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RESEARCH ARTICLE

Zarankiewicz's problem for semilinear hypergraphs

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Abstract

A bipartite graph $H = (V_1, V_2; E)$ with $|V_1| + |V_2| = n$ is *semilinear* if $V_i \subseteq \mathbb{R}^{d_i}$ for some d_i and the edge relation E consists of the pairs of points $(x_1, x_2) \in V_1 \times V_2$ satisfying a fixed Boolean combination of s linear equalities and inequalities in $d_1 + d_2$ variables for some s. We show that for a fixed k, the number of edges in a $K_{k,k}$-free semilinear H is almost linear in n, namely $|E| = O_{s,k,\varepsilon}\left(n^{1+\varepsilon}\right)$ for any $\varepsilon > 0$; and more generally, $|E| = O_{s,k,r,\varepsilon}\left(n^{r-1+\varepsilon}\right)$ for a $K_{k,\dots,k}$-free semilinear r-partite r-uniform hypergraph.

As an application, we obtain the following incidence bound: given n_1 points and n_2 open boxes with axis-parallel sides in $\mathbb{R}^d$ such that their incidence graph is $K_{k,k}$-free, there can be at most $O_{k,\varepsilon}\left(n^{1+\varepsilon}\right)$ incidences. The same bound holds if instead of boxes, one takes polytopes cut out by the translates of an arbitrary fixed finite set of half-spaces.

We also obtain matching upper and (superlinear) lower bounds in the case of dyadic boxes on the plane, and point out some connections to the model-theoretic trichotomy in o-minimal structures (showing that the failure of an almost-linear bound for some definable graph allows one to recover the field operations from that graph in a definable manner).

Contents

1. Introduction

We fix $r \in \mathbb{N}_{\geq 2}$ and let $H = (V_1, \ldots, V_r; E)$ be an r-partite and r-uniform hypergraph (or just an *r-hypergraph* for brevity) with vertex sets $V_1, \ldots, V_r$ having $|V_i| = n_i$, (hyper-) edge set E and a total number $n = \sum_{i=1}^{r} n_i$ of vertices.

Zarankiewicz's problem asks for the maximum number of edges in such a hypergraph H (as a function of $n_1, \ldots, n_r$) assuming that it does not contain the complete r-hypergraph $K_{k,\ldots,k}$ with $k > 0$ a fixed number of vertices in each part. The following classical upper bound is due to Kővári, Sós and Turán [14] for $r = 2$ and Erdős [9] for a general r: if H is $K_{k,\ldots,k}$-free, then $|E| = O_{r,k}\left(n^{r-\frac{1}{k^{r-1}}}\right)$. A probabilistic construction in [9] also shows that the exponent cannot be substantially improved.

However, stronger bounds are known for restricted families of hypergraphs arising in geometric settings. For example, if H is the incidence graph of a set of n_1 points and n_2 lines in $\mathbb{R}^2$, then H is $K_{2,2}$-free, and the Kővári–Sós–Turán Theorem implies $|E| = O\left(n^{3/2}\right)$. The Szemerédi–Trotter Theorem [20] improves this and gives the optimal bound $|E| = O\left(n^{4/3}\right)$. More generally, [12] gives improved bounds for *semialgebraic* graphs of bounded description complexity. This is generalised to semialgebraic hypergraphs in [8]. In a different direction, the results in [12] are generalised to graphs definable in o-minimal structures in [2] and, more generally, in distal structures in [4].

A related highly nontrivial problem is to understand when the bounds offered by the results in the preceding paragraph are sharp. When H is the incidence graph of n_1 points and n_2 circles of unit radius in $\mathbb{R}^2$, the best known upper bound is $|E| = O\left(n^{4/3}\right)$, proven in [19] and also implied by the general bound for semialgebraic graphs. Any improvement to this bound will be a step toward resolving the long-standing unit-distance conjecture of Erdős (an almost-linear bound of the form $|E| = O\left(n^{1+c/\log\log n}\right)$ will positively resolve it).

This paper was originally motivated by the following incidence problem: Let H be the incidence graph of a set of n_1 points and a set of n_2 solid rectangles *with axis-parallel sides* (which we refer to as *boxes*) in $\mathbb{R}^2$. Assuming that H is $K_{2,2}$-free – that is, no two points belong to two rectangles simultaneously – what is the maximum number of incidences $|E|$? In the following theorem, we obtain an almost-linear bound (which is much stronger than the bound implied by the aforementioned general result for semialgebraic graphs) and demonstrate that it is close to optimal:

Theorem (A). 1. *For any set P of n_1 points in $\mathbb{R}^2$ and any set R of n_2 boxes in $\mathbb{R}^2$, if the incidence graph on $P \times R$ is $K_{k,k}$-free, then it contains at most $O_k\left(n \log^4(n)\right)$ incidences (Corollary 2.38 with $d = 2$).*

2. *If all boxes in R are dyadic (i.e., direct products of intervals of the form $[s2^t, (s+1)2^t)$ for some integers s, t), then the number of incidences is at most $O_k\left(n \frac{\log(100+n_1)}{\log\log(100+n_1)}\right)$ (Theorem 4.7).*

3. *For an arbitrarily large n, there exists a set of n points and n dyadic boxes in $\mathbb{R}^2$ so that the incidence graph is $K_{2,2}$-free and the number of incidences is $\Omega\left(n \frac{\log(n)}{\log\log(n)}\right)$ (Proposition 3.5).*

Problem 1.1. *While the bound for dyadic boxes is tight, we leave it as an open problem to close the gap between the upper and lower bounds for arbitrary boxes.*

Remark 1.2. A related result in [11] demonstrates that every $K_{k,k}$-free intersection graph of n convex sets on the plane satisfies $|E| = O_k(n)$. Note that in Theorem (B) we consider a $K_{k,k}$-free *bipartite* graph, so in particular there is no restriction on the intersection graph of the boxes in R.

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Theorem (A.1) admits the following generalisation to higher dimensions and more general polytopes:

Theorem (B). 1. *For any set P of n_1 points and any set B of n_2 boxes in $\mathbb{R}^d$, if the incidence graph on $P \times B$ is $K_{k,k}$-free, then it contains at most $O_{d,k}\left(n \log^{2d} n\right)$ incidences (Corollary 2.38).*

2. *More generally, given finitely many half-spaces $H_1, \ldots, H_s$ in $\mathbb{R}^d$, let $\mathcal{F}$ be the family of all possible polytopes in $\mathbb{R}^d$ cut out by arbitrary translates of $H_1, \ldots, H_s$. Then for any set P of n_1 points in $\mathbb{R}^d$ and any set F of n_2 polytopes in $\mathcal{F}$, if the incidence graph on $P \times F$ is $K_{k,k}$-free, then it contains at most $O_{k,s}\left(n \log^s n\right)$ incidences (Corollary 2.37).*

Problem 1.3. *What is the optimal bound on the power of* $\log n$ *in Theorem (B)? In particular, does it actually have to grow with the dimension d?*

Remark 1.4. A bound similar to Theorem (B.1) and an improved bound for Theorem (A.1) in the $K_{2,2}$-free case are established independently by Tomon and Zakharov in [22], in which they also use our Theorem (A.3) to provide a counterexample to a conjecture of Alon et al. [1] about the number of edges in a graph of bounded separation dimension, as well as to a conjecture of Kostochka from [13]. Some further Ramsey properties of semilinear graphs are demonstrated by Tomon in [21].

The upper bounds in Theorems (A.1) and (B) are obtained as immediate applications of a general upper bound for Zarankiewicz's problem for *semilinear* hypergraphs of bounded description complexity.

Definition 1.5. Let V be an ordered vector space over an ordered division ring R (e.g., $\mathbb{R}$ viewed as a vector space over itself). A set $X \subseteq V^d$ is *semilinear, of description complexity* (s,t), if X is a union of at most t sets of the form

$$\left\{\bar{x} \in V^d : f_1(\bar{x}) \leq 0, \ldots, f_p(\bar{x}) \leq 0, f_{p+1}(\bar{x}) < 0, \ldots, f_s(\bar{x}) < 0\right\},$$

where $p \leq s \in \mathbb{N}$ and each $f_i : V^d \to V$ is a *linear* function – that is, of the form

$$f(x_1, \ldots, x_d) = \lambda_1 x_1 + \cdots + \lambda_d x_d + a$$

for some $\lambda_i \in R$ and $a \in V$.

We focus on the case $V = R = \mathbb{R}$ in the introduction, when these are precisely the semialgebraic sets that can be defined using only *linear* polynomials.

Remark 1.6. By a standard quantifier elimination result [23, §7], every set definable in an ordered vector space over an ordered division ring, in the sense of model theory, is semilinear (equivalently, a projection of a semilinear set is a finite union of semilinear sets).

Definition 1.7. We say that an r-hypergraph H is *semilinear, of description complexity* (s,t), if there exist some $d_i \in \mathbb{N}, V_i \subseteq \mathbb{R}^{d_i}$ and a semilinear set $X \subseteq \mathbb{R}^d = \prod_{i \in [r]} \mathbb{R}^{d_i}$ of description complexity (s,t) so that H is isomorphic to the r-hypergraph $\left(V_1, \ldots, V_r; X \cap \prod_{i \in [r]} V_i\right)$.

We stress that there is no restriction on the dimensions d_i in this definition. We obtain the following general upper bound:

Theorem (C). *If H is a semilinear r-hypergraph of description complexity (s,t) and H is $K_{k,\ldots,k}$-free, then*

$$|E| = O_{r,s,t,k}\left(n^{r-1} \log^{s\left(2^{r-1}-1\right)}(n)\right).$$

In particular, $|E| = O_{r,s,t,k,\varepsilon}\left(n^{r-1+\varepsilon}\right)$ for any $\varepsilon > 0$ in this case. For a more precise statement, see Corollary 2.36 (in particular, the dependence of the constant in $O_{r,s,t,k}$ on k is at most linear).

Remark 1.8. It is demonstrated in [17] that a similar bound holds in the situation when H is the intersection hypergraph of $(d-1)$-dimensional simplices in $\mathbb{R}^d$.

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One can get rid of the logarithmic factor entirely by restricting to the family of all finite r-hypergraphs induced by a given $K_{k,\dots,k}$-free semilinear relation (as opposed to all $K_{k,\dots,k}$-free r-hypergraphs induced by a given arbitrary semilinear relation, as in Theorem (C)).

Theorem (D). *Assume that $X \subseteq \mathbb{R}^d = \prod_{i\in[r]} \mathbb{R}^{d_i}$ is semilinear and X does not contain the direct product of r infinite sets (e.g., if X is $K_{k,\dots,k}$-free for some k). Then for any r-hypergraph H of the form $\left(V_1,\dots,V_r; X\cap\prod_{i\in[r]} V_i\right)$ for some finite $V_i \subseteq \mathbb{R}^{d_i}$, we have $|E| = O_X\left(n^{r-1}\right)$.*

This is Corollary 5.12 and follows from a more general Theorem 5.6 connecting linear Zarankiewicz bounds to a model-theoretic notion of *linearity* of a first-order structure (in the sense that the matroid given by the algebraic closure operator behaves like the linear span in a vector space, as opposed to the algebraic closure in an algebraically closed field – see Definition 5.3).

In particular, for every $K_{k,k}$-free semilinear relation $X \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ (equivalently, X definable with parameters in the first-order structure $(\mathbb{R}, <, +)$ by Remark 1.6) we have $|X \cap (V_1 \times V_2)| = O(n)$ for all $V_i \subseteq \mathbb{R}_i^{d_i}$, $|V_i| = n_i$, $n = n_1 + n_2$. One the other hand, by optimality of the Szemerédi–Trotter bound, for the semialgebraic $K_{2,2}$-free point-line incidence graph $X = \left\{(x_1, x_2; y_1, y_2) \in \mathbb{R}^4 : x_2 = y_1x_1 + y_2\right\} \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ we have $|X \cap (V_1 \times V_2)| = \Omega\left(n^{\frac{4}{3}}\right)$. Note that in order to define it we use both addition and multiplication – that is, the field structure. This is not coincidental; as a consequence of the trichotomy theorem in o-minimal structures [18], we observe that the failure of a linear Zarankiewicz bound always allows us to recover the field in a definable way (Corollary 5.11). In the semialgebraic case, we have the following corollary that is easy to state (Corollary 5.14):

Theorem (E). *Assume that $X \subseteq \mathbb{R}^d = \prod_{i\in[r]} \mathbb{R}^{d_i}$ for some $r, d_i \in \mathbb{N}$ is semialgebraic and $K_{k,\dots,k}$-free, but $|X \cap \prod_{i\in[r]} V_i| \neq O\left(n^{r-1}\right)$. Then the graph of multiplication $\times\restriction_{[0,1]}$ restricted to the unit box is definable in $(\mathbb{R}, <, +, X)$.*

We conclude with a brief overview of the paper.

In Section 2 we introduce a more general class of hypergraphs definable in terms of *coordinate-wise monotone* functions (Definition 2.1) and prove an upper Zarankiewicz bound for it (Theorem 2.17). Theorems (A.1), (B) and (C) are then deduced from it in Section 2.5.

In Section 3 we prove Theorem (A.3) by establishing a lower bound on the number of incidences between points and dyadic boxes on the plane, demonstrating that the logarithmic factor is unavoidable (Proposition 3.5).

In Section 4, we establish Theorem (A.2) by obtaining a stronger bound on the number of incidences with *dyadic* boxes on the plane (Theorem 4.7). We use a different argument, relying on a certain partial order specific to the dyadic case, to reduce from $\log^4(n)$ given by the general theorem to $\log(n)$. Up to a constant factor, this implies the same bound for incidences with general boxes when one counts only incidences that are bounded away from the border (Remark 4.8).

Finally, in Section 5 we prove a general Zarankiewicz bound for definable relations in weakly locally modular geometric first-order structures (Theorem 5.6), deduce Theorem (D) from it (Corollary 5.12) and observe how to recover a real closed field from the failure of Theorem (D) in the o-minimal case (Corollary 5.11).

2. Upper bounds

2.1. Coordinate-wise monotone functions and basic sets

For an integer $r \in \mathbb{N}_{>0}$, by *an r-grid* (or a *grid*, if r is clear from the context) we mean a cartesian product $B = B_1 \times \cdots \times B_r$ of some sets $B_1, \dots, B_r$. As usual, $[r]$ denotes the set $\{1, 2, \dots, r\}$.

If $B = B_1 \times \cdots \times B_r$ is a grid, then by a *subgrid* we mean a subset $C \subseteq B$ of the form $C = C_1 \times \cdots \times C_r$ for some $C_i \subseteq B_i$.

Let B be an r-grid, S an arbitrary set and $f : B \to S$ a function. For $i \in [r]$, set

$$B^i = B_1 \times \cdots \times B_{i-1} \times B_{i+1} \times \cdots \times B_r,$$

and let $\pi_i : B \to B_i$ and $\pi^i : B \to B^i$ be the projection maps.

For $a \in B^i$ and $b \in B_i$, we write $a \oplus_i b$ for the element $c \in B$ with $\pi^i(c) = a$ and $\pi_i(c) = b$. In particular, when $i = r$, $a \oplus_r b = (a, b)$.

Definition 2.1. Let B be an r-grid and $(S, <)$ a linearly ordered set. A function $f : B \to S$ is *coordinate-wise monotone* if for any $i \in [r]$, $a, a' \in B^i$ and $b, b' \in B_i$, we have

$$f(a \oplus_i b) \leq f(a \oplus_i b') \iff f(a' \oplus_i b) \leq f(a' \oplus_i b').$$

Remark 2.2. Let $B = B_1 \times \cdots \times B_r$ be an r-grid and Γ an ordered abelian group. We say that a function $f : B \to \Gamma$ is *quasi-linear* if there exist some functions $f_i : B_i \to \Gamma$, $i \in [r]$, such that

$$f(x_1, \ldots, x_r) = f_1(x_1) + \cdots + f_r(x_r).$$

Then every quasi-linear function is coordinate-wise monotone (as $f(a \oplus_i b) \leq f(a \oplus_i b') \Leftrightarrow f_i(b) \leq f_i(b')$ for any $a \in B^i$).

Example 2.3. Suppose that V is an ordered vector space over an ordered division ring R, $d_i \in \mathbb{N}$ for $i \in [r]$, and $f : V^{d_1} \times \cdots \times V^{d_r} \to V$ is a linear function. Then f is obviously quasi-linear, and hence coordinate-wise monotone.

Remark 2.4. Let B be a grid and $C \subseteq B$ a subgrid. If $f : B \to S$ is a coordinate-wise monotone function, then the restriction $f \restriction C$ is a coordinate-wise monotone function on C.

Definition 2.5. Let B be an r-grid. A subset $X \subseteq B$ is *a basic set* if there exists a linearly ordered set $(S, <)$, a coordinate-wise monotone function $f : B \to S$ and $l \in S$ such that $X = \{b \in B : f(b) < l\}$.

Remark 2.6. If $r = 1$, then every subset of $B = B_1$ is basic.

Remark 2.7. If $X \subseteq B$ is given by $X = \{b \in B : f(b) \leq l\}$ for some coordinate-wise monotone function $f : B \to S$, then X is a basic set as well. Indeed, we can just add a new element l' to S so that it is a successor of l, and then $X = \{b \in B : f(b) < l'\}$.

Similarly, the sets $\{b \in B : f(b) > l\}$, $\{b \in B : f(b) \geq l\}$ are basic, by inverting the order on S.

We have the following 'coordinate-splitting' presentation for basic sets:

Proposition 2.8. *Let $B = B_1 \times \cdots \times B_r$ be an r-grid and $X \subseteq B$ a basic set. Then there is a linearly ordered set $(S, <)$, a coordinate-wise monotone function $f^r : B^r \to S$ and a function $f_r : B_r \to S$ such that $X = \{b^r \oplus_r b_r : f^r(b^r) < f_r(b_r)\}$.*

Remark 2.9. The converse of this proposition is also true: an arbitrary linear order $(S, <)$ can be realised as a subset of some ordered abelian group $(G, +, <)$ with the induced ordering (we can take $G := \mathbb{Q}$ when S is at most countable); then define $f : B \to S$ by setting

$$f(b^r \oplus_r b_r) := f^r(b^r) - f_r(b_r), \text{ and } l := 0.$$

Proof of Proposition 2.8. Assume that we are given a coordinate-wise monotone function $f : B \to S$ and $l \in S$ with $X = \{b \in B : f(b) < l\}$.

For $i \in [r]$, let $\leq_i$ be the preorder on B_i induced by f – namely, for $b, b' \in B_i$ we set $b \leq_i b'$ if and only if for some (equivalently, any) $a \in B^i$ we have $f(a \oplus_i b) \leq f(a \oplus_i b')$.

Quotienting B_i by the equivalence relation corresponding to the preorder $\leq_i$ if needed, we may assume that each $\leq_i$ is actually a linear order.

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Let $<^r$ be the partial order on B^r with $(b_1, \ldots, b_{r-1}) <^r \left(b'_1, \ldots, b'_{r-1}\right)$ if and only if

$$(b_1, \ldots, b_{r-1}) \neq \left(b'_1, \ldots, b'_{r-1}\right) \text{ and } b_j \leq_j b'_j \text{ for all } j \in [r-1].$$

Define $T := B^r \dot{\cup} B_r$, where $\dot{\cup}$ denotes the disjoint union. Clearly $<^r$ is a *strict* partial order on T – that is, a transitive and antisymmetric (hence irreflexive) relation.

For any $b^r \in B^r$ and $b_r \in B_r$, we define

$$b^r \triangleleft b_r \text{ if } f(b^r \oplus_r b_r) < l, \text{ and } b_r \triangleleft b^r \text{ otherwise.}$$

Claim 2.10. *Set $a_1, a_2 \in B^r$ and $b_1, b_2 \in B_r$.*

1. *If $a_1 \triangleleft b_1 \triangleleft a_2 \triangleleft b_2$, then $b_2 <_r b_1$ and $a_1 \triangleleft b_2$.*
2. *If $b_1 \triangleleft a_1 \triangleleft b_2 \triangleleft a_2$, then $b_2 <_r b_1$ and $b_1 \triangleleft a_2$.*

Proof. (1). We have $f(a_2 \oplus_r b_1) \geq l$ and $f(a_2 \oplus_r b_2) < l$, hence $b_2 <_r b_1$. Since $f(a_1 \oplus_r b_1) < l$ and $b_2 <_r b_1$ we also have $f(a_1 \oplus_r b_2) < l$.

(2) is similar. □

Let $\triangleleft^t$ be the transitive closure of $\triangleleft$. It follows from the preceding claim that $\triangleleft^t = \triangleleft \cup \triangleleft \circ \triangleleft$. More explicitly, for $b_1, b_2 \in B_r$, we have $b_1 \triangleleft^t b_2$ if $b_2 <_r b_1$, and for $a_1, a_2 \in B^r$, we have $a_1 \triangleleft^t a_2$ if $f(a_1 \oplus b) < l < f(a_2 \oplus b)$ for some $b \in B_r$. It is not hard to see then that $\triangleleft^t$ is antisymmetric, and hence it is a strict partial order on T.

Claim 2.11. *The union $<^r \cup \triangleleft^t$ is a strict partial order on T.*

Proof. We first show transitivity. Note that $<^r$ and $\triangleleft^t$ are both transitive, so it suffices to show for $x, y, z \in T$ that if either $x <^r y \triangleleft^t z$ or $x \triangleleft^t y <^r z$, then $x \triangleleft^t z$. Furthermore, since $\triangleleft^t = \triangleleft \cup \triangleleft \circ \triangleleft$, we may restrict our attention to the following cases: If $a_1 <^r a_2 \triangleleft b$ with $a_1, a_2 \in B^r$ and $b \in B_r$, then $f(a_1 \oplus_r b) < f(a_2 \oplus_r b) < l$, and so $a_1 \triangleleft b$. If $b \triangleleft a_1 <^r a_2$ with $a_1, a_2 \in B^r$ and $b \in B_r$, then $f(a_2 \oplus_r b) > f(a_1 \oplus_r b) \geq l$, and so $b \triangleleft a_2$.

To check antisymmetry, assume $a_1 <^r a_2$ and $a_2 \triangleleft^t a_1$. Since $a_1, a_2 \in B^r$, we have $a_2 \triangleleft b \triangleleft a_1$ for some $b \in B_r$. We have $f(a_1 \oplus_r b) \geq l > f(a_2 \oplus_r b)$, contradicting $a_1 <^r a_2$. □

Finally, let $\prec$ be an arbitrary linear order on $T = B^r \dot{\cup} B_r$ extending $<^r \cup \triangleleft^t$. Since $\prec$ extends $\triangleleft$, for $a \in B^r$ and $b \in B_r$ we have $(a, b) \in X$ if and only if $a \prec b$.

We take $f^r : B^r \longrightarrow T$ and $f_r : B_r \longrightarrow T$ to be the identity maps. Since $\prec$ extends $<^r$, the map f^r is coordinate-wise monotone. □

2.2. *Main theorem*

Definition 2.12. Let $B = B_1 \times \cdots \times B_r$ be an r-grid.

1. Given $s \in \mathbb{N}$, we say that a set $X \subseteq B$ has *grid-complexity* s (in B) if X is the intersection of B with at most s basic subsets of B.

   We say that X has *finite grid-complexity* if it has grid-complexity s for some $s \in \mathbb{N}$.
2. For integers $k_1, \ldots, k_r$ we say that $X \subseteq B$ is $K_{k_1,\ldots,k_r}$*-free* if X does not contain a subgrid $C_1 \times \cdots \times C_r \subseteq S$ with $|C_i| = k_i$.

In particular, B itself is the only subset of B of grid-complexity 0.

Example 2.13. Suppose that V is an ordered vector space over an ordered division ring, $d = d_1 + \cdots + d_r \in \mathbb{N}$ and

$$X = \left\{\bar{x} \in V^d : f_1(\bar{x}) \leq 0, \ldots, f_p(\bar{x}) \leq 0, f_{p+1}(\bar{x}) < 0, \ldots, f_s(\bar{x}) < 0\right\},$$

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for some linear functions $f_i : V^d \to V, i \in [s]$. Then each f_i is coordinate-wise monotone (Example 2.3), and hence each of the sets

$$\{\bar{x} \in V^d : f_i(\bar{x}) < 0\}, \{\bar{x} \in V^d : f_i(\bar{x}) \leq 0\}$$

is a basic subset of the grid $V^{d_1} \times \cdots \times V^{d_r}$ (the latter by Remark 2.7), and $X \subseteq V^{d_1} \times \cdots \times V^{d_r}$ as an intersection of these s basic sets has grid-complexity s.

Remark 2.14. 1. Let B be an r-grid and $A \subseteq B$ a subset of B of grid-complexity s. If $C \subseteq B$ is a subgrid containing A, then A is also a subset of C of grid-complexity s.
2. In particular, if $A \subseteq B$ is a subset of grid-complexity s, then A is a subset of grid-complexity s of the grid $A_1 \times \cdots \times A_r$, where $A_i := \pi_i(A)$ is the projection of A on B_i (it is the smallest subgrid of B containing A).

Definition 2.15. Let $B = B_1 \times \cdots \times B_r$ be a finite r-grid and set $n_i := |B_i|$. For $j \in \{0, \ldots, r\}$, we will denote by $\delta^r_j(B)$ the integer

$$\delta^r_j(B) := \sum_{i_1 < i_2 < \cdots < i_j \in [r]} n_{i_1} \cdot n_{i_2} \cdot \cdots \cdot n_{i_j}.$$

Example 2.16. We have $\delta^r_0(B) = 1$, $\delta^r_1(B) = n_1 + \cdots + n_r$, $\delta^r_r(B) = n_1 n_2 \cdots n_r$.

We can now state the main theorem:

Theorem 2.17. *For all integers $r \geq 2, s \geq 0, k \geq 2$, there are $\alpha = \alpha(r, s, k) \in \mathbb{R}$ and $\beta = \beta(r, s) \in \mathbb{N}$ such that for any finite r-grid B and $K_{k,\ldots,k}$-free subset $A \subseteq B$ of grid-complexity s, we have*

$$|A| \leq \alpha \delta^r_{r-1}(B) \log^\beta \left(\delta^r_{r-1}(B) + 1\right).$$

Moreover, we can take $\beta(r, s) := s\left(2^{r-1} - 1\right)$.

Remark 2.18. Inspecting the proof in Sections 2.3 and 2.4, it can be verified that the dependence of α on k in Theorem 2.17 s at most linear.

Remark 2.19. We use $\log^\beta\left(\delta^r_{r-1}(B) + 1\right)$ instead of $\log^\beta\left(\delta^r_{r-1}(B)\right)$ to include the case $\delta^r_{r-1}(B) \leq 1$.

Remark 2.20. If, in Theorem 2.17, A is only assumed to be a union of at most t sets of grid-complexity s, then the same bound holds with $\alpha' := t \cdot \alpha$ (if $A = \bigcup_{i \in [t]} A_i$ is $K_{k,\ldots,k}$-free, then each A_i is also $K_{k,\ldots,k}$-free, so we can apply Theorem 2.17 to each A_i and bound $|A|$ by the sum of their bounds).

Definition 2.21. Let $B = B_1 \times \cdots \times B_r$ be a grid. We extend the definition of δ^r_j to arbitrary finite subsets of B as follows: let $A \subseteq B$ be a finite subset, and let $A_i := \pi_i(A)$, $i \in [r]$, be the projections of A. We define $\delta^r_j(A) := \delta^r_j(A_1 \times \cdots \times A_r)$.

If B is a finite r-grid and $A \subseteq B$, then obviously $\delta^r_j(A) \leq \delta^r_j(B)$. Thus Theorem 2.17 is equivalent to the following:

Proposition 2.22. *For all integers $r \geq 2, s \geq 0, k \geq 2$, there are $\alpha = \alpha(r, s, k) \in \mathbb{R}$ and $\beta = s\left(2^{r-1} - 1\right) \in \mathbb{N}$ such that for any r-grid B and $K_{k,\ldots,k}$-free finite subset $A \subseteq B$ of grid-complexity $\leq s$, we have*

$$|A| \leq \alpha \delta^r_{r-1}(A) \log^\beta \left(\delta^r_{r-1}(A) + 1\right).$$

Definition 2.23. For $r \geq 1, s \geq 0, k \geq 2$ and $n \in \mathbb{N}$, let $F_{r,k}(s, n)$ be the maximal size of a $K_{k,\ldots,k}$-free subset A of grid-complexity s of some r-grid B with $\delta^r_{r-1}(B) \leq n$.

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Then Proposition 2.22 can be restated as follows:

Proposition 2.24. *For all integers $r \geq 2, s \geq 0, k \geq 2$, there are $\alpha = \alpha(r, s, k) \in \mathbb{R}$ and $\beta = \beta(r, s) \in \mathbb{N}$ such that*

$$F_{r,k}(s, n) \leq \alpha n \log^{\beta}(n+1).$$

Remark 2.25. Notice that $F_{r,k}(s, 0) = 0$.

In the rest of the section we prove Proposition 2.24 by induction on r, where for each r it is proved by induction on s. We will use the following simple recurrence bound:

Fact 2.26. Let $\mu\colon \mathbb{N} \to \mathbb{N}$ be a function satisfying $\mu(0) = 0$ and $\mu(n) \leq 2\mu(\lfloor n/2 \rfloor) + \alpha n \log^{\beta}(n+1)$ for some $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{N}$. Then $\mu(n) \leq \alpha' n \log^{\beta+1}(n+1)$ for some $\alpha' = \alpha'(\alpha, \beta) \in \mathbb{R}$.

2.3. *The base case* $r = 2$

Let $B = B_1 \times B_2$ be a finite grid and $A \subseteq B$ a subset of grid-complexity s. We will proceed by induction on s.

If $s = 0$, then $A = B_1 \times B_2$. If A is $K_{k,k}$-free, then one of the sets B_1, B_2 must have size at most k. Hence $|A| \leq k(|B_1| + |B_2|) = k\delta_1^2(B)$.

Thus

$$F_{2,k}(0, n) \leq kn.$$

Remark 2.27. The same argument shows that $F_{r,k}(0, n) \leq kn$ for all $r \geq 2$.

Assume now that the theorem is proved for $r = 2$ and all $s' < s$. Define $n_1 := |B_1|$, $n_2 := |B_2|$ and $n := \delta_1^2(B) = n_1 + n_2$.

We choose basic sets $X_1, \ldots, X_s \subseteq B$ such that $A = B \cap \bigcap_{j \in [s]} X_j$.

By Proposition 2.8, we can choose a finite linear order $(S, <)$ and functions $f_1\colon B_1 \to S$ and $f_2\colon B_2 \to S$ so that

$$X_s = \{(x_1, x_2) \in B_1 \times B_2 \colon f_1(x_1) < f_2(x_2)\}.$$

For $l \in S$, $i \in \{1, 2\}$ and $\square \in \{<, =, >, \leq, \geq\}$, let

$$B_i^{\square l} = \{b \in B_i \colon f_i(b) \square l\}.$$

We choose $h \in S$ such that

$$\left|B_1^{<h}\right| + \left|B_2^{<h}\right| \leq n/2 \text{ and } \left|B_1^{>h}\right| + \left|B_2^{>h}\right| \leq n/2.$$

For example, we can take h to be the minimal element in $f_1(B_1) \cup f_2(B_2)$ with $\left|B_1^{\leq h}\right| + \left|B_2^{\leq h}\right| \geq n/2$. Then

$$X_s = \left[\left(B_1^{<h} \times B_2^{<h}\right) \cap X_s\right] \cup \left[\left(B_1^{>h} \times B_2^{>h}\right) \cap X_s\right] \cup \left(B_1^{<h} \times B_2^{\geq h}\right) \cup \left(B_1^{=h} \times B_2^{>h}\right).$$

Hence we conclude

$$F_{2,k}(s, n) \leq 2F_{2,k}(s, \lfloor n/2 \rfloor) + 2F_{2,k}(s-1, n).$$

Applying the induction hypothesis on s and using Fact 2.26 and Remark 2.25, we obtain $F_{2,k}(s, n) \leq \alpha n (\log n)^{\beta}$ for some $\alpha = \alpha(s, k) \in \mathbb{R}$ and $\beta = \beta(s) \in \mathbb{N}$.

This finishes the base case $r = 2$.

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2.4. Induction step

We fix $r \in \mathbb{N}_{\geq 3}$ and assume that Proposition 2.24 holds for all pairs (r', s) with $r' < r$ and $s \in \mathbb{N}$.

Definition 2.28. Let $B = B_1 \times \cdots \times B_r$ be a finite r-grid.

1. For integers $t, u \in \mathbb{N}$, we say that a subset $A \subseteq B$ is *of split grid-complexity* (t, u) if there are basic sets $X_1, \ldots, X_u \subseteq B$, a subset $A^r \subseteq B_1 \times \cdots \times B_{r-1}$ of grid-complexity t and a subset $A_r \subseteq B_r$ such that $A = (A^r \times A_r) \cap \bigcap_{i \in [u]} X_i$.
2. For $t, u \geq 0, k \geq 2$ and $n \in \mathbb{N}$, let $G_k(t, u, n)$ be the maximal size of a $K_{k,\ldots,k}$-free subset A of an r-grid B of split grid-complexity (t, u) with $\delta^r_{r-1}(B) \leq n$.

Remark 2.29. 1. Note that A_r has grid-complexity at most 1, which is the reason we do not include a parameter for the grid-complexity of A_r in the split grid-complexity of A.
2. If $A \subseteq B$ is of grid-complexity s, then it is of split grid-complexity $(0, s)$.
3. If $A \subseteq B$ is of split grid-complexity (t, u), then it is of grid-complexity $t + u$.

For the rest of the proof, we abuse notation slightly and refer to the split grid-complexity of a set as simply the grid-complexity. To complete the induction step we will prove the following proposition:

Proposition 2.30. *For any integers $t, u \geq 0, k \geq 2, r \geq 3$, there are $\alpha' = \alpha'(r, k, t, u) \in \mathbb{R}$ and $\beta' = \beta'(r, k, t, u) \in \mathbb{N}$ such that*

$$G_k(t, u, n) \leq \alpha' n \log^{\beta'}(n + 1).$$

We will use the following notations throughout the section:

- $B = B_1 \times \cdots \times B_r$ is a finite grid with $n = \delta^r_{r-1}(B)$;
- $A \subseteq B$ is a subset of grid-complexity (t, u);
- B^r is the $(r-1)$-grid $B^r := B_1 \times \cdots \times B_{r-1}$;
- $A^r \subseteq B^r$ is a subset of grid-complexity t, $A_r \subseteq B_r$, and $X_1, \ldots X_u \subseteq B$ are basic subsets such that $A = (A^r \times A_r) \cap \bigcap_{i \in [u]} X_i$.

We proceed by induction on u.

The base case $u = 0$ of Proposition 2.30.

In this case, $A = A^r \times A_r$. If A is $K_{k,\ldots,k}$-free, then either A^r is $K_{k,\ldots,k}$-free or $|A_r| < k$.

In the first case, by the induction hypothesis on r, there are $\alpha = \alpha(r-1, t, k)$ and $\beta = \beta(r-1, t)$ such that $|A^r| \leq \alpha \delta^{r-1}_{r-2}(B^r) \log^{\beta}\left(\delta^{r-1}_{r-2}(B^r) + 1\right)$. In the second case, we have $|A| \leq |B^r| k = \delta^{r-1}_{r-1}(B^r) k$.

Since $n = \delta^r_{r-1}(B) = \delta^{r-1}_{r-1}(B^r) + \delta^{r-1}_{r-2}(B^r)|B_r|$, the conclusion of the proposition follows with $\alpha' := \alpha, \beta' := \beta$.

Induction step of Proposition 2.30.

We assume now that the proposition holds for all pairs (t, u') with $u' < u$ and $t \in \mathbb{N}$.

Given a tuple $x = (x_1, \ldots, x_r) \in B$, we set $x^r := (x_1, \ldots, x_{r-1})$. By Proposition 2.8, we can choose a finite linear order $(S, <)$, a coordinate-wise monotone function $f^r : B^r \to S$ and a function $f_r : B_r \to S$ so that

$$X_u = \{x^r \oplus_r x_r \in B^r \times B_r : f^r(x^r) < f_r(x_r)\}.$$

Moreover, by Remark 2.9 we may assume without loss of generality that the coordinate-wise monotone function defining X_u is given by

$$f(x^r \oplus_r x_r) = f^r(x^r) - f_r(x_r).$$

Definition 2.31. Given an arbitrary set $C^r \subseteq B^r$, we say that a set $H^r \subseteq C^r$ is an *f^r-strip in C^r* if

$$H^r = \{x^r \in C^r : l_1 \triangleleft_1 f^r(x^r) \triangleleft_2 l_2\}$$

for some $l_1, l_2 \in S$, $\lhd_1, \lhd_2 \in \{<, \leq\}$. Likewise, given an arbitrary set $C_r \subseteq B_r$, we say that $H_r \subseteq C_r$ is an *f_r-strip in C_r* if

$$H_r = \{x_r \in C_r : l_1 \lhd_1 f_r(x_r) \lhd_2 l_2\}$$

for some $l_1, l_2 \in S$, $\lhd_1, \lhd_2 \in \{<, \leq\}$. If $C^r = A^r$ or $C_r = A_r$, we simply say an *f^r-strip* or *f_r-strip*, respectively.

Remark 2.32. Note the following:

1. A^r is an f^r-strip, and A_r is an f_r-strip.
2. Every f^r-strip is a subset of the $(r-1)$-grid B^r of grid-complexity $t+2$ (using Remark 2.7).
3. The intersection of any two f^r-strips is an f^r-strip; the same conclusion holds for f_r-strips.

Definition 2.33. 1. We say that a subset $H \subseteq B$ is an *f-grid* if $H = H^r \times H_r$, where $H^r \subseteq B^r$ is an f^r-strip in B^r and $H_r \subseteq B_r$ is an f_r-strip in B_r.
2. If $H = H^r \times H_r$ is an f-grid, we set

$$\Delta(H) := |H^r| + \delta^{r-1}_{r-2}(H^r)|H_r| \text{ (see Definition 2.21 for } \delta^{r-1}_{r-2}).$$

Note that if H is a sub-grid of B, then $\Delta(H) = \delta^r_{r-1}(H)$.
3. For an f-grid H, we will denote by A_H the set $A \cap H$.

The induction step for Proposition 2.30 will follow from the following proposition:

Proposition 2.34. *For all integers $k \geq 2, r \geq 3$, there are $\alpha' = \alpha'(r, k, t, u) \in \mathbb{R}$ and $\beta' = \beta'(r, t, u) \in \mathbb{N}$ such that, for any f-grid H, if the set A_H is $K_{k,\dots,k}$-free then*

$$|A_H| \leq \alpha' \Delta(H) \log^{\beta'}(\Delta(H) + 1).$$

We should stress that in this proposition, α' and β' do not depend on f^r, f_r, B, A^r, and A_r, but they may depend on our fixed t and u.

Given Proposition 2.34, we can apply it to the f-grid $H := A^r \times A_r$ (so $A_H = A$) and get

$$|A| \leq \alpha' \Delta(H) \log^{\beta'}(\Delta(H) + 1).$$

It is easy to see that $\Delta(A^r \times A_r) \leq \delta^r_{r-1}(B)$, and hence Proposition 2.30 follows with the same α' and β'.

We proceed with the proof of Proposition 2.34:

Proof of Proposition 2.34. Fix $m \in \mathbb{N}$, and let $L(m)$ be the maximal size of a $K_{k,\dots,k}$-free set A_H among all f-grids $H \subseteq B$ with $\Delta(H) \leq m$. We need to show that for some $\alpha' = \alpha'(k) \in \mathbb{R}$ and $\beta' \in \mathbb{N}$ we have

$$L(m) \leq \alpha' m \log^{\beta'}(m + 1).$$

Let $H = H^r \times H_r$ be an f-grid with $\Delta(H) \leq m$.

For $l \in S$ and $\square \in \{<, =, >, \leq, \geq\}$, define

$$H^{r,\square l} := \{x^r \in H^r : f^r(x^r) \square l\}$$

and

$$H_r^{\square l} := \{x_r \in H_r : f_r(x_r) \square l\}.$$

Note that for every $l \in S$, $H^{r,\square l}$ is an f^r-strip in H^r, $H_r^{\square l}$ is an f_r-strip in H_r and their product is an f-grid.

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Claim 2.35. *There is $h \in S$ such that*

$$\Delta\left(H^{r,<h} \times H_r^{<h}\right) \leq m/2 \text{ and } \Delta\left(H^{r,>h} \times H_r^{>h}\right) \leq m/2.$$

Proof. Set $\delta := \delta_{r-2}^{r-1}(H^r)$.

Let h be the minimal element in $f^r(H^r) \cup f_r(H_r)$ with

$$\left|H^{r,\leq h}\right| + \delta\left|H_r^{\leq h}\right| \geq m/2.$$

Then $\left|H^{r,<h}\right| + \delta\left|H_r^{<h}\right| \leq m/2$ and $\left|H^{r,>h}\right| + \delta\left|H_r^{>h}\right| \leq m/2$. Since $H^{r,<h}, H^{r,>h} \subseteq H^r$, we have $\delta_{r-2}^{r-1}\left(H^{r,<h}\right), \delta_{r-2}^{r-1}\left(H^{r,>h}\right) \leq \delta$. The claim follows. □

Let h be as in the claim. It is not hard to see that the following hold:

$$\left(H^{r,\leq h} \times H_r^{\geq h}\right) \cap X_u = \left(H^{r,<h} \times H_r^{\geq h}\right) \cup \left(H^{r,=h} \times H_r^{>h}\right),$$
$$\left(H^{r,\geq h} \times H_r^{\leq h}\right) \cap X_u = \emptyset.$$

It follows that

$$A_H \cap X_u = \left[\left(H^{r,<h} \times H_r^{<h}\right) \cap X_u\right] \cup \left[\left(H^{r,>h} \times H_r^{>h}\right) \cap X_u\right] \cup \left(H^{r,<h} \times H_r^{\geq h}\right) \cup \left(H^{r,=h} \times H_r^{>h}\right).$$

Hence, by the choice of h and using Remark 2.32(2),

$$L(m) \leq 2L(\lfloor m/2 \rfloor) + 2G_k(t+2, u-1, m).$$

Applying the induction hypothesis on u and using Fact 2.26, we obtain $L(m) \leq \alpha' m \log^{\beta'}(m+1)$ for some $\alpha' = \alpha'(k) \in \mathbb{R}$ and $\beta' \in \mathbb{N}$.

This finishes the proof of Proposition 2.34, and hence of the induction step of Proposition 2.24. □

Finally, inspecting the proof, we have shown the following:

1. $\beta(2, s) \leq s$ for all $s \in \mathbb{N}$.
2. $\beta'(r, t, 0) \leq \beta(r-1, t)$ for all $r \geq 3$ and $t \in \mathbb{N}$.
3. $\beta'(r, t, u) \leq \beta'(r, t+2, u-1) + 1$ for all $r \geq 3, t \geq 0, u \geq 1$.

Iterating (3), for every $r \geq 3, s \geq 1$ we have $\beta(r, s) \leq \beta'(r, 0, s) \leq \beta'(r, 2s, 0) + s$. Hence, by (2), $\beta(r, s) \leq \beta(r-1, 2s) + s$ for every $r \geq 3$ and $s \geq 1$. Iterating this, we get $\beta(r, s) \leq \beta\left(2, 2^{r-2}s\right) + s\sum_{i=0}^{r-3} 2^i$. Using (1), this implies $\beta(r, s) \leq s\sum_{i=0}^{r-2} 2^i = s\left(2^{r-1} - 1\right)$ for all $r \geq 3, s \geq 1$. Hence, by Remark 2.27 and (1) again, $\beta(r, s) \leq s\left(2^{r-1} - 1\right)$ for all $r \geq 2, s \geq 0$.

2.5. *Some applications*

We observe several immediate applications of Theorem 2.17, starting with the following bound for semilinear hypergraphs:

Corollary 2.36. *For every $r, s, t, k \in \mathbb{N}, r \geq 2$, there exist some $\alpha = \alpha(r, s, t, k) \in \mathbb{R}$ and $\beta(r, s) := s\left(2^{r-1} - 1\right)$ satisfying the following: for any semilinear $K_{k,\ldots,k}$-free r-hypergraph $H = (V_1, \ldots, V_r; E)$ of description complexity (s, t) (see Definition 1.7), taking $V := \prod_{i \in [r]} V_i$ we have*

$$|E| \leq \alpha \delta_{r-1}^r(V) \log^\beta\left(\delta_{r-1}^r(V) + 1\right).$$

Proof. By assumption, the edge relation E can be defined by a union of t sets, each of which is defined by s linear equalities and inequalities, hence of grid-complexity $\leq s$ (see Example 2.13). The conclusion follows by Theorem 2.17 and Remark 2.20. □

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As a special case with $r = 2$, this implies a bound for the following incidence problem:

Corollary 2.37. *For every $s, k \in \mathbb{N}$ there exists some $\alpha = \alpha(s,k) \in \mathbb{R}$ satisfying the following:*

Let $d \in \mathbb{N}$ and $H_1, \ldots, H_s \subseteq \mathbb{R}^d$ be finitely many (closed or open) half-spaces in $\mathbb{R}^d$. Let $\mathcal{F}$ be the (infinite) family of all possible polytopes in $\mathbb{R}^d$ cut out by arbitrary translates of $H_1, \ldots, H_s$.

For any set P of n_1 points in $\mathbb{R}^d$ and any set F of n_2 polytopes in $\mathcal{F}$, if the incidence graph on $P \times F$ is $K_{k,k}$-free, then it contains at most $\alpha n \log^s n$ incidences.

Proof. We can write

$$H_i = \left\{ \bar{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d : \sum_{j \in [d]} a_{i,j} x_j \square_i b_i \right\},$$

where $a_{i,j}, b_i \in \mathbb{R}$ and $\square_i \in \{>, \geq\}$ for $i \in [s], j \in [d]$ depending on whether H_i is an open or a closed half-space.

Every polytope $F \in \mathcal{F}$ is of the form $\bigcap_{i \in [s]} (\bar{y}_i + H_i)$ for some $(\bar{y}_1, \ldots, \bar{y}_s) \in \mathbb{R}^{sd}$, where $\bar{y}_i + H_i$ is the translate of H_i by the vector $\bar{y}_i = (y_{i,1}, \ldots, y_{i,d}) \in \mathbb{R}^d$ – that is,

$$\bar{y}_i + H_i = \left\{ \bar{x} \in \mathbb{R}^d : \sum_{j \in [d]} a_{i,j} x_j + \sum_{j \in [d]} (-a_{i,j})\, y_j \square_i b_i \right\}.$$

Then the incidence relation between points in $\mathbb{R}^d$ and polytopes in $\mathcal{F}$ can be identified with the semilinear set

$$\left\{ \left(\bar{x}; (y_{i,j})_{i \in [s], j \in [d]}\right) \in \mathbb{R}^d \times \mathbb{R}^{sd} : \bigwedge_{i \in [s]} \sum_{j \in [d]} a_{i,j} x_j + \sum_{j \in [d]} (-a_{i,j})\, y_{i,j} \square_i b_i \right\}$$

defined by s linear inequalities. The conclusion now follows by Corollary 2.36 with $r = 2$. □

In particular, we get a bound for the original question that motivated this paper.

Corollary 2.38. *Let $\mathcal{F}_d$ be the family of all (closed or open) boxes in $\mathbb{R}^d$. Then for every k there exists some $\alpha = \alpha(d,k)$ satisfying the following: for any set P of n_1 points in $\mathbb{R}^d$ and any set F of n_2 boxes in $\mathcal{F}_d$, if the incidence graph on $P \times F$ is $K_{k,k}$-free, then it contains at most $\alpha n \log^{2d} n$ incidences.*

Proof. This is immediate from Corollary 2.37, since we have $2d$ half-spaces in $\mathbb{R}^d$ such that every box in $\mathbb{R}^d$ is cut out by the intersection of their translates. □

3. Lower bounds

While we do not know if the bound $\beta(2,s) \leq s$ in Theorem 2.17 is optimal, in this section we show that at least the logarithmic factor is unavoidable already for the incidence relation between points and dyadic boxes in $\mathbb{R}^2$.

We describe a slightly more general construction first. Fix $d \in \mathbb{N}_{>0}$.

Definition 3.1. Given finite tuples $\bar{p} = (p_1, \ldots, p_n), \bar{q} = (q_1, \ldots, q_n)$ and $\bar{r} = (r_1, \ldots, r_m)$ with $p_i, q_i, r_i \in \mathbb{R}^d$ – say $p_i = (p_{i,1}, \ldots, p_{i,d}), q_i = (q_{i,1}, \ldots, q_{i,d}), r_i = (r_{i,1}, \ldots, r_{i,d})$ – we say that *$\bar{p}$ and $\bar{q}$ have the same order-type over $\bar{r}$* if

$$p_{i,j} \square p_{i',j'} \iff q_{i,j} \square q_{i',j'} \text{ and}$$
$$p_{i,j} \square r_{k,j'} \iff q_{i,j} \square r_{k,j'}$$

for all $\square \in \{<, >, =\}$, $1 \leq i, i' \leq n$, $1 \leq j, j' \leq d$ and $1 \leq k \leq m$.

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In other words, the tuples $(p_{i,j} : 1 \le i \le n, 1 \le j \le d)$ and $(q_{i,j} : 1 \le i \le n, 1 \le j \le d)$ have the same quantifier-free type over the set $\{r_{i,j} : 1 \le i \le m, 1 \le j \le d\}$ in the structure $(\mathbb{R}, <)$.

Remark 3.2. Assume that $P = \{p_1, \dots, p_n\} \subseteq \mathbb{R}^d$ is a finite set of points and B is a finite set of d-dimensional open boxes with axis-parallel sides, with I incidences between P and B.

1. By perturbing P and B slightly, we may assume that for every $1 \le j \le d$, all points in P have pairwise distinct jth coordinates $p_{1,j}, \dots, p_{n,j}$, and none of the points in P belongs to the border of any of the boxes in B, while the incidence graph between P and B remains unchanged.
2. Let $\bar{r}$ be the tuple listing all corners of all boxes in B. If $P' = \{p'_1, \dots, p'_n\} \subseteq \mathbb{R}^d$ is an arbitrary set of points with the same order-type as P over $\bar{r}$, then the incidence graph on $P \times B$ is isomorphic to the incidence graph on $P' \times B$.

We have the following lemma for combining point-box incidence configurations in a higher-dimensional space:

Lemma 3.3. *Given any $d, n_1, n_2, n'_1, n'_2, m, m' \in \mathbb{N}_{>0}$, assume that:*

1. *there exists a set of points $P^{d-1} \subseteq \mathbb{R}^{d-1}$ with $|P^{d-1}| = n_1$ and a set of $(d-1)$-dimensional boxes B^{d-1} with $|B^{d-1}| = n_2$, with m incidences between them and the incidence graph $K_{2,2}$-free; and*
2. *there exists a set of points $P^d \subseteq \mathbb{R}^d$ with $|P^d| = n'_1$ and a set of d-dimensional boxes B^d with $|B^d| = n'_2$, with m' incidences between them and the incidence graph $K_{2,2}$-free.*

Then there exists a set of points $P \subseteq \mathbb{R}^d$ with $|P| = n_1 n'_1$ and a set of d-dimensional boxes B with $|B| = n_1 n'_2 + n'_1 n_2$, so that there are $n_1 m' + m n'_1$ incidences between P and B and their incidence graph is still $K_{2,2}$-free.

Proof. By Remark 3.2(1) we may assume that for every $1 \le j \le d$, all points in P^d have pairwise distinct jth coordinates; for every $1 \le j \le d-1$, all points in P^{d-1} have pairwise distinct jth coordinates; and none of the points is on the border of any of the boxes. Write P^{d-1} as $p_1, \dots, p_{n_1}$. Let $\bar{r}$ be the tuple listing all corners of all boxes in B^{d-1}.

Using this, for each p_i we can choose a very small $(d-1)$-dimensional box β_i with $p_i \in \beta_i$ and such that for any choice of points $p'_i \in \beta_i$, $1 \le i \le n_1$, we have that $(p'_1, \dots, p'_{n_1})$ has the same order-type as $(p_1, \dots, p_{n_1})$ over $\bar{r}$. In particular, every β_i is pairwise disjoint, and the incidence graph between P^{d-1} and B^{d-1} is isomorphic to the incidence graph between $(p'_i, \dots, p'_{n_1})$ and B^{d-1} by Remark 3.2(2).

Contracting and translating while keeping the dth coordinate unchanged, for each $1 \le i \le n_1$ we can find a copy (P^d_i, B^d_i) of the configuration (P^d, B^d) entirely contained in the box $\beta_i \times \mathbb{R}$ – that is,

- all points in P^d_i and boxes in B^d_i are contained in $\beta_i \times \mathbb{R}$;
- the incidence graph on (P^d_i, B^d_i) is isomorphic to the incidence graph on (P^d, B^d); and
- for all i, the dth coordinate of every point in P^d_i is the same as the dth coordinate of the corresponding point in P^d.

Set $P := \bigcup_{1 \le i \le n_1} P^d_i$ and $B' := \bigcup_{1 \le i \le n_1} B^d_i$; then $|P| = n_1 n'_1$, $|B'| = n_1 n'_2$ and there are $n_1 m'$ incidences between P and B'.

Write P^d as $q_1, \dots, q_{n'_1}$ and B^{d-1} as $c_1, \dots, c_{n_2}$. As all of the dth coordinates of the points in P^d are pairwise disjoint, for each $1 \le j \le n'_1$ we can choose a small interval $I_j \subseteq \mathbb{R}$ with $q_{j,d} \in I_j$ and such that all of the intervals I_j, $1 \le j \le n'_1$, are pairwise disjoint. For each $1 \le j \le n'_1$ and $c_l \in B^{d-1}$, we consider the d-dimensional box $c_{j,l} := c_l \times I_j$. Define $B_j := \{c_{j,l} : 1 \le l \le n_2\}$. For each $1 \le i \le n_1$ and $1 \le j \le n'_1$, $(\beta_i \times \mathbb{R}) \cap (\mathbb{R}^{d-1} \times I_j)$ contains exactly one point $q_{i,j}$ (given by the copy of q_j in P^d_i), and the projection $q'_{i,j}$ of $q_{i,j}$ onto the first $d-1$ coordinates is in β_i. Hence the incidence graph between P and B_j is isomorphic to the incidence graph between P^{d-1} and B^{d-1} by the choice of the β_is, and in particular the number of incidences is m.

Finally, define $B := B' \cup \bigcup_{1 \le j \le n'_1} B_j$; then $|B| = n_1 n'_2 + n'_1 n_2$. Note that $c_{j,l} \cap c_{j',l'} = \emptyset$ for $j \ne j'$ and any l, l' – that is, no box in B_j intersects any of the boxes in $B_{j'}$ for $j \ne j'$. It is now not hard to check that the incidence graph between P and B is $K_{2,2}$-free, by construction and the assumptions of $K_{2,2}$-freeness of (P^d, B^d) and (P^{d-1}, B^{d-1}), and that there are $n_1m' + mn_1'$ incidences between P and B. □

Remark 3.4. It follows from the proof that if all the boxes in B^{d-1} and B^d are dyadic (see Definition 4.6), then we can choose the boxes in B to be dyadic as well.

Proposition 3.5. *For any $\ell \in \mathbb{N}$, there exist a set P of ℓ^ℓ points and a set B of ℓ^ℓ dyadic boxes in $\mathbb{R}^2$ such that their incidence graph is $K_{2,2}$-free and the number of incidences is $\ell\ell^\ell$.*

In particular, substituting $n := \ell^\ell$, this shows that the number of incidences grows as $\Omega\left(n\frac{\log n}{\log\log n}\right)$.

Proof. Given d, assume that there exist $K_{2,2}$-free 'point–dyadic box' configurations satisfying Lemma 3.3(1) and (2) for some parameters $d, n_1, n_2, n_1', n_2', m, m'$. Then for any $j \in \mathbb{N}$, we can iterate the lemma j times and find a $K_{2,2}$-free 'point–dyadic box' configuration in $\mathbb{R}^d$ with $n_1^j n_1'$ points, $n_1^j n_2' + jn_1^{j-1}n_1'n_2$ dyadic boxes (Remark 3.4) and $n_1^j m' + jn_1^{j-1}n_1'm$ incidences.

In particular, let $d = 2$ and let ℓ be arbitrary. We can start with $n_1 = \ell, n_2 = 1, m = \ell$ (one dyadic interval containing n_1 points in $\mathbb{R}$) and $n_1' = 1, n_2' = 0, m' = 0$ (one point and zero dyadic boxes in $\mathbb{R}^2$). Taking $j := \ell$, we then find a $K_{2,2}$-free configuration with ℓ^ℓ points, ℓ^ℓ dyadic boxes and $\ell\ell^\ell$ incidences. Hence for $n := k^k$, we have n points, n boxes and $\Omega\left(n\frac{\log n}{\log\log n}\right)$ incidences. □

Remark 3.6. We remark that the construction in Lemma 3.3 cannot produce a $K_{2,2}$-free configuration with more than $O\left(n\frac{\log n}{\log\log n}\right)$ incidences in $\mathbb{R}^d$ for any d.

Indeed, using the 'coordinates' $\left(\log n_1', \frac{n_2'}{n_1'}, \frac{m'}{n_1'}\right)$ instead of (n_1', n_2', m'), where the coordinates correspond to the number of points, boxes and incidences, respectively, the lemma says that if $\left(\log n_1, \frac{n_2}{n_1}, \frac{m}{n_1}\right)$ is attainable in $d-1$ dimensions and $\left(\log n_1', \frac{n_2'}{n_1'}, \frac{m'}{n_1'}\right)$ is attainable in d dimensions, then $\left(\log n_1' + \log n_1, \frac{n_2'}{n_1'} + \frac{n_2}{n_1}, \frac{m'}{n_1'} + \frac{m}{n_1}\right)$ is attainable in d dimensions. Thus, one adds the vector $\left(\frac{n_2}{n_1}, \frac{m}{n_1}\right)$ to $\left(\frac{n_2'}{n_1'}, \frac{m'}{n_1'}\right)$. We want to maximise the second coordinate of this vector while keeping the first coordinate below 1, and the optimal way to do this essentially is to add n_1 times the vector $\left(\frac{1}{n_1}, 1\right)$, which increases $\log n_1'$ by $n_1 \log n_1$ and gives the $\frac{\log n}{\log\log n}$ lower bound.

We thus ask whether in the 'point-box' incidence bound in $\mathbb{R}^d$ the power of $\log n$ has to grow with the dimension d (see Problem 1.3).

4. Dyadic rectangles

In this section we strengthen the bound on the number of incidences with rectangles on the plane with axis-parallel sides given by Corollary 2.38 – that is, $O_k\left(n\log^4 n\right)$ – in the special case of *dyadic* rectangles, using a different argument (which relies on a certain partial order specific to the dyadic case).

4.1. Locally d-linear orders

Throughout this section, let $(P, \leq)$ be a partially ordered set of size at most n_1, and let L be a collection of subsets of P (possibly with repetitions) of size at most n_2. As before, we let $n = n_1 + n_2$.

Definition 4.1. We say that a set $S \subseteq P$ is *d-linear* if it contains no antichains of size greater than d, and $(P, \leq)$ is *locally d-linear* if any interval $[a, b] = \{x \in P : a \leq x \leq b\}$ is d-linear.

Note that d-linearity is preserved under removing points from P.

Definition 4.2. The collection L is said to be a *$K_{k,k}$-free arrangement* if for any $a_1 \neq \cdots \neq a_k \in P$, there are at most $k-1$ sets from L containing all of them simultaneously.

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Observe that if one removes any number of points from P or removes any number of sets from L, one still obtains a $K_{k,k}$-free arrangement. We now state the main theorem of this section:

Theorem 4.3. *Suppose $(P, <)$ is locally d-linear and L is a $K_{k,k}$-free arrangement of d-linear subsets of P. Then*

$$\sum_{\ell \in L} |\ell| = O_{d,k}\left(n \frac{\log(100 + n_1)}{\log \log(100 + n_1)}\right).$$

To prove this theorem, we first need some definitions and a lemma. If $x \in P$, define a *parent* of x to be an element $y \in P$ with $y > x$ and no element between x and y, and similarly define a *child* of x to be an element $z \in P$ with $z < x$ and no element between z and x. We say that z is a *strict t-descendant* of x if there are some elements $z_0 = x > z_1 > \cdots > z_t = z$ such that z_{i+1} is a child of z_i, and that z is a *t-descendant* of x if it is a strict s-descendant for some $0 \le s \le t$.

Lemma 4.4. *Fix $d, k \in \mathbb{N}$. Let L be a $K_{k,k}$-free arrangement of d-linear subsets of P and let $m > 0$. Let P' denote the set of all elements in P which have a $(k-1)$-descendant with more than m children. Then*

$$\sum_{\ell \in L} |\ell| \le \sum_{\ell \in L} |\ell \cap P'| + d(k-1)|L| + (k-1)m^{k-1}(|P| - |P'|).$$

Proof. Let $P'' := P \backslash P'$ denote the set of elements $x \in P$ such that every $(k-1)$-descendant of x has at most m children. Then we can rearrange the desired inequality as

$$\sum_{\ell \in L} |\ell \cap P''| \le d(k-1)|L| + (k-1)m^{k-1}|P''|.$$

The quantity $\sum_{\ell \in L} |\ell \cap P''|$ is counting incidences (x, ℓ) where $\ell \in L$ and $x \in P'' \cap \ell$.

Given $\ell \in L$, call a point $x \in \ell$ low if it has no descending chain of length $k-1$ under it in ℓ. Every ℓ can contain at most $d(k-1)$ low points. Indeed, as ℓ is d-linear, it has at most d minimal elements. Removing them, we obtain a d-linear set $\ell_1 \subseteq \ell$ such that every point in it contains an element under it in ℓ, and ℓ_1 itself has at most d minimal elements. Remove them to obtain a d-linear set $\ell_2 \subseteq \ell_1$ such that each point in it contains a descending chain of length 2 under it in ℓ, and so on.

Hence each $\ell \in L$ contributes at most $d(k-1)$ incidences with its low points, giving a total contribution of at most $d(k-1)|L|$ to the sum. If x is not a low point on ℓ, then there are some $z_1 < \cdots < z_{k-1} < x$ in ℓ, with each one a child of the next one. As L is a $K_{k,k}$-free arrangement, among the sets $\ell \in L$ there are at most $k-1$ containing all these points. By the definition of P'', for each $x \in P''$ there are at most m^{k-1} choices for such tuples $(z_1, \ldots, z_{k-1})$. Hence x is incident to at most $(k-1)m^{k-1}$ sets $\ell \in L$ for which it is not low, and the total number of contributions of incidences in this case is at most $(k-1)m^{k-1}|P''|$, so the claim follows. □

Now we prove Theorem 4.3. Let t be a natural number to be chosen later and $m > 0$ be another parameter to be chosen later. Define the subsets

$$P = P_0 \supset P_1 \supset \cdots \supset P_t$$

of P by defining $P_0 := P$, and for each $i = 0, \ldots, t-1$, defining P_{i+1} to be the set of points in P_i that have a $(k-1)$-descendant with more than m children in $(P_i, <)$. By Lemma 4.4, we have

$$\sum_{\ell \in L} |\ell \cap P_i| \le \sum_{\ell \in L} |\ell \cap P_{i+1}| + d(k-1)|L| + (k-1)m^{k-1}(|P_i| - |P_{i+1}|)$$

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for all $i = 0, \dots, t-1$, and hence on telescoping,

$$\sum_{\ell \in L} |\ell| \leq \sum_{\ell \in L} |\ell \cap P_t| + d(k-1)t|L| + (k-1)m^{k-1}n_1.$$

Claim 4.5. *Let x be a point in P_t. Then it has at least $\frac{m^t}{(kd^k)^{t-1}}$ distinct descendants in P.*

Proof. By definition of P_t there is some $(k-1)$-descendant $x' \in P_{t-1}$ of x which has at least m children in P_{t-1}. Let $S_{t-1} \subseteq P_{t-1}$ denote the set of children of x', so $|S_{t-1}| \geq m$. By reverse induction for $i = t-1, t-2, \dots, 0$, we choose sets $S_i \subseteq P_i$ of descendants of x so that $|S_{i-1}| \geq \frac{|S_i| m}{kd^k}$. Then $|S_0| \geq \frac{m^t}{(kd^k)^{t-1}}$, as wanted.

Let S_i be given. By the definition of P_i and the pigeonhole principle, there is some $0 \leq s \leq k-1$ and $S_i' \subseteq S_i$ such that $\left|S_i'\right| \geq \frac{|S_i|}{k}$ and every $y \in S_i'$ has a strict s-descendant $z_y \in P_{i-1}$ with at least m children in P_{i-1}. Fix a path I_y of length s connecting y to z_y, and for $0 \leq r \leq s$ let z_y^r denote the rth element on the path I_y (so $z_y^0 = y$, $z_y^s = z_y$ and z_y^{r+1} is a child of z_y^r). Define $I^r := \left\{z_y^r : y \in S_i'\right\}$, so $I^0 = S_i'$. Then $\left|I^{r+1}\right| \geq \frac{|I^r|}{d}$ (otherwise there is some element $z \in I^{r+1}$ which has at least $d+1$ different parents in I^r, which would then form an antichain of size $d+1$, contradicting the local d-linearity of P). Hence

$$|I^s| \geq \frac{\left|I^0\right|}{d^s} \geq \frac{\left|S_i'\right|}{d^{k-1}} \geq \frac{|S_i|}{kd^{k-1}}.$$

Now by hypothesis every element in I^s has at least m children in P_{i-1}; denote the set of all the children of the elements in I^s by $S_{i-1} \subseteq P_{i-1}$. Then, again by d-linearity, $|S_{i-1}| \geq \frac{|I^s| m}{d} \geq \frac{|S_i| m}{kd^k}$. □

Thus if we choose m, t such that

$$\left(\frac{m}{kd^k}\right)^t > n_1,$$

then we will get a contradiction, unless P_t is empty. We conclude, for such m and t, that

$$\sum_{\ell \in L} |\ell| \leq d(k-1)t|L| + (k-1)m^{k-1}n_1.$$

If we take $m := \left(\frac{c \log(100+n_1)}{\log\log(100+n_1)}\right)^{\frac{1}{k-1}}$ and t to be the integer part of $\frac{c\log(100+n_1)}{\log\log(100+n_1)}$, and assume that c is sufficiently large relative to k and d, then the claim follows.

4.2. *Reduction for dyadic rectangles*

Definition 4.6. 1. Define a *dyadic interval* to be a half-open interval I of the form $I = [s2^t, (s+1)2^t)$ for integers s, t; we use $|I| = 2^t$ to denote the length of such an interval.
2. Define a *dyadic box* in $\mathbb{R}^d$ (a *dyadic rectangle* when $d = 2$) to be a product $I_1 \times \cdots \times I_d$ of dyadic intervals.

Note that if two dyadic intervals intersect, then one must be contained in the other.

Theorem 4.7. *Fix $k \in \mathbb{N}$. Assume we have a collection P of n_1 points in $\mathbb{R}^2$ and a collection R of n_2 dyadic rectangles in $\mathbb{R}^2$, with the property that the incidence graph contains no $K_{k,k}$, and $n = n_1 + n_2$. Then the number of incidences $(p, I \times J)$ with $p \in P$ and $p \in I \times J \in R$ is at most*

$$O_k\left(n \frac{\log(100+n_1)}{\log\log(100+n_1)}\right).$$

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Proof. Suppose that we have some nested dyadic rectangles $D_1 \supseteq D_2 \supseteq \cdots \supseteq D_k$ in R. As the incidence graph is $K_{k,k}$-free by hypothesis, D_k may contain at most $(k-1)$ points from P. Removing all such rectangles repeatedly, we lose only $(k-1)n_2$ incidences, and thus may assume that any nested sequence in R is of length at most $k-1$. In particular, any rectangle can be repeated at most $k-1$ times in R. Then, possibly increasing the number of incidences by a multiple of $(k-1)$, we may assume that there are no repetitions in R.

We now define a relation $\leq$ on R by declaring $I \times J \leq I' \times J'$ if $I \subseteq I'$ and $J \supseteq J'$. This is a locally $(k-1)$-linear partial order (by the previous paragraph: antisymmetry holds, as there are no repetitions in R, and using the fact that all rectangles are dyadic, any antichain of size k inside an interval would give a nested sequence of rectangles of length k).

For each point p in P, let ℓ_p be a subset of R consisting of all those rectangles in R that contain p; then ℓ_p is a $(k-1)$-linear set (again, any antichain gives a nested sequence of rectangles of the same length). Finally, $p \in R \iff R \in \ell_p$, hence the collection $\left\{\ell_p : p \in P\right\}$ is a $K_{k,k}$-free arrangement and the claim now follows from Theorem 4.3 with $d := k-1$. □

Remark 4.8. For a nondyadic rectangle R, let $0.99R$ denote the rectangle with the same centre as R but whose lengths and heights have been shrunk by a factor of 0.99. Define a *good incidence* to be a pair (p, R) where p is a point lying in $0.99R$, not just in R. Then the dyadic bound in Theorem 4.7 implies that for a family of arbitrary (not necessarily dyadic) rectangles with no $K_{k,k}$s, one still gets the $O\left(\frac{n \log n}{\log \log n}\right)$-type bound for the number of good incidences.

The reason is as follows. First, we can randomly translate and dilate (nonisotropically, with the horizontal and vertical coordinates dilated separately) the configuration of points and rectangles by some translation parameter and a pair of dilation parameters (s, t) for each of the coordinates. While there is no invariant probability measure on the space of dilations, one can, for instance, pick a large number N (much larger than the number of points and rectangles, etc.) and dilate horizontally by a random dilation between $1/N$ and N (using, say, the dt/t Haar measure), making (with positive probability) the horizontal side length close to a power of 2; then a vertical dilation will achieve a similar effect for the vertical side length; and then one can translate by a random amount in $[-N, N]^2$ (chosen uniformly at random), placing the rectangle very close to a dyadic one with positive probability. If R is a rectangle that is randomly dilated and translated in this way, then with probability $> 10^{-10}$, there will be a dyadic rectangle R' stuck between R and $0.99R$. If the original rectangles have no $K_{k,k}$, then neither will these new dyadic rectangles. The expected number of incidences amongst the dyadic rectangles is at least 10^{-10} times the number of good incidences amongst the original rectangles. Hence any incidence bound we get on dyadic rectangles implies the corresponding bound for good incidences for nondyadic rectangles (losing a factor of 10^{10}).

5. A connection to model-theoretic linearity

In this section we obtain a stronger bound in Theorem 2.17 (without the logarithmic factor) under a stronger assumption that the whole semilinear relation X is $K_{k,\ldots,k}$-free (Corollary 5.12). And we show that if this stronger bound does not hold for a given semialgebraic relation, then the field operations can be recovered from this relation (see Corollary 5.14 for the precise statement). These results are deduced in Section 5.2 from a more general model-theoretic theorem proved in Section 5.1.

5.1. Main theorem

We recall some standard model-theoretic notation and definitions, and refer to [15] for a general introduction to model theory and [3] for further details on geometric structures.

Recall that acl denotes the algebraic closure operator – that is, if $\mathcal{M} = (M, \ldots)$ is a first-order structure, $A \subseteq M$ and a is a finite tuple in M, then $a \in \operatorname{acl}(A)$ if it belongs to some finite A-definable subset of $M^{|a|}$ (this generalises the linear span in vector spaces and algebraic closure in fields).

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Throughout this section, we follow the standard model-theoretic notation: depending on the context, writing BC denotes either the union of two subsets B, C of M or the tuple obtained by concatenating the (possibly infinite) tuples B, C of elements of M.

Definition 5.1. A complete first-order theory T in a language $\mathcal{L}$ is *geometric* if for any model $\mathcal{M} = (M, \dots) \models T$ we have the following:

1. The algebraic closure in $\mathcal{M}$ satisfies the *exchange principle*:
   if a, b are singletons in $\mathcal{M}$, $A \subseteq M$ and $b \in \operatorname{acl}(A, a) \setminus \operatorname{acl}(A)$, then $a \in \operatorname{acl}(A, b)$.
2. *T eliminates the $\exists^\infty$ quantifier*:
   for every $\mathcal{L}$-formula $\varphi(x, y)$ with x a single variable and y a tuple of variables, there exists some $k \in \mathbb{N}$ such that for every $b \in M^{|y|}$, if $\varphi(x, b)$ has more than k solutions in M, then it has infinitely many solutions in M.

In models of a geometric theory, the algebraic closure operator acl gives rise to a matroid, and given a a finite tuple in M and $A \subseteq M$, $\dim(a/A)$ is the minimal cardinality of a subtuple a' of a such that $\operatorname{acl}(a \cup A) = \operatorname{acl}(a' \cup A)$ (in an algebraically closed field, this is just the transcendence degree of a over the field generated by A). Finally, given a finite tuple a and sets $C, B \subseteq M$, we write$a \mathop{\smile\!\!\!\!\!\mid}_C B$ to denote that $\dim(a/BC) = \dim(a/C)$.

Remark 5.2. If T is geometric, then it is easy to check that $\mathop{\smile\!\!\!\!\!\mid}$ is an *independence relation* – that is, it satisfies the following properties for all tuples a, a', b, b', d and $C, D \subseteq M$:

- $a \mathop{\smile\!\!\!\!\!\mid}_C b \iff \operatorname{acl}(a, C) \mathop{\smile\!\!\!\!\!\mid}_C \operatorname{acl}(b, C)$.
- (Extension) If $a \mathop{\smile\!\!\!\!\!\mid}_C b$ and d is arbitrary, then there exists some a' such that $a' \mathop{\smile\!\!\!\!\!\mid}_C bd$ and $a' \equiv_{Cb} a$ (which means that a' belongs to exactly the same Cb-definable subsets of $M^{|a|}$ as a).
- (Monotonicity) $aa' \mathop{\smile\!\!\!\!\!\mid}_C bb' \implies a \mathop{\smile\!\!\!\!\!\mid}_C b$.
- (Symmetry) $a \mathop{\smile\!\!\!\!\!\mid}_C b \implies b \mathop{\smile\!\!\!\!\!\mid}_C a$.
- (Transitivity) $a \mathop{\smile\!\!\!\!\!\mid}_D bb' \iff a \mathop{\smile\!\!\!\!\!\mid}_{Db} b'$ and $a \mathop{\smile\!\!\!\!\!\mid}_D b$.
- (Nondegeneracy) If $a \mathop{\smile\!\!\!\!\!\mid}_C b$ and $d \in \operatorname{acl}(a, C) \cap \operatorname{acl}(b, C)$, then $d \in \operatorname{acl}(C)$.

The following property expresses that the matroid defined by the algebraic closure is *linear*, in the sense that the closure operator behaves more like the span in vector spaces, as opposed to algebraic closure in fields:

Definition 5.3 ([3, Definition 2.1].). A geometric theory T is *weakly locally modular* if for any saturated $\mathcal{M} \models T$ and A, B small subsets of $\mathcal{M}$ there exists some small set $C \mathop{\smile\!\!\!\!\!\mid}_\emptyset AB$ such that $A \mathop{\smile\!\!\!\!\!\mid}_{\operatorname{acl}(AC) \cap \operatorname{acl}(BC)} B$.

Recall that a linearly ordered structure $\mathcal{M} = (M, <, \dots)$ is *o-minimal* if every definable subset of M is a finite union of intervals (see, e.g., [23]).

Example 5.4 ([3, Section 3.2].). An o-minimal structure is linear (i.e., any normal interpretable family of plane curves in T has dimension ≤ 1) if and only if it is weakly locally modular.

In particular, every theory of an ordered vector space over an ordered division ring is weakly locally modular (so Theorem 5.6 applies to semilinear relations).

The following is a key model-theoretic lemma:

Lemma 5.5. *Assume that T is geometric and weakly locally modular, and $\mathcal{M} = (M, \dots) \models T$ is $\aleph_1$-saturated. Assume that $E \subseteq M^{d_1} \times \cdots \times M^{d_r}$ is an r-ary relation defined by a formula with parameters in a finite tuple b, and E contains no r-grid $A = \prod_{i \in [r]} A_i$ with each $A_i \subseteq M^{d_i}$ infinite. Then for any $(a_1, \dots, a_r) \in E$ there exists some $i \in [r]$ such that $a_i \in \operatorname{acl}\left(\{a_j : j \in [r] \setminus \{i\}\}, b\right)$.*

Proof. Assume the lemma is untrue; then there exist some $(a_1, \dots, a_r)$ in $\mathcal{M}$ such that $(a_1, \dots, a_r) \in E$, but $a_i \notin \operatorname{acl}(a_{\neq i}, b)$ for every $i \in [r]$, where $a_{\neq i} := \{a_j : j \in [r] \setminus \{i\}\}$.

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By weak local modularity, for each $i \in [r]$ there exists some small set $C_i \subseteq \mathcal{M}$ such that

$$C_i \mathop{\downfree}_{\emptyset} \{a_1, \ldots, a_r\} \cup \{b\} \text{ and } a_i \mathop{\downfree}_{\operatorname{acl}(a_i, C_i) \cap \operatorname{acl}(a_{\neq i}, b, C_i)} a_{\neq i} b.$$

By extension of $\downfree$, we may assume that $C_i \downfree_{\emptyset} a_1, \ldots, a_r, b, C_{<i}$ for all $i \in [r]$. Hence by transitivity, $C \downfree_{\emptyset} a_1, \ldots, a_r, b$, where $C := \bigcup_{i \in [r]} C_i$.

Set $D := \bigcap_{i \in [r]} \operatorname{acl}(a_{\neq i}, b, C)$.

Claim A. *For every $i \in [r]$, $a_i \downfree_D a_{\neq i}$.*

Proof. Fix $i \in [r]$. As $C \downfree_{\emptyset} a_1, \ldots, a_r, b$ and $a_i \downfree_{\operatorname{acl}(a_i, C_i) \cap \operatorname{acl}(a_{\neq i}, b, C_i)} a_{\neq i} b$, by symmetry and transitivity we have

$$a_i \mathop{\downfree}_{\operatorname{acl}(a_i, C_i) \cap \operatorname{acl}(a_{\neq i}, b, C_i)} a_{\neq i} b C.$$

Note that $\operatorname{acl}(a_i, C_i) \subseteq \operatorname{acl}\left(a_{\neq j}, C\right)$ for every $i \neq j \in [r]$, and hence $\operatorname{acl}(a_i, C_i) \cap \operatorname{acl}(a_{\neq i}, b, C_i) \subseteq D$, and clearly $D \subseteq \operatorname{acl}(a_{\neq i}, b, C)$. Hence $a_i \downfree_D a_{\neq i} bC$, and in particular $a_i \downfree_D a_{\neq i}$. □

Claim B. *For every $i \in [r]$, $a_i \notin \operatorname{acl}(D)$.*

Proof. Fix $i \in [r]$. Then $\operatorname{acl}(D) \subseteq \operatorname{acl}(a_{\neq i}, b, C)$ by definition. But as $C \downfree_{a_{\neq i} b} a_i$ by transitivity, if $a_i \in \operatorname{acl}(a_{\neq i}, b, C)$ then we would get $a_i \in \operatorname{acl}(a_{\neq i}, b)$, contradicting the assumption. □

By induction we will choose sequences of tuples $\bar{\alpha}_i = \left(a_i^t\right)_{t \in \mathbb{N}}, i \in [r]$, in $\mathcal{M}$ such that for every $i \in [r]$ we have:

1. $a_i^t \equiv_{D \bar{\alpha}_{<i} a_{>i}} a_i$ for all $t \in \mathbb{N}$;
2. $a_i^t \neq a_i^s$ (as tuples) for all $s \neq t \in \mathbb{N}$;
3. $\bar{\alpha}_i \downfree_D \bar{\alpha}_{<i} a_{>i}$.

Fix $i \in [r]$ and assume that we already chose some sequences $\bar{\alpha}_j$ for $1 \leq j < i$ satisfying (1)–(3).

Claim C. *We have $a_i \downfree_D \bar{\alpha}_{<i} a_{>i}$.*

Proof. If $i = 1$, this claim becomes $a_i \downfree_D a_{\neq i}$, hence holds by Claim (A). So assume $i \geq 2$. We will show by induction that for each $l = 1, \ldots, i-1$ we have

$$\bar{\alpha}_{i-1} \cdots \bar{\alpha}_{i-l} \mathop{\downfree}_{D} \bar{\alpha}_{<i-l} a_{>i-1}.$$

For $l = 1$ this is equivalent to $\bar{\alpha}_{i-1} \downfree_D \bar{\alpha}_{<i-1} a_{>i-1}$, which holds by (3) for $i-1$. So we assume this holds for $l < i-1$ – that is, we have $\bar{\alpha}_{i-1} \cdots \bar{\alpha}_{i-l} \downfree_D \bar{\alpha}_{<i-l} a_{>i-1}$ – and show it for $l+1$. By assumption and transitivity, we have

$$\bar{\alpha}_{i-1} \cdots \bar{\alpha}_{i-l} \mathop{\downfree}_{D \bar{\alpha}_{i-(l+1)}} \bar{\alpha}_{<i-(l+1)} a_{>i-1}.$$

Also, $\bar{\alpha}_{i-(l+1)} \downfree_D \bar{\alpha}_{<i-(l+1)} a_{>i-1}$ by (3) for $i-(l+1) < i$. Then by transitivity again, $\bar{\alpha}_{i-1} \cdots \bar{\alpha}_{i-l} \bar{\alpha}_{i-(l+1)} \downfree_D \bar{\alpha}_{<i-(l+1)} a_{>i-1}$, which concludes the inductive step.

In particular, for $l = i-1$ we get $\bar{\alpha}_{<i} \downfree_D a_{>i-1}$ – that is, $\bar{\alpha}_{<i} \downfree_D a_i a_{>i}$. By transitivity and Claim (A), this implies $\bar{\alpha}_{<i} a_{>i} \downfree_D a_i$, and we conclude by symmetry. □

Using Claim (C) and extension of $\downfree$, we can choose a sequence $\bar{\alpha}_i = \left(a_i^t\right)_{t \in \mathbb{N}}$ so that $a_i^t \equiv_{D \bar{\alpha}_{<i} a_{>i}} a_i$ and $a_i^t \downfree_D \bar{\alpha}_{<i} a_{>i} a_i^{<t}$ for every $t \in \mathbb{N}$. By Claim (B) we have $a_i \notin \operatorname{acl}(D)$, hence $a_i^t \notin \operatorname{acl}(D)$, hence $a_i^t \notin \operatorname{acl}\left(\bar{\alpha}_{<i}, a_{>i}, a_i^{<t}\right)$, so in particular all the tuples $\left(a_i^t\right)_{t \in \mathbb{N}}$ are pairwise-distinct and $\bar{\alpha}_i$ satisfies (1)

and (2). By symmetry and transitivity of $\perp$, we get $\bar{\alpha}_i \perp_D \bar{\alpha}_{<i} a_{>i}$. This concludes the inductive step in the construction of the sequences.

Finally, as (1) holds for all $i \in [r]$ and b is contained in D, it follows that $\left(a_1^{t_1}, \ldots, a_r^{t_r}\right) \equiv_b (a_1, \ldots, a_r)$, and hence $\left(a_1^{t_1}, \ldots, a_r^{t_r}\right) \in E$ for every $(t_1, \ldots, t_r) \in \mathbb{N}^r$. By (1), each of the sets $\left\{a_i^t : t \in \mathbb{N}\right\}, i \in [r]$, is infinite – contradicting the assumption on E. This concludes the proof of the lemma. □

Theorem 5.6. *Assume that T is a geometric, weakly locally modular theory, and $\mathcal{M} \models T$. Assume that $r \in \mathbb{N}_{\geq 2}$ and $\varphi(\bar{x}_1, \ldots, \bar{x}_r, \bar{y})$ is an $\mathcal{L}$-formula without parameters, with $|\bar{x}_i| = d_i, |\bar{y}| = e$. Then there exists some $\alpha = \alpha(\varphi) \in \mathbb{R}_{>0}$ satisfying the following:*

Given $b \in M^e$, consider the r-ary relation

$$E_b := \left\{(a_1, \ldots, a_r) \in M^{d_1} \times \cdots \times M^{d_r} : \mathcal{M} \models \varphi(a_1, \ldots, a_r, b)\right\}.$$

Then for every $b \in M^e$, exactly one of the following two cases must occur:

1. *E_b is not $K_{k,\ldots,k}$-free for any $k \in \mathbb{N}$, or*
2. *for any finite r-grid $B \subseteq \prod_{i \in [r]} M^{d_i}$, we have*

$$|E_b \cap B| \leq \alpha \delta_{r-1}^r(B).$$

Proof. Assume that $\mathcal{N} = (N, \ldots)$ is an elementary extension of $\mathcal{M}$ and $b \in M^e$. Then for a fixed $k \in \mathbb{N}$,

$$E_b = \left\{(a_1, \ldots, a_r) \in M^{d_1} \times \cdots \times M^{d_r} : \mathcal{M} \models \varphi(a_1, \ldots, a_r, b)\right\}$$

is $K_{k,\ldots,k}$-free if and only if

$$E'_b = \left\{(a_1, \ldots, a_r) \in N^{d_1} \times \cdots \times N^{d_r} : \mathcal{N} \models \varphi(a_1, \ldots, a_r, b)\right\}$$

is $K_{k,\ldots,k}$-free, as this can be expressed by a first-order formula $\psi(y)$ and $\mathcal{M} \models \psi(b) \iff \mathcal{N} \models \psi(b)$. Similarly, for a fixed $\alpha \in \mathbb{R}$, $|E_b \cap B| \leq \alpha \delta_{r-1}^r(B)$ for every finite r-grid $B \subseteq \prod_{i \in [r]} M^{d_i}$ if and only if $\left|E'_b \cap B\right| \leq \alpha \delta_{r-1}^r(B)$ for every finite r-grid $B \subseteq \prod_{i \in [r]} N^{d_i}$ (as for all fixed sizes of $B_1, \ldots, B_r$, this condition can be expressed by a first-order formula). Hence, passing to an elementary extension, we may assume that $\mathcal{M}$ is $\aleph_1$-saturated.

As T eliminates $\exists^\infty$, there exists some $m = m(\varphi) \in \mathbb{N}$ such that for any $i \in [r]$, $b \in M^e$ and tuple $\bar{a} := \left(a_j \in M^{d_j} : j \in [r] \setminus \{i\}\right)$, the fibre

$$E^i_{\bar{a};b} := \left\{a^* \in M^{d_i} : \mathcal{M} \models \varphi\left(a_1, \ldots, a_{i-1}, a^*, a_{i+1}, \ldots, a_r; b\right)\right\}$$

is finite if and only if it has size $\leq m$.

Given an arbitrary $b \in M^e$ such that E_b is $K_{k,\ldots,k}$-free, Lemma 5.5 and compactness imply that for every tuple $(a_1, \ldots, a_r) \in E_b$, there exists some $i \in [r]$ such that the fibre $E^i_{\bar{a};b}$ is finite, hence $\left|E^i_{\bar{a};b}\right| \leq m$. This easily implies that for any finite r-grid $B \subseteq \prod_{i \in [r]} M^{d_i}$, we have $|E_b \cap B| \leq m\delta_{r-1}^r(B)$. □

Remark 5.7. In the binary case, a similar observation was made by Evans in the context of certain *stable* theories [10, Proposition 3.1].

Restricting to *distal* structures, we can relax the assumption ' E_b is $K_{k,\ldots,k}$-free for some k' to ' E_b does not contain a direct product of infinite sets' in Theorem 5.6 (we refer to, e.g., the introduction in [6] or [4] for a general discussion of model-theoretic distality and its connections to combinatorics).

Corollary 5.8. *Assume that T is a distal, geometric, weakly locally modular theory, $\mathcal{M} \models T$, $r \in \mathbb{N}_{\geq 2}$ and $\varphi(\bar{x}_1, \ldots, \bar{x}_r, \bar{y})$ is an $\mathcal{L}$-formula without parameters, with $|\bar{x}_i| = d_i, |\bar{y}| = e$. Then there exists some $\alpha = \alpha(\varphi) \in \mathbb{R}_{>0}$ satisfying the following:*

Assume that $b \in M^e$ and the r-ary relation E_b does not contain an r-grid $A = \prod_{i \in [r]} A_i$ with each $A_i \subseteq M^{d_i}$ infinite. Then $|E_b \cap B| \leq \alpha \delta^r_{r-1}(B)$ for any finite r-grid B.

Proof. By [4, Theorem 5.12], if $\mathcal{M}$ is a distal structure with elimination of $\exists^\infty$, then there exists some $k = k(\varphi) \in \mathbb{N}$ such that for every $b \in M^e$, E_b is not $K_{k,\ldots,k}$-free if and only if $\prod_{i \in [r]} A_i \subseteq E_b$ for some infinite $A_i \subseteq M^{d_i}$. The conclusion now follows by Theorem 5.6. □

Remark 5.9. Weaker bounds for noncartesian relations definable in arbitrary distal theories are established in [7, 5].

Now we show that in the o-minimal case, this result actually characterises weak local modularity. By the trichotomy theorem in o-minimal structures [18], we have the following equivalence:

Fact 5.10. Let $\mathcal{M}$ be an o-minimal ($\aleph_1$-)saturated structure. The following are equivalent:

- ∘ $\mathcal{M}$ is not linear (see Example 5.4).
- ∘ $\mathcal{M}$ is not weakly locally modular.
- ∘ There exists a real closed field definable in $\mathcal{M}$.

Corollary 5.11. *Let $\mathcal{M}$ be an o-minimal structure. The following are equivalent:*

1. *$\mathcal{M}$ is weakly locally modular.*
2. *Corollary 5.8 holds in $\mathcal{M}$.*
3. *For every $d_1, d_2 \in \mathbb{N}$ and every definable (with parameters) $X \subseteq M^{d_1} \times M^{d_2}$, if X is $K_{k,k}$-free for some $k \in \mathbb{N}$, then there exist some $\beta < \frac{4}{3}$ and α such that for any n and $B_i \subseteq M^{d_i}$ with $|B_i| = n$, we have*

$$|X \cap B_1 \times B_2| \leq \alpha n^\beta.$$

4. *There is no infinite field definable in $\mathcal{M}$.*

Proof. (1) ⇒ (2) by Corollary 5.8, and (2) ⇒ (3) is obvious.

For (3) ⇒ (4), assume that $\mathcal{R}$ is an infinite field definable in $\mathcal{M}$, $\mathrm{char}(\mathcal{R}) = 0$ by o-minimality. Then the point-line incidence relation on $\mathcal{R}^2$ corresponds to a $K_{2,2}$-free definable relation $E \subseteq \mathcal{M}^d \times \mathcal{M}^d$ for some d. By the standard lower bound for Szemerédi–Trotter, the number of incidences satisfies $\Omega\left(n^{4/3}\right)$, hence E cannot satisfy (3).

For (4) ⇒ (1), if $\mathcal{M}$ is not weakly locally modular, then by Fact 5.10 a real closed field $\mathcal{R}$ is definable in $\mathcal{M}$. □

5.2. *Applications to semialgebraic relations*

Corollary 5.12. *Assume that $X \subseteq \mathbb{R}^d = \prod_{i \in [r]} \mathbb{R}^{d_i}$ is semilinear and X does not contain a direct product of r infinite sets (e.g., if X is $K_{k,\ldots,k}$-free for some k). Then there exists some $\alpha = \alpha(X)$ such that for any r-hypergraph H of the form $\left(V_1, \ldots, V_r; X \cap \prod_{i \in [r]} V_i\right)$ for some finite $V_i \subseteq \mathbb{R}^{d_i}$, with $\sum_{i=1}^r |V_i| = n$, we have $|E| \leq \alpha n^{r-1}$.*

Proof. As every o-minimal structure is distal and every semilinear relation is definable in an ordered vector space over $\mathbb{R}$ which is o-minimal and locally modular (Example 5.4), the result follows by Corollary 5.8. □

We recall the following special case of the trichotomy theorem in o-minimal structures restricted to semialgebraic relations:

Fact 5.13. ([16, Theorem 1.3]). Let $X \subseteq \mathbb{R}^n$ be a semialgebraic but not semilinear set. Then $\times \restriction_{[0,1]^2}$ (i.e., the graph of multiplication restricted to the unit box) is definable in the first-order structure $(\mathbb{R}, <, +, X)$.

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Using this fact, we have the following more explicit variant of Corollary 5.11 in the semialgebraic case:

Corollary 5.14. *Let $X \subseteq \mathbb{R}^d$ be a semialgebraic set, and consider the first-order structure $\mathcal{M} = (\mathbb{R}, <, +, X)$. Then the following are equivalent:*

1. *For any $r \in \mathbb{N}$ and any r-ary relation $Y \subseteq \prod_{i\in[r]} \mathbb{R}^{d_i}$ not containing an r-grid $A = \prod_{i\in[r]} A_i$ with each $A_i \subseteq \mathbb{R}^{d_i}$ infinite, there exists some $\alpha \in \mathbb{R}$ such that $|Y \cap B| \leq \alpha\delta^r_{r-1}(B)$ for every finite r-grid B.*
2. *For every $d_1, d_2 \in \mathbb{N}$ and $Y \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ definable (with parameters) in $\mathcal{M}$, if Y is $K_{k,k}$-free for some $k \in \mathbb{N}$, then there exist some $\beta < \frac{4}{3}$ and α such that for any n and $B_i \subseteq \mathbb{R}^{d_i}$ with $|B_i| = n$, we have*
$$|X \cap B_1 \times B_2| \leq \alpha n^{\beta}.$$
3. *$\times \restriction_{[0,1]^2}$ is not definable in $\mathcal{M}$.*

Proof. (1) $\Rightarrow$ (2) is obvious.

For (2) $\Rightarrow$ (3), using $\times \restriction_{[0,1]^2}$ the $K_{2,2}$-free point-line incidence relation in $\mathbb{R}^2$ is definable restricted to $[0,1]^2$, and the standard configurations witnessing the lower bound in Szemerédi–Trotter can be scaled down to the unit box.

For (3) $\Rightarrow$ (1), assume that (1) does not hold in $(\mathbb{R}, <, +, X)$. Then necessarily some Y definable in $(\mathbb{R}, <, +, X)$ is not semilinear, by Corollary 5.12. By Fact 5.13, if Y is not semilinear, then $\times \restriction_{[0,1]^2}$ is definable in the structure $(\mathbb{R}, <, +, Y)$, hence in $(\mathbb{R}, <, +, X)$. □

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