lambda_{g_2}(\alpha) \frac{\Lambda_{g_2}(1) \Lambda_{g_2}(\alpha)}{1 - \psi_0}\) with \((-1)^{g_2} \int_{\mathcal{M}_{g_2,1}} \psi_{3g_2 - 2}\).

We define

\[
A_{g,a}(\alpha) = \sum_{g_1 + g_2 = g} \int_{\mathcal{M}_{g_1,n+1}} \psi_i a_i \Lambda_{g_1}(1) \Lambda_{g_1}(\alpha) \frac{\Lambda_{g_2}(1) \Lambda_{g_2}(\alpha)}{1 - \psi_0} \cdot \int_{\mathcal{M}_{g_2,1}} \psi_{3g_2 - 2}.
\]

Then, we have

\[
A_{g,a}(\alpha) = \begin{cases} 
0 & |a| < g \\
\langle \prod_{i=1}^n \tau_{a_i}(\omega) | y \rangle_{g,1}^{\mathbb{Z}} & |a| = g
\end{cases}
\]

By the definition of \(\Lambda_{g}(t)\) we see that \(\Lambda_{g}(1) \Lambda_{g}(- (\alpha + 1))\) is a polynomial in \(\alpha\) of degree \(g\), which actually determines the degree of \(A_{g,a}(\alpha)\).

We now present a proof for the main result.

Proof [of Theorem 0.1] We begin by stating the well known fact

\[
1 + \sum_{g > 0} t^g \int_{\mathcal{M}_{g,1}} \psi_{3g - 2} = \exp \left( \frac{t}{24} \right)
\]

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proven in section 3.1 of [FP00a]. Now, we consider the product of exp $\frac{t}{\psi_0}$ and

$$
\sum_{g \geq 0} t^g \left( \int_{\overline{M}_{g,n+1}} \frac{\Lambda^N_{g}(1) \Lambda^N_{g}(\alpha)}{1 - \psi_0} \prod_{i=1}^n (2a_i + 1)! (-4\psi_1)^{a_i} \right)
$$

to obtain a new power series whose coefficients in degree $g$ are given by

$$
\sum_{g_1 + g_2 = g} \int_{\overline{M}_{g_1,n+1}} \prod_{i=1}^n (2a_i + 1)! (-4)^{a_i} \prod_{i=1}^n \psi_i^{a_i} \Lambda^N_{g_1}(1) \Lambda^N_{g_1}(\alpha) \frac{1}{1 - \psi_0} \cdot \int_{\overline{M}_{g_2,i}} \psi_0^{3g_2 - 2}
$$

This is exactly $A_{g,a}(\alpha) \cdot \prod_{i=1}^n (2a_i + 1)! (-4)^{a_i}$. Hence, we can rewrite the power series $P_a(\alpha, t)$ in the form

$$
P_a(\alpha, t) = \prod_{i=1}^n (2a_i + 1)! (-4)^{a_i} \sum_{g \geq 0} t^g A_{g,a}(\alpha)
$$

As it is computed in the start of Section 2 we have that the numbers $A_{g,a}(\alpha)$ vanish when $g > |\alpha|$. Hence, we get that all coefficients of the power series $P_a(\alpha, t)$ vanish when $g > |\alpha|$, i.e. $P_a(\alpha, t)$ is a polynomial of degree $|\alpha|$. Furthermore, the top coefficient of $P_a(\alpha, t)$, i.e. the coefficient of $t^{|\alpha|}$ is given by

$$
\langle \prod_{i=1}^n (-4)^{a_i} (2a_i + 1)! \tau_i(\omega)|y\rangle_{|\alpha|,1}^{\psi_1}
$$

This value is computed in [KL11] and is actually equal to 1. In particular, the number $\prod_{i=1}^n (-4)^{a_i} (2a_i + 1)!$ is here to make the polynomial monic.

We now prove several other properties of the polynomials $P_a$.

**Proposition 2.1** The constant term $c_0$ of $P_a(\alpha, t)$ is non zero if and only if $n = 1$ where then $c_0 = (-1)^a \prod_{i=1}^n (-4)^{a_i} (2a_i + 1)!$ or if $n > 1$ and $\sum_{i=1}^n a_i \leq n - 2$ where then

$$
c_0 = \prod_{i=1}^n (-4)^{a_i} (2a_i + 1)! \frac{(n - 2)!}{a_1! \cdots (n - 2 - \sum a_i)!}
$$

**Proof** We only compute the integrals appearing in the constant term of this polynomial since then we only have to multiply with $\prod_{i=1}^n (2a_i + 1)! (-4)^{a_i}$. The integral in the constant term of $P_a(\alpha, t)$ is given by $\int_{\overline{M}_{0,n+1}} \frac{\prod_{i=1}^n \psi_i^{a_i}}{1 - \psi_0}$. When $n = 1$, using the convention $\int_{\overline{M}_{0,2}} \frac{\psi_1^{a_1}}{1 - \psi_0} = (-1)^a$ we get that

$$
c_0 = (-1)^a \prod_{i=1}^n (-4)^{a_i} (2a_i + 1)!
$$
When \( n > 1 \), if \( \sum_{i=1}^{n} a_i > n - 2 \), then \( c_0 \) is zero for dimensional reasons. Otherwise, we have

\[
\int_{\mathcal{M}_{0,n+1}} \prod_{i=1}^{n} \frac{\psi^{a_i}_i}{1 - \psi} = \int_{\mathcal{M}_{0,n+1}} \psi^{n-2-\sum a_i}_0 \prod_{i=1}^{n} \frac{\psi^{a_i}_i}{1 - \psi} = \frac{(n-2)!}{a_1! \cdots (n-2 - \sum a_i)!}.
\]

**Proposition 2.2**  Let \( n \geq 3 \). Then we have the following rules:

(i) **[String equation]**

\[
P_{(a_1,\ldots,a_{n-1},0)}(\alpha,t) = P_{(a_1,\ldots,a_{n-1})}(\alpha,t) - \sum_{i=1}^{n} (8a_i + 4)P_{(a_1,\ldots,a_{i-1},\ldots,a_{n-1})}(\alpha,t)
\]

(ii) **[Dilaton equation]**

\[
P_{(a_1,\ldots,a_{n-1},1)}(\alpha,t) = (t - 12n + 24)P_{(a_1,\ldots,a_{n-1})}(\alpha,t) - 24tP'_{(a_1,\ldots,a_{n-1})}(\alpha,t)
\]

**Proof**  We define the power series

\[
\bar{P}_a(\alpha,t) = \sum_{g \geq 0} t^g \left( \int_{\mathcal{M}_{g,n+1}} \prod_{i=1}^{n} \psi^{a_i}_i \frac{\Lambda^\vee_g(1)\Lambda^\vee_g(\alpha)}{1 - \psi} \right)
\]

Note that the following equation holds.

\[
P_a(\alpha,t) = \prod_{i=1}^{n} (2a_i + 1)!!(-4)^{a_i} \bar{P}_a(\alpha,t) \exp \left( \frac{t}{24} \right)
\]

We can rewrite the coefficients of \( \bar{P}_a(\alpha,t) \) as

\[
\sum_{k=0}^{g} \sum_{j=0}^{g} (-1)^{k+j} (a+1)^{g-j} \int_{\mathcal{M}_{g,n+1}} \prod_{i=1}^{n} \psi^{a_i}_i \Lambda_k \Lambda_j \frac{1}{1 - \psi}
\]

(i) Applying the String equation for Hodge integrals we obtain the following formula

\[
\bar{P}_{(a_1,\ldots,a_{n-1},0)}(\alpha,t) = \bar{P}_{(a_1,\ldots,a_{n-1})}(\alpha,t) + \sum_{i=1}^{n} \bar{P}_{(a_1,\ldots,a_{i-1},\ldots,a_{n-1})}(\alpha,t)
\]

Hence, multiplying with \( \prod_{i=1}^{n} (2a_i + 1)!!(-4)^{a_i} \exp \left( \frac{t}{24} \right) \) we obtain the desired result after a straightforward calculation.

(ii) Applying Dilaton equation for Hodge integrals we obtain the following formula

\[
\bar{P}_{(a_1,\ldots,a_{n-1},1)}(\alpha,t) = 2 \sum_{g \geq 0} gt^g \int_{\mathcal{M}_{g,n+1}} \prod_{i=1}^{n-1} \psi^{a_i}_i \frac{\Lambda^\vee_g(1)\Lambda^\vee_g(\alpha)}{1 - \psi} + (n-2)\bar{P}_{(a_1,\ldots,a_{n-1})}(\alpha,t)
\]
Note that the first term of the sum is equal to \( 2t\tilde{P}'_{(a_1,\ldots,a_{n-1})}(\alpha,t) \). Now, multiplying both sides of the equation above with

\[
\prod_{i=1}^{n-1} (2a_i + 1)(-4)^{a_i} \exp \left( \frac{t}{24} \right)
\]

we have

\[
\frac{-1}{12} P_{(a_1,\ldots,a_{n-1},1)}(\alpha,t) = (n-2)P_{(a_1,\ldots,a_{n-1})}(\alpha,t)
\]

\[
+2t\left( \prod_{i=1}^{n-1} (2a_i + 1)!! \right) \tilde{P}'_{(a_1,\ldots,a_{n-1})}(\alpha,t) e^{t/24}
\]

\[
= (n-2)P_{(a_1,\ldots,a_{n-1})}(\alpha,t)
\]

\[
+2t(P'_{(a_1,\ldots,a_{n-1})}(\alpha,t) - \frac{1}{24} P_{(a_1,\ldots,a_{n-1})}(\alpha,t)).
\]

Finally clearing the denominators we obtain the desired result.

\[\blacksquare\]

We recall Mumford’s relation \( \Lambda^\vee_\mathcal{X}(1) \cdot \Lambda^\vee_\mathcal{X}(-1) = 1 \) (see [Mum83]). In particular, \( P_\alpha(-1,t) \) is defined by integrals of \( \psi \)-classes.

**Corollary 2.3** For any vector \( \alpha \in \mathbb{Z}^n_{\geq 0} \), the power series

\[
P_\alpha(-1,t) = \prod_{i=1}^{n} (2a_i + 1)!!(-4)^{a_i} \exp \left( \frac{t}{24} \right) \cdot \sum_{g \geq 0} (-t)^g \int_{\mathcal{M}_{g,n+1}} \prod_{i=1}^{n} \frac{\psi_i^{a_i}}{1 - \psi_0}
\]

is a polynomial of degree \( |\alpha| \).

In this case the polynomiality as well as a closed expression were proved in [LX11].

**References**


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