ON A PROBLEM OF ORE ON MAXIMAL TREES

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We consider only graphs without loops or multiple edges. Pertinent definitions are given below. For notation and other definitions we generally follow Ore [1].

A connected graph \( G = (X, E) \) is said to have the property \( P \) if for every maximal tree \( T \) of \( G \) there exists a vertex \( a_T \) of \( G \) such that distance between \( a_T \) and \( x \) is same in \( T \) as in \( G \) for every \( x \) in \( X \). The following problem has been posed by Ore (see [1], page 103, problem 4): Determine the graphs with property \( P \). This paper presents a solution to the above problem in the finite case.

**Theorem 1:** A finite biconnected graph \( G = (X, E) \) has the property \( P \) if and only if it is a cycle (type I) or a complete bipartite graph \( K(V, X - V) \) with \( |V| = 2 \) and \( |X - V| \geq 2 \) (type II).

**Proof.** It is easy to check that the graphs mentioned in the statement of the theorem have the property \( P \).

Conversely, let \( G \) be a finite biconnected graph with property \( P \) and \( d(G) \) its diameter. If \( d(G) = 1 \), then, \( G \) is a triangle. So assume that \( d(G) \geq 2 \). We note the following facts.

1. If \( T \) is a maximal tree of \( G \) then \( d(T) \leq 2d(G) \) and further if \( d(T) = 2d(G) \) then \( a_T \) (given by the property \( P \)) is the unique centre of \( T \).
2. Every subgraph of \( G \) which is a tree can be extended to a maximal tree of \( G \).

Let \( x_0, y_0 \) be vertices of \( G \) such that \( d_G(x_0, y_0) = d(G) \). Since \( G \) is biconnected there is a simple circuit \( \mu \) containing \( x_0, y_0 \) (Theorem 5.4.3 of [1]). Without loss of generality assume that \( \mu = [x_0, x_1, x_2, \ldots, x_t, x_0] \). Clearly the length of \( \mu \), \( L(\mu) \) is greater than or equal to \( 2d(G) \). We show that \( L(\mu) \leq 2d(G) + 1 \). Suppose not, then consider the subgraph \( \mu[x_0, x_t] \) whose length \( \geq 2d(G) + 1 \). By (2) and (1) we get a contradiction.

**Case (i).** \( L(\mu) = 2d(G) + 1 \). Let \( A = \{x_0, x_1, \ldots, x_t\} \). Then \( X = A \). For otherwise let \( y \) be a vertex of \( X - A \) adjacent to some vertex of \( A \), say \( x_i \). Consider the subgraph \( \xi = (y, x_i) + \mu[x_i, x_0] + \mu[x_0, x_{i-1}] \) whose diameter \( \geq 2d(G) + 1 \). By (2) and (1) we get a contradiction.
Now $G = \mu$. Otherwise, let $(x_i, x_j)$ be an edge of $G$, where $j$ is different from $i-1$ and $i+1$. Consider the subgraph $T = \mu[x_{i+1}, x_j] + (x_j, x_i) + \mu[x_0, x_{i-1}]$. $T$ is a maximal tree of $G$ and $d(T) = 2d(G)$. Since $G$ has the property $P$, by (2), $a_T$ is the unique centre of $T$, but here it is not, a contradiction. Hence $G$ is a cycle (Type 1).

**Case (ii).** $L(\mu) = 2d(G)$. Let $A = \{x_0, x_1, \ldots, x_t\}$. Define $B_i = \{y : y \in X - A$ and $y$ is adjacent to $x_i \in G\}$, for every $i$, $0 \leq i \leq t$ and $B = \bigcup_{i=0}^{t} B_i$. If $B$ is empty $G = \mu$ (as in case (i)). Assume that $B$ is non-empty. We show that $B$ is an independent set in $G$. Let if possible, $x, y$ be vertices in $B$ and $(x, y)$ be an edge of $G$ with $y$ in $B_0$. Then consider the following subgraph

$$\xi = (x, y) + \mu[x_0, x_0] + \mu[x_0, x_{t-1}]$$

of $G$ whose length is $2d(G) + 1$; by (2) and (1) this leads to a contradiction. Further, if $z$ is in $B_i$, $(z, x_{i+1})$, $(z, x_{i-1})$ are not edges of $G$. Since $B$ is an independent set and $G$ is biconnected, $z$ is joined to $x_j$ for some $j$, $0 \leq j \leq t$ and $i \notin \{i-1, i+1\}$. If $d(G) > 2$ consider the subgraph

$$\xi = [x_{j+1}, x_{j+2}, \ldots, x_l, z, x_j, x_{j-1}, \ldots, x_{i+1}]$$

By (2) this can be extended to a maximal tree $T$ of $G$ and $d(T) = 2d(G)$ but $a_T$ is not the unique centre of $T$—a contradiction. Hence $d(G) = 2$ so $\mu = [x_0, x_1, x_2, x_3, x_0]$. Since $B$ is nonempty at least one of $B_i$, $0 \leq i \leq 3$ is nonempty. Assume that $B_0$ is non empty. Now if $x$ is in $X - A$ it belongs to $B_0$ and $B_2$. Let $V = \{x_0, x_2\}$ then $G = K(V, X-V)$, the complete bipartite graph, with $|X-V| \geq 2$ (type II). This completes the proof of theorem 1.

**Theorem 2.** A finite connected graph with property $P$ on $n$ vertices is a tree or consists of a subgraph $H$ on $n_0$ ($3 \leq n_0 \leq n$) vertices of type I or type II to which trees with a total of $n - n_0$ edges are attached at some vertices of $H$.

**Proof.** Let $x$ be a cut vertex of $G$. It can be easily shown that at most one leaf with respect to $x$ of $G$ is not a tree. Now theorem 2 follows from theorem 1.

**Remark.** Perhaps it is true that $G = K(V, X-V)$, the complete bipartite graph with $|V| = 2$, is the only biconnected graph with property $P$ if $X$ is infinite.

**Reference**


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