ASYMPTOTIC FORMULAE FOR SOLUTIONS OF LINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS WITH A LARGE PARAMETER

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ABSTRACT

Uniform asymptotic formulae are obtained for solutions of the differential equation

$$\frac{d^2 w}{dz^2} = u^{2p} f(u, \theta, z) w,$$

for large positive values of the parameter u. Here p is a positive integer, θ an arbitrary parameter and z a complex variable whose domain of variation may be unbounded. The function $f(u, \theta, z)$ is a regular function of z having an asymptotic expansion of the form

$$f(u, \theta, z) \sim 1 + \sum_{s=1}^{\infty} \frac{f_s(\theta, z)}{u^s}$$

for large u.

The results obtained include and extend those of earlier writers which are applicable to this equation.

Introductory Note

F. W. J. OLVER[1]

Roger Thorne was killed in a road accident on May 19th, 1959, and this paper is based on manuscripts he left behind. I have redrafted some of his proofs to make them more clear and made certain other changes of detail; in particular, I have added some new material at the beginning of § 7. The essential results, however, are due to Thorne alone.

The results given in the paper apply to the first of three cases A, B and D defined in § 1 below, but Thorne clearly intended to extend the work to the other two cases. Some material by him on case B is in fact available, and I hope to incorporate this in a paper in due course.

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1. Introduction and Summary

This investigation is an attempt to unify work which has been carried out in recent years by several writers, particularly T. M. Cherry, A. Erdélyi, R. E. Langer and F. W. J. Olver, on the solution of homogeneous linear second-order differential equations of the form

$$\frac{d^2 w}{dz^2} = u^{2p} f(u, z) w, \tag{1.1}$$

for large values of the real or complex parameter u. Here p is a positive integer, and the function $f(u, z)$ can be expanded for large $|u|$ in descending powers of u:

$$f(u, z) \sim f_0(z) + u^{-1} f_1(z) + u^{-2} f_2(z) + \cdots . \tag{1.2}$$

The object is to determine asymptotic representations of solutions of (1.1) which hold uniformly with respect to the independent variable z throughout a domain in the complex plane or an interval of the real axis.

The points in the z-plane which determine the asymptotic character of the solutions are called the *transition points* of (1.1). Clearly these include the singularities, regular and irregular, of $f(u, z)$ regarded as a function of z. The other kind of transition points are the zeros of $f_0(z)$, for near these points the leading term in (1.2) fails to dominate the behaviour of $f(u, z)$. Zeros of $f_0(z)$ are also called *turning points*.

In the simplest case, the z-domain (or interval) is free from transition points and asymptotic solutions can be derived in terms of exponential functions. In other cases it may be necessary to use higher transcendental functions. In introducing higher functions, however, it is important to realize that the most fundamental advances use only transcendental functions of a *single* variable, for the asymptotic expressions obtained reduce the computation of the solution w, which depends on the *two* variables u and z, to the evaluation of a few functions of a single variable.

Adopting this criterion Olver [9] showed that when $p = 1$ and all the $f_s(z)$ vanish identically save for $f_0(z)$ and $f_2(z)$, the three most important cases of (1.1) are those for which

$$f_0(z) = 1, \qquad f_0(z) = z, \qquad f_0(z) = 1/z, \tag{1.3}$$

and $f(u, z)$ is a regular function of z, except that it may have a regular singularity at the origin in the third case. All cases of (1.1) whose solutions can be represented asymptotically in terms of functions of a single variable can be

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expressed in one of these forms with the aid of transformations due to Liouville [6] and Langer [3, 4].

Following Olver, we designate these cases A, B and D respectively, and we extend this classification to include all values of the positive integer p and the functions $f_s(z)$ $(s \geqq 1)$. The cases correspond to problems in which (A) the z-domain **D**, say, is free from transition points, (B) **D** includes a simple turning point, (D) **D** includes a regular singularity, a simple pole of $f_0(z)$ which may also be a double pole of $f_s(z)$ for $s \geqq 1$. The basic functions in terms of which the asymptotic solutions are constructed are respectively exponential functions, Airy functions and Bessel functions of fixed order.

A further case considered by Olver, case C, occurs when **D** contains a double pole of $f_0(z)$. By projecting the pole to infinity, however, this problem can be reduced to one for which $f_0(z) = 1$ and so included in case A (see [7]). Similarly [11], the case in which **D** contains a simple turning point and a double pole may be included in case B. An example of a case in which the asymptotic solutions involve functions of more than one variable is provided by Thorne [12]: "case E". Here **D** contains a double pole of $f_0(z)$ and two simple turning points symmetrically placed on either side of the branch cut for the solutions which emanates from the pole. The expansions are constructed in terms of Bessel functions of variable order.

Using this classification, we now describe briefly the main features of the works of the authors mentioned in the opening paragraph.

The relevant paper by Langer is [5] and it applies to cases A and B with $p = 1$. Algorithms are given for determining formal series solutions of the form

$$(1.4)\qquad w_j(u, z) = P_j(u, z) \sum_{s=0}^{\infty} \frac{A_s(z)}{u^s} + \frac{1}{u^2} \frac{\partial P_j}{\partial z} \sum_{s=0}^{\infty} \frac{B_s(z)}{u^s} \qquad (j = 1, 2),$$

in which $P_j(u, z)$ is an exponential function in case A and a Bessel function of order one-third (equivalently, an Airy function) in case B. The coefficients $A_s(z)$ and $B_s(z)$ are given by recurrence relations. Langer shows that when z lies in a bounded real interval, solutions $w_{m,j}(u, z)$ of (1.1) exist such that

$$(1.5)\qquad w_{m,j}(u, z) = P_j(u, z) \sum_{s=0}^{m} \left\{\frac{A_s(z)}{u^s} + O\left(\frac{1}{u^{m+1}}\right)\right\} + \frac{1}{u^2} \frac{\partial P_j}{\partial z} \left\{\sum_{s=0}^{m} \frac{B_s(z)}{u^s} + O\left(\frac{1}{u^{m+1}}\right)\right\}$$

for large complex values of u, where m is an arbitrary positive integer and the O's are uniform with respect to z.

The solutions $w_{m,j}(u, z)$ depend on the integer m, accordingly (1.5) is not an asymptotic expansion of a function in the usual sense. Results of this kind we shall call *asymptotic formulae*.

Erdélyi's contribution to the theory ([2], §§ 4.2, 4.3) was to extend Lan-

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ger's form of result for case A to all values of the positive integer p, again for real bounded[2] values of z.

Cherry [1] considers case B with the conditions $p = 1$, $f_s(z) = 0$ ($s = 1, 3, 5, \cdots$). He establishes uniform asymptotic formulae of the form

$$(1.6) \qquad w_{m,j}(u, z) = \left(\frac{\partial \Phi_m}{\partial z}\right)^{-\frac{1}{2}} P_j\{u^{\frac{2}{3}} \Phi_m(u, z)\}\{1 + O(u^{-2m-1})\} \quad (j = 1, 2, 3),$$

for large complex u, and z in a complex star-domain, the P_j being Airy functions. Here $\Phi_m(u, z)$ is a regular function of z depending on the arbitrary integer m, which can be expanded in a convergent series

$$(1.7) \qquad \Phi_m(u, z) = z + \sum_{s=1}^{m} \frac{\phi_s(z)}{u^{2s}} + \sum_{s=m+1}^{\infty} \frac{\phi_{s,m}(z)}{u^{2s}},$$

the coefficients in which are given by recurrence relations. Equation (1.6) holds in unbounded sectors of the z-plane in which the condition [3]

$$(1.8) \qquad f(u, z) = \alpha(u)z + \beta(u) + O(|z|^{-1})$$

is fulfilled, $\alpha(u)$ and $\beta(u)$ being functions independent of z, which can be expanded in convergent series of the form

$$(1.9) \qquad \alpha(u) = u^2 + \sum_{s=0}^{\infty} \alpha_s u^{-2s}, \qquad \beta(u) = \sum_{s=0}^{\infty} \beta_s u^{-2s}.$$

Cherry further shows that when the z-domain is bounded, the asymptotic formulae can be applied to prove the existence of solutions independent of m having asymptotic *expansions* of the form

$$(1.10) \qquad w_j(u, z) \sim P_j(u^{\frac{2}{3}} z) \sum_{s=0}^{\infty} \frac{A_s(z)}{u^s} + \frac{P_j'(u^{\frac{2}{3}} z)}{u^{\frac{4}{3}}} \sum_{s=0}^{\infty} \frac{B_s(z)}{u^s}.$$

Olver [10] considers the three cases A, B and D, each with the conditions $p = 1$, $f_1(z) = 0$. For large complex u and complex z asymptotic expansions of the form (1.10) are established directly for case B, with similar results for cases A and D. The z-regions are unbounded where the condition

$$(1.11) \qquad f_s(z) = O(|z|^{-\alpha}) \quad (s \geqq 2)$$

is fulfilled, α being a constant such that

$$(1.12) \quad \alpha > 1 \quad \text{(case A)}, \qquad \alpha > \tfrac{1}{2} \quad \text{(case B)}, \qquad \alpha > \tfrac{3}{2} \quad \text{(case D)}.$$

The condition (1.11) is stronger than Cherry's condition (1.8) above in case B

[2] A further extension, to infinite intervals, was made by the same author in Technical Report No. 6, NR 043-121 (1955) of the California Institute of Technology.

[3] In an unpublished paper, H. F. Bohnenblust summarizing research undertaken at the California Institute of Technology has shown that (1.8) may be replaced by $f(u, z) = O(|z|)$ as $|z| \to \infty$.

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for the existence of asymptotic formulae, but less restrictive than Cherry's own requirement (z bounded) for the existence of asymptotic expansions.

Another generalization of Olver is to extend the z-regions of validity by permitting them to depend on $\arg u$.

In brief, the main comparisons between the works of the four writers are as follows. Langer's and Erdélyi's results concern only real bounded values of the independent variable; those of Cherry and Olver apply to the complex plane, and the last-named writer allows much freedom in the definition of the regions of validity. Langer and Erdélyi establish asymptotic formulae, Olver asymptotic expansions, whilst Cherry investigates both of these forms. Olver is the only writer to establish asymptotic expansions in unbounded z-regions, but Cherry's asymptotic formulae hold in unbounded regions and his conditions admitting this result are not included in the conditions of Olver. Langer, Cherry and Olver confine their investigations to the case $p = 1$; the last two writers also require $f_1(z)$ to vanish. Erdélyi has provided the only extension so far to all values of the positive integer p; his results apply to case A with z real and bounded.

In this paper we develop for case A a theory which is more general than hitherto available, and includes all the results of the previous authors in so far as they are relevant to this case. It is hoped that a similar theory may be developable for cases B and D.

Using the notation of Olver, we take as our standard form of differential equation

$$\frac{d^2 w}{dz^2} = u^{2p} f(u, \theta, z) w, \tag{1.13}$$

in which p is a fixed positive integer, z a complex variable, θ an arbitrary parameter and u a large parameter. The function $f(u, \theta, z)$ is a regular function of z and has an asymptotic expansion of the form

$$f(u, \theta, z) \sim 1 + u^{-1} f_1(\theta, z) + u^{-2} f_2(\theta, z) + \cdots \tag{1.14}$$

for large u, uniformly valid with respect to θ and z. We consider only positive values of u, since the phase of complex u can always be incorporated in the parameter θ (see [10], § 1).

The principal result we shall obtain is that solutions $w_j(u, \theta, z)$ $(j = 1, 2)$ of (1.13) exist such that

$$w_j(u, \theta, z) = \left(\frac{\partial \Phi}{\partial z}\right)^{-\frac{1}{2}} \exp\{(-)^{j-1} u^p \Phi(u, \theta, z)\}\{1 + O(u^{-p-m-1})\} \tag{1.15}$$

as $u \to +\infty$, uniformly with respect to θ and z, where $\Phi(u, \theta, z)$ is a regular function of z having an asymptotic expansion of the form

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$$\Phi(u, \theta, z) \sim z + \sum_{s=1}^{\infty} \frac{\phi_s(\theta, z)}{u^s}, \tag{1.16}$$

the coefficients $\phi_s(\theta, z)$ being given by recurrence relations. Here m is an arbitrary integer, but Φ and the w_j are independent of m. Even so, we regard (1.15) as an asymptotic formula rather than an asymptotic expansion, because the remainder term associated with (1.16) occurs exponentially in (1.15). This point is more fully discussed at the beginning of § 7 below.

The expression (1.15) is of Cherry's type (1.6) above and is valid in z-domains of the kind formulated by Olver ([10], § 2). They extend to infinity in regions where the condition

$$f_s(\theta, z) = O(|z|^{-\alpha}) \quad (\alpha > 0, \ s \geqq 1), \tag{1.17}$$

together with similar conditions on the coefficients in the corresponding expansions of $\partial f/\partial z$ and $\partial^2 f/\partial z^2$, are satisfied. By imposing the additional restriction

$$\alpha > 1, \tag{1.18}$$

we find that it is possible to expand the right of (1.15) in descending powers of u to yield an asymptotic expansion of the $w_j(u, \theta, z)$ of Olver's form. The condition (1.17) corresponds to Cherry's condition (1.8) above in case B with $p = 1$, and (1.18) is exactly Olver's condition (1.11) above in case A with $p = 1$. This expanded form of our result in fact reduces to Olver's theorem A ([10], § 2) in the circumstances considered by him.

The paper is arranged as follows. In § 2 we give some preliminary lemmas. In § 3 the precise conditions adopted are stated, an algorithm is developed for constructing formal solutions of the differential equation in descending powers of u, and the main result given in existence form as theorem 1. The proof of theorem 1 follows in §§ 4—6; in § 4 we investigate by methods of Olver properties of the coefficients $\phi_s(\theta, z)$ appearing in (1.16); in § 5 we establish the existence of the function Φ having the expansion (1.16), and investigate some of its other properties; in § 6 the proof of theorem 1 is completed by solving a transformed form of the differential equation by successive approximation. The final section, § 7, concerns the expansion of (1.15) in asymptotic series, and the result obtained is stated in the form of a second existence theorem.

2. Preliminary Transformations

LEMMA 1. (i) *With the substitution*

$$y(z) = w(z) \exp \{-\tfrac{1}{2} \int p(z)\, dz\}, \tag{2.1}$$

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the differential equation

$$\frac{d^2 y}{dz^2} + p(z)\frac{dy}{dz} + q(z)y = 0 \tag{2.2}$$

becomes

$$\frac{d^2 w}{dz^2} + \left[q(z) - \frac{1}{4}\{p(z)\}^2 - \frac{1}{2}\frac{d}{dz}p(z)\right] w = 0. \tag{2.3}$$

(ii) *With the transformations* $\xi = \xi(z)$, $W = \dot{z}^{-\frac{1}{2}} w$ *where* $\dot{z} \equiv dz/d\xi$, *the differential equation* (1.13) *becomes*

$$\frac{d^2 W}{d\xi^2} = [u^{2p} f(u, \theta, z)\dot{z}^2 - \tfrac{1}{2}\{z, \xi\}]W, \tag{2.4}$$

where

$$\{z, \xi\} = \frac{2\dot{z}\,\dddot{z} - 3\ddot{z}^2}{2\dot{z}^2}. \tag{2.5}$$

The proof of this lemma is elementary. The expression $\{z, \xi\}$ is known as the *Schwarzian derivative* of z with respect to ξ.

Some properties of the Schwarzian are as follows.

$$\frac{d}{d\xi}\{z, \xi\} = \frac{\dot{z}^2 z^{iv} - 4\dot{z}\,\ddot{z}\,\dddot{z} + 3\ddot{z}^3}{\dot{z}^3}, \tag{2.6}$$

$$\frac{d^2}{d\xi^2}\{z, \xi\} = \frac{\dot{z}^3 z^{v} - 5\dot{z}^2\ddot{z} z^{iv} + 17\dot{z}\ddot{z}^2\dddot{z} - 4\dot{z}^2\dddot{z}^2 - 9\ddot{z}^4}{\dot{z}^4}, \tag{2.7}$$

$$\left(\frac{d\xi}{dz}\right)^2 \{z, \xi\} = -\{\xi, z\}, \tag{2.8}$$

$$\{z_1, \xi\} - \{z_2, \xi\} = \left(\frac{dz_2}{d\xi}\right)^2 \{z_1, z_2\}. \tag{2.9}$$

Equation (2.8) may be proved by considering the transformation inverse to that stated in part (ii) of the lemma, and equation (2.9) by considering the effect of two successive transformations.

3. Theorem 1: Statement and Conditions

We consider the differential equation

$$\frac{d^2 w}{dz^2} = u^{2p} f(u, \theta, z)w; \qquad f(u, \theta, z) = 1 + f^*(u, \theta, z), \tag{3.1}$$

where p is a positive integer, θ is a parameter which may take any one of a set of values $\mathbf{\Theta}$, real or complex, u is a large positive parameter and z is a complex variable lying in an open simply-connected domain $\mathbf{D}(\theta)$ which

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may be unbounded. The function $f(u, \theta, z)$ is, for each u and θ, a regular function of z in $\mathbf{D}(\theta)$ such that for $u \geqq u_0$ (> 0) there exists for each $m = 1, 2, \cdots$ an inequality

$$\left| f(u, \theta, z) - \sum_{s=0}^{m-1} \frac{f_s(\theta, z)}{u^s} \right| < \frac{k_m u^{-m}}{1 + |z|^\alpha}, \tag{3.2}$$

for all z in $\mathbf{D}(\theta)$, $\theta \in \mathbf{\Theta}$. Here $f_0(\theta, z) = 1$, and for $s \geqq 1$ the $f_s(\theta, z)$ are regular functions of z in $\mathbf{D}(\theta)$ which are independent of u except for the possible dependence of θ on u. The quantity α is a positive constant independent of all parameters and variables. The constants k_m are independent of u, θ and z but may depend on m; the symbol k_m will be used generically to denote such a constant and k will be used similarly to denote a constant independent of u, θ, z and m. The inequalities (3.2) show that an asymptotic expansion of the form (1.14) holds for $f(u, \theta, z)$, the remainder on truncating at the mth term being of the form $(1 + |z|^\alpha)^{-1} O(u^{-m})$.

We suppose further that for each $m \geqq 0$ and z in $\mathbf{D}(\theta)$

$$\left| f'(u, \theta, z) - \sum_{s=0}^{m-1} \frac{f_s'(\theta, z)}{u^s} \right| < \frac{k_m u^{-m}}{1 + |z|^{\frac{1}{2}+\beta}}, \tag{3.3}$$

$$\left| f''(u, \theta, z) - \sum_{s=0}^{m-1} \frac{f_s''(\theta, z)}{u^s} \right| < \frac{k_m u^{-m}}{1 + |z|^{1+\gamma}}, \tag{3.4}$$

where here and elsewhere primes denote differentiation with respect to z, and β and γ are positive constants independent of u, θ, z and m. We note that if, for example, $\mathbf{D}(\theta)$ consists of an infinite sector $\lambda_1 < \arg z < \lambda_2$, then in the subsector $\lambda_1 + \varepsilon < \arg z < \lambda_2 - \varepsilon$ the conditions (3.3) and (3.4) are satisfied with $\beta = \frac{1}{2} + \alpha$ and $\gamma = 1 + \alpha$, ε being an arbitrary positive number. This result follows by application of Cauchy's integral formulae for the derivatives of an analytic function.

Olver ([10], § 2) obtains formal solutions of (3.1) for the case $p = 1$, $f_1(\theta, z) \equiv 0$, by substituting in (3.1) series of the form

$$e^{uz} \sum_{s=0}^{\infty} \frac{A_s(\theta, z)}{u^s} \text{ and } e^{-uz} \sum_{s=0}^{\infty} \frac{(-)^s A_s^*(\theta, z)}{u^s} \tag{3.5}$$

(compare (1.4)), and then deriving recurrence relations for the coefficients $A_s(\theta, z)$ and $A_s^*(\theta, z)$. For the case $p > 1$ the formal series corresponding to the first of (3.5) has the form

$$\exp \{u^p \sum_{s=0}^{p} \phi_s(\theta, z) u^{-s}\} \sum_{s=0}^{\infty} \frac{A_s(\theta, z)}{u^s}$$

(see [2], § 4.2). We shall not attempt to establish directly the asymptotic nature of this series. We first prove in §§ 4—6 certain asymptotic formulae and then show in § 7 that with the extra condition $\alpha > 1$ these formulae can be expanded in the above form.

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The asymptotic formulae are obtained by comparing (3.1) with a differential equation satisfied by the exponential function, namely

$$(3.6)\qquad \frac{d^2}{d\phi^2}Y(u,\theta,\phi)=u^{2p}Y(u,\theta,\phi).$$

Applying the transformations

$$(3.7)\quad \phi\equiv\phi(u,\theta,z)=z+\sum_{s=1}^{\infty}\phi_s(\theta,z)u^{-s},\qquad y(u,\theta,z)=\left(\frac{d\phi}{dz}\right)^{-\frac{1}{2}}Y(u,\theta,\phi),$$

to (3.6), we see from lemma 1 of § 2 that $y(u,\theta,z)$ satisfies the equation

$$(3.8)\qquad \frac{d^2}{dz^2}y(z)=[u^{2p}\{\phi'(z)\}^2-\tfrac{1}{2}\{\phi(z),z\}]y(z),$$

where $\phi'(z)=d\phi(u,\theta,z)/dz$, the arguments u and θ being suppressed for convenience. Equations (3.1) and (3.8) are identical if

$$(3.9)\qquad u^{2p}\{\phi'(z)\}^2-\frac{1}{2}\frac{\phi'''(z)}{\phi'(z)}+\frac{3}{4}\left\{\frac{\phi''(z)}{\phi'(z)}\right\}^2-u^{2p}f(z)=0.$$

If it were possible to determine $\phi(z)$ from this equation, w would be expressible in terms of the known function $\{\phi'(z)\}^{-\frac{1}{2}}Y\{u,\theta,\phi(z)\}$.

In general we cannot solve (3.9) explicitly. Instead, we substitute the series (3.7), and formally equate powers of u. Suppressing the argument z, we derive the equations

$$(3.10)\quad 2\phi_1'=f_1,\quad 2\phi_2'=f_2-(\phi_1')^2,\quad \cdots,\quad 2\phi_s'=f_s-g_s\quad (s=1,2,\cdots),$$

where

$$(3.11)\qquad \text{(i)}\quad \text{for } s\leqq 2p,\qquad g_s=\sum_{r=1}^{s-1}\phi_r'\phi_{s-r}',$$

$$(3.12)\qquad \text{(ii)}\quad \text{for } s\geqq 2p+1,\qquad g_s=\sum_{r=1}^{s-1}\phi_r'\phi_{s-r}'-\tfrac{1}{2}h_{s-2p},$$

the quantities h_s being coefficients in the expansion of the Schwarzian:

$$(3.13)\qquad \{\phi,z\}=\frac{\phi'''}{\phi'}-\frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2=\sum_{s=1}^{\infty}\frac{h_s}{u^s}.$$

Since g_s depends only on the ϕ_r', ϕ_r'', ϕ_r''' of suffices r not exceeding $s-1$, we may determine successively $\phi_1,\phi_2,\phi_3,\cdots$ (except for arbitrary constants of integration) by integration of the equations (3.10). This gives

$$(3.14)\qquad \phi_s(\theta,z)=\tfrac{1}{2}\int^{z}\{f_s(\theta,t)-g_s(\theta,t)\}dt+\text{constant}.$$

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Since $\mathbf{D}(\theta)$ is simply connected and the $f_s(\theta, z)$ are regular there, it follows that each $\phi_s(\theta, z)$ is a regular single-valued function of z in $\mathbf{D}(\theta)$.

We shall establish an asymptotic property of the expression

$$\{\phi'(z)\}^{-\frac{1}{2}} Y\{u, \theta, \phi(z)\}$$

in regions of the z-plane which are essentially the same as those of the earlier expansions of Olver [10]. As a preliminary we introduce the quantities:

$$(3.15) \qquad \alpha_1 = \min(\alpha, 1+2\beta, 1+\gamma), \quad \beta_1 = \min(\beta, \tfrac{1}{2}+\gamma), \quad \gamma_1 = \min(2\beta, \gamma),$$

and

$$(3.16) \qquad \kappa = \max(1-\alpha, 0) \quad (\alpha \neq 1);$$

if $\alpha = 1$ we take κ to be an arbitrary positive number. We note in passing that

$$(3.17) \qquad \kappa = \max(1-\alpha_1, 0) \quad (\alpha_1 \neq 1),$$

because if $\alpha \leqq 1$, then $\alpha_1 = \alpha$.

We define $\mathbf{G}(\theta)$ as a closed subdomain of $\mathbf{D}(\theta)$ with the following properties.

(i) The distance between each point of $\mathbf{G}(\theta)$ and each boundary point of $\mathbf{D}(\theta)$ has a positive lower bound independent of θ.

(ii) For each value of θ, we can find a point $c(\theta)$ in $\mathbf{G}(\theta)$, not at infinity, and a path which lies wholly within $\mathbf{G}(\theta)$ joining $c(\theta)$ and an arbitrary point z in $\mathbf{G}(\theta)$ such that

$$(3.18) \qquad \int_{c(\theta)}^{z} \frac{|dt|}{1+|t|^{\alpha_1}} < k(1+|z|^{\kappa}).$$

We suppose that the integration constants in (3.14) have been chosen so that the quantities $\phi_s\{\theta, c(\theta)\}$ are bounded functions of θ; thus

$$(3.19) \qquad |\phi_s\{\theta, c(\theta)\}| < k_s.$$

We define

$$(3.20) \qquad \Phi_1(m; u, \theta, z) = z + \sum_{s=1}^{m} \frac{\phi_s(\theta, z)}{u^s},$$

and call Φ_1 *the first approximating function.* It is an approximate solution of equation (3.9) when u is large. In § 5 we introduce two further approximating functions $\Phi_2(u, \theta, z)$ and $\Phi_3(u, \theta, z)$. Each is a regular function of z in $\mathbf{G}(\theta)$, and for large positive u they share the properties

$$(3.21) \qquad \Phi_l(u, \theta, z) = z + \sum_{s=1}^{m} \frac{\phi_s(\theta, z)}{u^s} + (1+|z|^{\kappa})O(u^{-m-1}),$$

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$$\Phi_l'(u, \theta, z) = 1 + \sum_{s=1}^{m} \frac{\phi_s'(\theta, z)}{u^s} + \frac{O(u^{-m-1})}{1+|z|^{\alpha_1}}, \tag{3.22}$$

$$\Phi_l''(u, \theta, z) = \sum_{s=1}^{m} \frac{\phi_s''(\theta, z)}{u^s} + \frac{O(u^{-m-1})}{1+|z|^{\frac{1}{2}+\beta_1}}, \tag{3.23}$$

$$\Phi_l'''(u, \theta, z) = \sum_{s=1}^{m} \frac{\phi_s'''(\theta, z)}{u^s} + \frac{O(u^{-m-1})}{1+|z|^{1+\gamma_1}}, \tag{3.24}$$

where $l = 2, 3$; κ, α_1, β_1, γ_1 are defined by (3.16) and (3.15), and the O's are uniform with respect to θ in $\mathbf{\Theta}$ and z in $\mathbf{G}(\theta)$. Here m is an arbitrary positive integer or zero but Φ_2 and Φ_3, unlike Φ_1, are independent of m.

Writing now $\Phi(z)$ for $\Phi_3(u, \theta, z)$, we express the asymptotic character of the solutions of (3.1) by means of the following existence theorem, the proof of which is given in §§ 4—6.

THEOREM 1. *The differential equation* (3.1) *has solutions* $w_j(u, \theta, z)$ $(j = 1, 2)$ *with the properties*

$$w_j(u, \theta, z) = \{\Phi'(z)\}^{-\frac{1}{2}} \exp\{(-)^{j-1} u^p \Phi(z)\}\{1 + O(u^{-m})\}, \tag{3.25}$$

$$\begin{aligned} w_j'(u, \theta, z) = {} & [(-)^{j-1} u^p \{\Phi'(z)\}^{\frac{1}{2}} - \tfrac{1}{2}\Phi''(z)\{\Phi'(z)\}^{-\frac{3}{2}}] \\ & \times \exp\{(-)^{j-1} u^p \Phi(z)\}\{1 + O(u^{-m})\}, \end{aligned} \tag{3.26}$$

as $u \to +\infty$, *where the* O*'s are uniform with respect to* θ *in* $\mathbf{\Theta}$ *and* z *in* $\mathbf{H}_j(\theta)$. *Here* m *is an arbitrary positive integer, and the* $w_j(u, \theta, z)$ *are independent of* m.

The regions of validity $\mathbf{H}_j(\theta)$ are defined as follows. Let $\mathbf{G}_0(\theta)$ be the open domain remaining after removing from $\mathbf{G}(\theta)$ all its boundary points, and let $a_j(\theta)$ $(j = 1, 2)$ be prescribed points of $\mathbf{G}_0(\theta)$ or points at infinity on prescribed straight lines lying in $\mathbf{G}_0(\theta)$. In the latter event we suppose that

$$|\arg\{(-)^j a_j(\theta)\}| < \tfrac{1}{2}\pi - \varepsilon, \tag{3.27}$$

where ε $(< \frac{1}{4}\pi)$ is an arbitrary positive number independent of all variables. Then $\mathbf{H}_j(\theta)$ consists of the set of points z of $\mathbf{G}_0(\theta)$ which can be joined to $a_j(\theta)$ by a polygonal arc $\mathscr{P}$ having the following properties, t being a typical point of $\mathscr{P}$.

(a) $\mathscr{P}$ lies in $\mathbf{G}_0(\theta)$.

(b) If $a_j(\theta)$ is at infinity on a straight line $\mathscr{L}$, then $\mathscr{P}$ coincides with $\mathscr{L}$ for all sufficiently large $|t|$.

(c) (3.28)
$$\int_{\mathscr{P}} \frac{|dt|}{1+|t|^{1+\gamma_1}} < k,$$

where γ_1 is defined by (3.15).

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(d) The angle of slope of $\mathscr{P}$ with the positive real axis lies in the open interval $(-\frac{1}{2}\pi + \varepsilon, \frac{1}{2}\pi - \varepsilon)$ when $\mathscr{P}$ is traversed in the sense $a_j(\theta)$ to z if $j = 1$, and in the sense z to $a_j(\theta)$ if $j = 2$.

The regions $\mathbf{H}_j(\theta)$ *are domains*. For if z is a point of $\mathbf{H}_j(\theta)$, then clearly all points of the path $\mathscr{P}$ joining z to $a_j(\theta)$ belong to $\mathbf{H}_j(\theta)$. Any two points z_1, z_2 of $\mathbf{H}_j(\theta)$ can be joined by a polygonal arc consisting of the known paths joining z_1 and $a_j(\theta)$, and z_2 and $a_j(\theta)$. Therefore the set $\mathbf{H}_j(\theta)$ is connex. To determine whether $\mathbf{H}_1(\theta)$, for example, is open, we surround a given point z_1 of $\mathbf{H}_1(\theta)$ other than $a_1(\theta)$ by a square lying in $\mathbf{G}_0(\theta)$ with centre z_1 and side 2δ (> 0), and not containing $a_1(\theta)$. Condition (d) above shows that the path $\mathscr{P}$ from $a_1(\theta)$ to z_1 can intersect the boundary of the square only at a finite number of points, all of which are to the left of the point z_0 on the lower edge such that $\operatorname{Re} z_0 = \operatorname{Re} z_1 - \delta \tan \varepsilon$ (coincidence with z_0 being excluded). If t_1 is the extreme right of such points then it is evident from figure 1 that all points of the square included in the open sector $|\arg (z - t_1)| < \frac{1}{2}\pi - \varepsilon$ belong to $\mathbf{H}_1(\theta)$, moreover this sector does contain a neighbourhood of z_1. Therefore $\mathbf{H}_1(\theta)$ consists of an open set of points plus $a_1(\theta)$, which is of course a boundary point of the set if it is not at infinity. This proves the stated result.

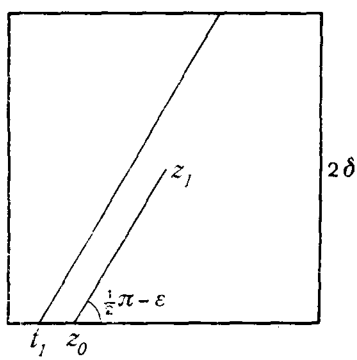

Figure 1

Remarks on Theorem 1.

(i) The definition of the regions $\mathbf{H}_j(\theta)$ adopted here is slightly more restrictive than the corresponding definition of Olver ([10], § 2). The present restrictions ensure that the $\mathbf{H}_j(\theta)$ are domains; as a consequence the proof of the existence theorem can be shortened (compare [10], page 490).

(ii) For other remarks on the regions of validity see [10], § 3, (i) and (ii).

(iii) When $a_j(\theta)$ is at infinity, it is an irregular singularity of the differential equation (3.1). From (3.25) we see that $w_j(u, \theta, z)$ is a *subdominant* solution at $a_j(\theta)$, in the terminology of Langer, in the sense that all independent solutions are exponentially large compared with $w_j(u, \theta, z)$ as $z \to a_j(\theta)$.

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Let $w(u, \theta, z)$ be any other solution subdominant at $a_j(\theta)$ whose limiting behaviour at $a_j(\theta)$ is completely known. Then the ratio $w(u, \theta, z)/w_j(u, \theta, z)$ is a function of u and θ which may be determined, at least asymptotically for large u, by letting $z \to a_j(\theta)$. Indeed, in applications of theorem 1, this is likely to be the commonest method of expressing given solutions of the differential equation in terms of the asymptotic solutions $w_1(u, \theta, z)$ and $w_2(u, \theta, z)$; see, for example, [1], §§ 4.4, 4.8 and [8], §§ 2, 4.

In passing, we may note that for some points z of $\mathbf{H}_j(\theta)$, for example the points on $\mathscr{L}$ in the neighbourhood of $a_j(\theta)$, an inequality of the form

$$\int_{a_j(\theta)}^{z} \frac{|dt|}{1 + |t|^{1+\gamma_1}} < \frac{k}{1 + |z|^{\gamma_1}} \tag{3.29}$$

is satisfied. For these points the error terms $O(u^{-m})$ in (3.25) and (3.26) may both be replaced by $(1 + |z|^{\gamma_1})^{-1}O(u^{-m})$; this result easily follows from the proof of theorem 1 given in § 6 on using (3.29) in place of (3.28).

(iv) Erdélyi ([2], § 4.3) has proved a result similar to as theorem 1 in the case when z is real and bounded.

4. Properties of $\phi_s(\theta, z)$

For convenience here and elsewhere in this paper we shall omit, on occasion, some or all of the arguments of the functions used; thus f or $f(z)$ may be written for $f(u, \theta, z)$ when these forms are not ambiguous.

Lemma 2. *For $z \in \mathbf{G}(\theta)$ and $s \geqq 1$,*

(4.1)

$$|\phi_s(\theta, z)| < k_s(1 + |z|^{\kappa}), \qquad |\phi_s'(\theta, z)| < \frac{k_s}{1 + |z|^{\alpha_1}}, \qquad |\phi_s''(\theta, z)| < \frac{k_s}{1 + |z|^{\frac{1}{2}+\beta_1}},$$

$$|\phi_s^{(n)}(\theta, z)| < \frac{k_s}{1 + |z|^{1+\gamma_1}} \quad (n = 3, 4, 5), \tag{4.2}$$

$$|\{\Phi_1'(z)\}^2 - \tfrac{1}{2}u^{-2p}\{\Phi_1(z), z\} - f(z)| < \frac{k_m u^{-m-1}}{1 + |z|^{\alpha_1}}. \tag{4.3}$$

Here $\alpha_1, \beta_1, \gamma_1, \kappa$ are defined by (3.15) and (3.16), and $\Phi_1(z) \equiv \Phi_1(m; u, \theta, z)$ by (3.20).

The first, second and third inequalities in (4.1) will be denoted by (4.1a), (4.1b) and (4.1c) respectively; a similar notation will be used elsewhere in this paper.

We prove the lemma for z lying in a subdomain of $\mathbf{D}(\theta)$ which includes $\mathbf{G}(\theta)$. Let $d > 0$ be the lower bound of the distances between points of $\mathbf{G}(\theta)$ and boundary points of $\mathbf{D}(\theta)$ (see the definition of $\mathbf{G}(\theta)$ in § 3) and let $\mathbf{G}(\theta, \delta)$ $(0 < \delta < d)$ be the domain consisting of $\mathbf{G}(\theta)$ together with each

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point of $\mathbf{D}(\theta)$ whose distance from the boundary points of $\mathbf{G}(\theta)$ does not exceed δ; then $\mathbf{G}(\theta, \delta)$ is a subdomain of $\mathbf{D}(\theta)$.

From (3.1b), (3.2), (3.3) and (3.4), we deduce that for $s = 1, 2, 3, \cdots$, and $z \in \mathbf{D}(\theta)$,

$$(4.4) \qquad |f^*(u, \theta, z)| < \frac{ku^{-1}}{1+|z|^{\alpha}}, \qquad |f_s(\theta, z)| < \frac{k_s}{1+|z|^{\alpha}};$$

$$(4.5) \qquad |f'(u, \theta, z)| < \frac{k}{1+|z|^{\frac{1}{2}+\beta}}, \qquad |f'_s(\theta, z)| < \frac{k_s}{1+|z|^{\frac{1}{2}+\beta}};$$

$$(4.6) \qquad |f''(u, \theta, z)| < \frac{k}{1+|z|^{1+\gamma}}, \qquad |f''_s(\theta, z)| < \frac{k_s}{1+|z|^{1+\gamma}};$$

compare [10], relations (5.1) to (5.4).

Lemma 2 is proved by induction. Let us assume that when $z \in \mathbf{G}(\theta, \delta)$

$$(4.7) \qquad |\phi'_r| < \frac{k_r}{1+|z|^{\alpha_1}}, \qquad |\phi''_{r-1}| < \frac{k_{r-1}}{1+|z|^{\frac{1}{2}+\beta_1}},$$

$$(4.8) \qquad |\phi^{(n)}_{r-1}| < \frac{k_{r-1}}{1+|z|^{1+\gamma_1}} \quad (n = 3, 4, 5),$$

for $r = 1, 2, \cdots, s$. Since $\phi'_1 = \frac{1}{2}f_1$ (see (3.10)) and $\phi''_0 = \phi'''_0 = \cdots = 0$, it is clear from (4.4b) that this assumption is correct when $s = 1$. We now show that if (4.7) and (4.8) hold for $r = 1, 2, \cdots, s$ for arbitrary $s \geqq 1$, then they also hold for $r = s + 1$, provided that $z \in \mathbf{G}(\theta, \eta)$, where η is an arbitrary number in the range $0 < \eta < \delta$.

From (3.11), (3.12) and (3.13), we see that g_{s+1} is a polynomial in the functions

$$\phi'_1, \phi'_2, \cdots, \phi'_s; \quad \phi''_1, \phi''_2, \cdots, \phi''_{s-2p+1}; \quad \phi'''_1, \phi'''_2, \cdots, \phi'''_{s-2p+1}.$$

Using (4.4), (4.7), (4.8) and the facts that $1 + \gamma_1 \geqq \alpha_1$, $1 + 2\beta_1 \geqq \alpha_1$, we conclude that

$$(4.9) \qquad |f_{s+1} - g_{s+1}| < \frac{k_s}{1+|z|^{\alpha_1}}.$$

This immediately establishes (4.7a) for $r = s + 1$.

Next, we have by differentiation of (3.10)

$$(4.10) \qquad 2\phi''_s = f'_s - g'_s \qquad 2\phi'''_s = f''_s - g''_s.$$

Calculating the differentiated forms of (3.11) to (3.13) by means of (2.6), (2.7), and using (4.7), (4.8) and the facts that $\beta_1 \leqq \frac{1}{2} + \gamma_1$, $\gamma_1 \leqq 2\beta_1$, we deduce that

$$(4.11) \qquad |g'_s| < \frac{1}{1+|z|^{\frac{1}{2}+\beta_1}}, \qquad |g''_s| < \frac{1}{1+|z|^{1+\gamma_1}}.$$

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Substitution of these inequalities and (4.5b), (4.6b) in (4.10) establishes (4.7b) with $r = s+1$, and (4.8) with $n = 3$, $r = s+1$.

To prove (4.8) for $n = 4$ and $r = s+1$, we use the Cauchy formula

$$\phi_s^{(4)}(\theta, z) = \frac{1}{2\pi i}\int_C \frac{\phi_s'''(\theta, t)}{(t-z)^2}\,dt, \tag{4.12}$$

where C is the circle $|t - z| = \delta - \eta$. For $0 < \eta < \delta$ the domain $\mathbf{G}(\theta, \eta)$ is contained in $\mathbf{G}(\theta, \delta)$, and if z lies in $\mathbf{G}(\theta, \eta)$ then t lies in $\mathbf{G}(\theta, \delta)$ and

$$\frac{1}{1+|t|^{1+\gamma_1}} < \frac{k}{1+|z|^{1+\gamma_1}}, \tag{4.13}$$

compare [10], (5.10) and (5.11). Hence when $z \in \mathbf{G}(\theta, \eta)$

$$|\phi_s^{(4)}(\theta, z)| < \frac{k_s}{2\pi}\int_C \frac{|dt|}{|t-z|^2(1+|t|^{1+\gamma_1})} < \frac{k\,k_s}{(\delta-\eta)(1+|z|^{1+\gamma_1})}. \tag{4.14}$$

The proof of (4.8) for $n = 5$ and $r = s+1$ is similar. This completes the proof of (4.1b), (4.1c) and (4.2).

Next, we have from (3.14)

$$\phi_s(\theta, z) = \tfrac{1}{2}\int_{c(\theta)}^{z} \{f_s(\theta, t) - g_s(\theta, t)\}dt + \phi_s\{\theta, c(\theta)\}. \tag{4.15}$$

Substitution of (4.9), with $s+1$ replaced by s, and use of (3.19) and (3.18) leads to (4.1a).

Finally, to establish (4.3) we substitute the right of equation (3.20) for Φ_1 in the expression

$$\{\Phi_1'(z)\}^2 - \tfrac{1}{2}u^{-2p}\{\Phi_1(z), z\} - f(z) \tag{4.16}$$

and expand in descending powers of u with the aid of (3.2). The terms in $1, u^{-1}, u^{-2}, \cdots, u^{-m}$ vanish as a consequence of (3.10) to (3.13), and using (4.1) and (4.2) we may verify without difficulty that the remainder can be expressed as $(1+|z|^{\alpha_1})^{-1}O(u^{-m-1})$. The proof of lemma 2 is now complete.

The function $\Phi_1(m; u, \theta, z)$ is an approximate solution of the differential equation (3.9); the inequality (4.3) is in fact an assessment of the approximation. Before proceeding to the proof of theorem 1, we shall improve upon this approximate solution in two respects. First, we construct a function $\Phi_2(u, \theta, z)$ which is independent of the integer m and satisfies (4.3) for all values of m. Then using $\Phi_2(u, \theta, z)$ we construct a third approximating function $\Phi_3(u, \theta, z)$, also independent of m, for which the inequality (4.3) is sharpened by having the factor $1+|z|^{\alpha_1}$ in the denominator replaced by $1+|z|^{1+\gamma_1}$.

5. The Second and Third Approximating Functions

In general the formal series $\sum \phi_s(\theta, z)u^{-s}$ diverges. In the first part of this section we construct a function $\Phi_2(u, \theta, z)$ which is a continuous function of

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u and a regular function of z in $\mathbf{G}(\theta)$, has the asymptotic expansion

$$(5.1) \qquad \Phi_2(u, \theta, z) \sim z + \sum_{s=1}^{\infty} \frac{\phi_s(\theta, z)}{u^s}$$

for large u, and satisfies (4.3) for any integer m. More precisely, we prove the following.

LEMMA 3. *When $z \in \mathbf{G}(\theta)$ and $\theta \in \mathbf{\Theta}$, equations* (3.21) *to* (3.24) *hold for $l = 2$, and*

$$(5.2) \qquad \Phi_2^{(n)}(u, \theta, z) = \sum_{s=1}^{m} \frac{\phi_s^{(n)}(\theta, z)}{u^s} + \frac{O(u^{-m-1})}{1 + |z|^{1+\gamma_1}} \quad (n = 4, 5),$$

$$(5.3) \qquad \{\Phi_2'(z)\}^2 - \tfrac{1}{2}u^{-2p}\{\Phi_2(z), z\} - f(z) = \frac{O(u^{-m-1})}{1 + |z|^{\alpha_1}}.$$

The method of constructing Φ_2 is similar to that given by Erdélyi ([2], § 1.7). Let $c_1, c_2, \cdots,$ be a set of positive constants such that

(5.4)

$$|\phi_s(\theta, z)| < c_s(1 + |z|^{\kappa}), \quad |\phi_s'(\theta, z)| < \frac{c_s}{1 + |z|^{\alpha_1}}, \quad |\phi_s''(\theta, z)| < \frac{c_s}{1 + |z|^{\frac{1}{2}+\beta_1}},$$

$$(5.5) \qquad |\phi_s^{(n)}(\theta, z)| < \frac{c_s}{1 + |z|^{1+\gamma_1}} \quad (n = 3, 4, 5),$$

when $z \in \mathbf{G}(\theta)$; compare lemma 2 of § 4. We suppose $u_1, u_2, \cdots,$ is a set of positive numbers having the properties

(5.6) (a) $u_{s+1} > u_s$, (b) $u_s \to \infty$ with s, (c) $c_{s+1} < \frac{1}{2}uc_s$ when $u > u_s$.

Such a set obviously exists.

Let $\nu_s(u)$ be a continuous function of u such that $0 \leqq \nu_s(u) \leqq 1$ and

$$(5.7) \qquad \nu_s(u) = 0 \quad (u \leqq u_s), \qquad \nu_s(u) = 1 \quad (u \geqq u_{s+1}).$$

We might, for example, take $\nu_s(u) = (u - u_s)/(u_{s+1} - u_s)$ when

$$u_s < u < u_{s+1}.$$

We now define

$$(5.8) \qquad \Phi_2(u, \theta, z) = z + \sum_{s=1}^{\infty} \frac{\nu_s(u)\phi_s(\theta, z)}{u^s}.$$

For any finite value of u, the $\nu_s(u)$ vanish for sufficiently large s and this series terminates. We note also that

$$(5.9) \qquad \left|\frac{\nu_s(u)\phi_s(\theta, z)}{u^s}\right| < \frac{c_1(1 + |z|^{\kappa})}{2^{s-1}u} \quad (s \geqq 1).$$

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For if $u \leqq u_s$ the left-hand side vanishes, and if $u > u_s$ we have from (5.4a) and (5.6c)

$$(5.10) \qquad \frac{|\phi_s(\theta, z)|}{1 + |z|^\kappa} < c_s < (\tfrac{1}{2}u)\, c_{s-1} < \cdots < (\tfrac{1}{2}u)^{s-1} c_1.$$

Hence the series (5.8) converges uniformly for $u \geqq u_1$.

To prove the asymptotic property (3.21) for $l = 2$, we first observe that

$$(5.11) \qquad \left| \frac{\nu_s(u)\,\phi_s(\theta, z)}{u^s} \right| < \frac{c_{m+1}(1 + |z|^\kappa)}{2^{s-m-1}\, u^{m+1}} \quad (s \geqq m + 1),$$

the proof of this inequality being similar to that of (5.9). Hence if $u \geqq u_{m+1}$, where m is an arbitrary positive integer or zero,

$$(5.12) \qquad \begin{aligned} \left| \Phi_2(u, \theta, z) - z - \sum_{s=1}^{m} \frac{\phi_s(\theta, z)}{u^s} \right| &= \left| \sum_{s=m+1}^{\infty} \frac{\nu_s(u)\,\phi_s(\theta, z)}{u^s} \right| \\ &< \sum_{s=m+1}^{\infty} \frac{c_{m+1}(1 + |z|^\kappa)}{2^{s-m-1}\, u^{m+1}} = \frac{2c_{m+1}}{u^{m+1}}(1 + |z|^\kappa) = k_m u^{-m-1}(1 + |z|^\kappa). \end{aligned}$$

The inequalities (3.22), (3.23), (3.24) and (5.2) may be established in a similar manner. The remaining result (5.3) follows from them in the same way that (4.3) followed from (4.1) and (4.2).

Henceforth we shall use the notation u_m in the same generic sense as k_m to denote a number depending on the integer m, but independent of u, θ and z.

The third approximating function

We define $\Phi_3(u, \theta, z)$ to be the solution of the differential equation [4]

$$(5.13) \qquad \{\Phi_3'(u, \theta, z)\}^2 = \tfrac{1}{2}u^{-2p}\{\Phi_2(u, \theta, z), z\} + f(u, \theta, z)$$

with the condition

$$(5.14) \qquad \Phi_3\{u, \theta, c(\theta)\} = \Phi_2\{u, \theta, c(\theta)\}.$$

On integration, we have

$$(5.15) \quad \Phi_3(u, \theta, z) = \int_{c(\theta)}^{z} [\tfrac{1}{2}u^{-2p}\{\Phi_2(u, \theta, t), t\} + f(u, \theta, t)]^{\frac{1}{2}} dt + \Phi_2\{u, \theta, c(\theta)\}.$$

We suppose that the fractional power in the integrand takes its principal value. From (3.2) with $m = 1$, we have

$$(5.16) \quad f(u, \theta, t) = 1 + (1 + |t|^\alpha)^{-1} O(u^{-1}) = 1 + O(u^{-1}) \quad (t \in \mathbf{G}(\theta)),$$

and from (3.22), (3.23) and (3.24) with $l = 2$ and $m = 0$, we readily deduce that

$$(5.17) \quad \{\Phi_2(u, \theta, t), t\} = (1 + |t|^{\alpha_1})^{-1} O(u^{-1}) = O(u^{-1}) \quad (t \in \mathbf{G}(\theta)).$$

[4] The form of (5.13) was suggested to the author by an equation in the unpublished paper of H. F. Bohnenblust mentioned in § 1.

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Hence the contents of the square brackets in (5.15) is $1 + O(u^{-1})$, uniformly for t in $\mathbf{G}(\theta)$, and so does not vanish when u is large. Thus $\Phi_3(u, \theta, z)$ is a regular function of z in $\mathbf{G}(\theta)$ if, as we now suppose, u is sufficiently large.

LEMMA 4. *When $z \in \mathbf{G}(\theta)$ and $\theta \in \mathbf{\Theta}$, equations (3.21) to (3.24) hold for $l = 3$, and*

$$\{\Phi_3'(z)\}^2 - \tfrac{1}{2}u^{-2p}\{\Phi_3(z), z\} - f(z) = \frac{O(u^{-2p-m-1})}{1 + |z|^{1+\gamma_1}}. \tag{5.18}$$

Let us set

$$E(u, \theta, z) = \{\Phi_2'(z)\}^{-2}[-\{\Phi_2'(z)\}^2 + \tfrac{1}{2}u^{-2p}\{\Phi_2(z), z\} + f(z)]. \tag{5.19}$$

From (3.22) with $l = 2$ and $m = 0$, we see that $\{\Phi_2'(z)\}^{-1}$ is bounded for all sufficiently large u. Combining this result with (5.3), we obtain

$$|E(u, \theta, z)| < \frac{k_m u^{-m-1}}{1 + |z|^{\alpha_1}} \quad (z \in \mathbf{G}(\theta)). \tag{5.20}$$

We now write (5.15) in the form

$$\begin{aligned} \Phi_3(u, \theta, z) &= \int_{c(\theta)}^{z} \Phi_2'(u, \theta, t)\{1 + E(u, \theta, t)\}^{\frac{1}{2}} dt + \Phi_2\{u, \theta, c(\theta)\} \\ &= \Phi_2(u, \theta, z) + \int_{c(\theta)}^{z} \Phi_2'(u, \theta, t)[\{1 + E(u, \theta, t)\}^{\frac{1}{2}} - 1]dt. \end{aligned} \tag{5.21}$$

Now

$$|\{1 + E(t)\}^{\frac{1}{2}} - 1| = \left|\frac{E(t)}{\{1 + E(t)\}^{\frac{1}{2}} + 1}\right| < \frac{k_m u^{-m-1}}{1 + |t|^{\alpha_1}}, \tag{5.22}$$

in consequence of (5.20). Substituting this result and a bound k for $\Phi_2'(u, \theta, t)$ in (5.21) and then using (3.18), we obtain immediately (3.21) with $l = 3$. The equation (3.22) with $l = 3$, is proved in a similar way using the differentiated form of (5.21).

Next, differentiating (5.13) with the aid of (2.6) and subtracting $2\Phi_2'\Phi_2''$ from both sides, we obtain

$$2\Phi_3'\Phi_3'' - 2\Phi_2'\Phi_2'' = \tfrac{1}{2}u^{-2p}\frac{\Phi_2'^2\Phi_2^{iv} - 4\Phi_2'\Phi_2''\Phi_2''' + 3\Phi_2''^3}{\Phi_2'^3} + f' - 2\Phi_2'\Phi_2''. \tag{5.23}$$

If we substitute on the right by means of (5.2), (3.22) to (3.24), and (3.3) with m replaced by $m + 1$, the terms in $1, u^{-1}, \cdots, u^{-m}$ must cancel as a consequence of the differentiated forms of equations (3.10) to (3.13), and the remainder is expressible in the form $(1 + |z|^{\frac{1}{2}+\beta_1})^{-1}O(u^{-m-1})$, since $1 + \gamma_1 \geqq \frac{1}{2} + \beta_1$. Hence

$$\Phi_3'' = \frac{\Phi_2'}{\Phi_3'}\Phi_2'' + \frac{1}{\Phi_3'}\frac{O(u^{-m-1})}{1 + |z|^{\frac{1}{2}+\beta_1}}. \tag{5.24}$$

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From (3.22) we see that

$$\frac{\Phi_2'}{\Phi_3'} = 1 + \frac{O(u^{-m-1})}{1+|z|^{\alpha_1}} = 1 + O(u^{-m-1}), \tag{5.25}$$

and substituting this result and (3.23) with $l = 2$, and using the fact that $(\Phi_3')^{-1}$ is bounded, we obtain (3.23) with $l = 3$.

The equation (3.24) with $l = 3$ may be established in a similar manner.

To prove the remaining result (5.18), we find from (5.13)

$$\{\Phi_3'(z)\}^2 - \tfrac{1}{2}u^{-2p}\{\Phi_3(z), z\} - f(z) = \tfrac{1}{2}u^{-2p}[\{\Phi_2(z), z\} - \{\Phi_3(z), z\}]. \tag{5.26}$$

From the relations (3.22), (3.23) and (3.24) ,we see that the expansions in descending powers of u of $\{\Phi_2(z), z\}$ and $\{\Phi_3(z), z\}$ are identical up to and including the term in u^{-m}, and that

$$|\{\Phi_2(z), z\} - \{\Phi_3(z), z\}| < \frac{k_m u^{-m-1}}{1+|z|^{1+\gamma_1}}, \tag{5.27}$$

since $1 + 2\beta_1 \geqq 1 + \gamma_1$. This completes the proof of lemma 4.

Notes on other approximating functions

If we had by-passed the function $\Phi_2(z)$ and constructed the function $\Phi_3(z)$ directly from $\Phi_1(z)$ by substituting $\Phi_1(z)$ for $\Phi_2(z)$ in (5.13), the key result (5.18) would hold, but $\Phi_3(z)$ would contain the integer m in its definition and this would result in our obtaining solutions of (3.1) which depend on m. Another way of modifying (5.13) is used by Cherry ([1], page 234) in case B; the approximating function he obtains also depends on m.

6. Proof of Theorem 1

We shall give a proof only for the case $j = 1$. A similar proof holds for $j = 2$.

In order to compare the solutions of equation (3.1) with those of (3.6), we rearrange the former equation as

$$\frac{d^2}{dz^2}w(z) - u^{2p}[\{\Phi'(z)\}^2 - \tfrac{1}{2}u^{-2p}\{\Phi(z), z\}]w(z) = u^{2p}F(z)w(z), \tag{6.1}$$

where here and elsewhere in this section $\Phi(z) \equiv \Phi_3(u, \theta, z)$ and

$$F(z) = f(z) - \{\Phi'(z)\}^2 + \tfrac{1}{2}u^{-2p}\{\Phi(z), z\}. \tag{6.2}$$

The equation

$$\frac{d^2}{dz^2}y(z) - u^{2p}[\{\Phi'(z)\}^2 - \tfrac{1}{2}u^{-2p}\{\Phi(z), z\}]y(z) = 0 \tag{6.3}$$

has the independent solutions

$$y_1(z) = \{\Phi'(z)\}^{-\frac{1}{2}}\exp\{u^p\Phi(z)\}, \qquad y_2(z) = \{\Phi'(z)\}^{-\frac{1}{2}}\exp\{-u^p\Phi(z)\}; \tag{6.4}$$

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this follows immediately from lemma 1 (ii) of § 2 on replacing the $f(u, \theta, z)$ occurring there by unity, and substituting Φ for z and z for ξ. We shall suppose the branches of the fractional powers in (6.4) are the principal ones. They are regular for sufficiently large u, since from (3.22) we see that $\Phi'(z) = 1 + O(u^{-1})$, uniformly with respect to z and θ.

We suppose first that the point $a_1(\theta)$ of the open domain $\mathbf{G}_0(\theta)$ introduced in § 3 is not at infinity. We define $w_1(u, \theta, z) \equiv w_1(z)$ to be the solution of the differential equation (3.1) which is regular in $\mathbf{D}(\theta)$ and satisfies the conditions

$$(6.5) \qquad w_1(u, \theta, a_1) = y_1(u, \theta, a_1), \qquad w_1'(u, \theta, a_1) = y_1'(u, \theta, a_1),$$

primes denoting, as elsewhere, derivatives with respect to z.

Subtracting (6.3) from (6.1), we obtain

$$(6.6) \qquad \begin{aligned} &\frac{d^2}{dz^2}\{w_1(z) - y_1(z)\} - u^{2p}[\{\Phi'(z)\}^2 - \tfrac{1}{2}u^{-2p}\{\Phi(z), z\}]\{w_1(z) - y_1(z)\} \\ &\qquad = u^{2p}F(z)\{w_1(z) - y_1(z)\} + u^{2p}F(z)y_1(z). \end{aligned}$$

This equation can also be written in the form

$$(6.7) \qquad \frac{d^2}{dz^2}\{w_1(z) - y_1(z)\} = u^{2p}f(z)\{w_1(z) - y_1(z)\} + u^{2p}F(z)y_1(z).$$

We establish the asymptotic property (3.25) by iterative solution of the inhomogeneous differential equation (6.6). Let z be a point of the domain $\mathbf{H}_1(\theta)$, defined in § 3. We shall need the following property of the polygonal arc $\mathscr{P}$ joining z and a_1.

Lemma 5. *As the point t moves along $\mathscr{P}$ from a_1 to z,* Re $\Phi(t)$ *is monotonic strictly increasing if u is sufficiently large.*

Condition (d) of the definition of $\mathscr{P}$ given in § 3, shows that the equation of a typical segment of $\mathscr{P}$ can be expressed in the form

$$(6.8) \qquad t = t(\tau) = \lambda + \tau e^{i\chi},$$

where λ, χ are constants, $-\frac{1}{2}\pi + \varepsilon < \chi < \frac{1}{2}\pi - \varepsilon$, and the real parameter τ ranges over a finite interval. If τ_1, τ_2 are points of this interval, then from the mean value theorem

$$(6.9) \qquad \Phi\{t(\tau_2)\} - \Phi\{t(\tau_1)\} = (\tau_2 - \tau_1)t'(\tau_3)\Phi'\{t(\tau_3)\},$$

where τ_3 lies in the interval (τ_1, τ_2). Using (6.8) and (3.22) with $m = 0$, we deduce that

$$(6.10) \qquad \Phi\{t(\tau_2)\} - \Phi\{t(\tau_1)\} = (\tau_2 - \tau_1)e^{i\chi}\{1 + O(u^{-1})\},$$

from which the lemma follows immediately.

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We define a sequence of functions $h_n(u, \theta, z) \equiv h_n(z)$ by $h_0(z) = 0$ and the differential equation

$$\begin{aligned}(6.11)\quad & h_n''(z) - u^{2p}[\{\Phi'(z)\}^2 - \tfrac{1}{2}u^{-2p}\{\Phi(z), z\}]\,h_n(z)\\ &\qquad = u^{2p}F(z)h_{n-1}(z) + u^{2p}F(z)\,y_1(z) \quad (n \geqq 1)\end{aligned}$$

(compare (6.6)), with the conditions

$$(6.12)\qquad h_n(a_1) = h_n'(a_1) = 0.$$

Solving (6.11) by the method of variation of parameters, using the Wronskian relation

$$(6.13)\qquad y_1(z)\,y_2'(z) - y_1'(z)\,y_2(z) = -2u^p,$$

we obtain

$$(6.14)\quad h_n(z) = \tfrac{1}{2}u^p \int_{a_1}^{z} \{y_1(z)\,y_2(t) - y_2(z)\,y_1(t)\}\{h_{n-1}(t) + y_1(t)\}F(t)\,dt.$$

We suppose that z is a point of $\mathbf{H}_1(\theta)$, and that the path in (6.14) is the polygonal arc $\mathscr{P}$ joining a_1 and z.

For $n = 1$, (6.14) becomes

$$(6.15)\qquad h_1(z) = \tfrac{1}{2}u^p \int_{a_1}^{z} \{y_1(z)\,y_2(t) - y_2(z)\,y_1(t)\}\,y_1(t)F(t)\,dt.$$

From (6.4) and the fact that $(\Phi')^{-1}$ is bounded, we see that

$$(6.16)\qquad |y_1(z)\,y_2(t)\,y_1(t)| < k\,|\exp\{u^p\Phi(z)\}|,$$

and

$$(6.17)\quad |y_2(z)\{y_1(t)\}^2| < k\,|\exp\{-u^p\Phi(z) + 2u^p\Phi(t)\}| \leqq k\,|\exp\{u^p\Phi(z)\}|,$$

in consequence of lemma 5. Also from (6.2) and (5.18), we have

$$(6.18)\qquad |F(t)| < \frac{k_m u^{-2p-m-1}}{1 + |t|^{1+\gamma_1}}$$

if $u > u_m$, where m is an arbitrary positive integer or zero. Substituting (6.16), (6.17) and (6.18) in (6.15), we derive

$$\begin{aligned}(6.19)\quad |h_1(z)| &< \frac{k\,k_m}{u^{p+m+1}}\,|\exp\{u^p\Phi(z)\}| \int_{a_1}^{z} \frac{|dt|}{1 + |t|^{1+\gamma_1}}\\ &< k^2 k_m u^{-p-m-1}\,|\exp\{u^p\Phi(z)\}|,\end{aligned}$$

in consequence of (3.28). It is evident that (6.19) also holds with z replaced by t, where t is any point of the path $\mathscr{P}$ joining a_1 to z, since all such points belong to $\mathbf{H}_1(\theta)$.

Again, from (6.14) we have

$$\begin{aligned}(6.20)\quad & h_{n+1}(z) - h_n(z)\\ &= \tfrac{1}{2}u^p \int_{a_1}^{z} \{y_1(z)\,y_2(t) - y_2(z)\,y_1(t)\}\{h_n(t) - h_{n-1}(t)\}F(t)\,dt \quad (n \geqq 1).\end{aligned}$$

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From (6.4) and (6.19) we see that

$$(6.21)\quad |y_1(z)\,y_2(t)\,h_1(t)|,\quad |y_2(z)\,y_1(t)\,h_1(t)| < k^3 k_m u^{-p-m-1}\,|\exp\{u^p\Phi(z)\}|$$

(compare (6.16) and (6.17)). Taking $n = 1$ in (6.20) and using (6.21), (6.18) and (3.28), we find that

$$(6.22)\qquad |h_2(z) - h_1(z)| < k^4 k_m^2 u^{-2p-2m-2}\,|\exp\{u^p\Phi(z)\}|,$$

where k, k_m are the same constants as in (6.19). Continuing the argument, we arrive at

$$(6.23)\quad |h_{n+1}(z) - h_n(z)| < k^{2n+2} k_m^{n+1} u^{-(n+1)(p+m+1)}\,|\exp\{u^p\Phi(z)\}| \quad (n \geqq 0).$$

Next, we have by differentiation of (6.15) and (6.20)

$$(6.24)\qquad h_1'(z) = \tfrac{1}{2}u^p \int_{a_1}^{z} \{y_1'(z)\,y_2(t) - y_2'(z)\,y_1(t)\}\,y_1(t)\,F(t)\,dt,$$

$$(6.25)\quad \begin{aligned} &h_{n+1}'(z) - h_n'(z) \\ &= \tfrac{1}{2}u^p \int_{a_1}^{z} \{y_1'(z)\,y_2(t) - y_2'(z)\,y_1(t)\}\{h_n(t) - h_{n-1}(t)\}\,F(t)\,dt \quad (n \geqq 1). \end{aligned}$$

Now from (6.4)

$$(6.26)\qquad y_1'(z) = u^p[\{\Phi'(z)\}^{\frac{1}{2}} - \tfrac{1}{2}u^{-p}\Phi''(z)\{\Phi'(z)\}^{-\frac{3}{2}}]\exp\{u^p\Phi(z)\},$$

$$(6.27)\quad y_2'(z) = -u^p[\{\Phi'(z)\}^{\frac{1}{2}} + \tfrac{1}{2}u^{-p}\Phi''(z)\{\Phi'(z)\}^{-\frac{3}{2}}]\exp\{-u^p\Phi(z)\}.$$

The contents of the square brackets in these two equations are bounded, as a consequence of (3.22) and (3.23). Hence

$$(6.28)\qquad |y_1'(z)\,y_2(t)\,y_1(t)|,\quad |y_2'(z)\{y_1(t)\}^2| < ku^p\,|\exp\{u^p\Phi(z)\}|$$

(compare (6.16) and (6.17)). Substituting these inequalities and (6.23) in (6.24) and (6.25), and again using (6.18) and (3.28), we obtain

$$(6.29)\quad |h_{n+1}'(z) - h_n'(z)| < k^{2n+2} k_m^{n+1} u^p u^{-(n+1)(p+m+1)}\,|\exp\{u^p\Phi(z)\}| \quad (n \geqq 0).$$

The inequalities (6.23) and (6.29) show that the series

$$(6.30)\qquad \sum_{n=0}^{\infty}\{h_{n+1}(z) - h_n(z)\},\qquad \sum_{n=0}^{\infty}\{h_{n+1}'(z) - h_n'(z)\}$$

converge uniformly with respect to z in any bounded subregion of $\mathbf{H}_1(\theta)$ for all sufficiently large u. Since $\mathbf{H}_1(\theta)$ is a domain, term-by-term differentiation of the two series is valid at interior points. Hence from (6.11) it follows that at the interior points of $\mathbf{H}_1(\theta)$ the sum (6.30a) satisfies the same inhomogeneous differential equation (6.6) as $w_1(z) - y_1(z)$. Moreover, the sums (6.30a) and (6.30b) are continuous at all points of $\mathbf{H}_1(\theta)$, hence from (6.5) and (6.12) it follows that

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$$(6.31)\qquad \left.\begin{aligned} w_1(z) - y_1(z) &= \sum_{n=0}^{\infty} \{h_{n+1}(z) - h_n(z)\} \\ w_1'(z) - y_1'(z) &= \sum_{n=0}^{\infty} \{h_{n+1}'(z) - h_n'(z)\} \end{aligned}\right\} \quad (z \in \mathbf{H}_1(\theta)).$$

Again, if $u > (2k^2k_m)^{1/(p+m+1)}$ we deduce from (6.23) and (6.29) that

$$(6.32)\qquad |w_1(z) - y_1(z)| < 2k^2 k_m u^{-p-m-1} |\exp\{u^p \Phi(z)\}|,$$

$$(6.33)\qquad |w_1'(z) - y_1'(z)| < 2k^2 k_m u^{-m-1} |\exp\{u^p \Phi(z)\}|.$$

These results establish (3.25) and (3.26) for $j = 1$ and $a_1(\theta)$ not at infinity.

If $a_j(\theta)$ is a point at infinity on a straight line $\mathscr{L}$ lying in $\mathbf{G}_0(\theta)$, we modify the proof as follows. The function $h_1(z)$ is now defined by (6.15), the convergence of the integral at its lower limit being assured in consequence of (6.16), (6.17), (6.18) and (3.28). Clearly the inequality (6.19) again holds when $z \in \mathbf{H}_1(\theta)$. Higher members of the sequence $h_n(z)$ we define by (6.14), the convergence of each integral at its lower limit being readily proved by induction. The inequalities (6.23) and (6.29) again hold, so that the series (6.30) converge uniformly with respect to z in any bounded subregion of $\mathbf{H}_1(\theta)$. The function $w_1(z)$ is taken to be the sum of (6.30a) and $y_1(z)$; clearly it satisfies (6.1) and (3.1), and it is independent of the integer m. The inequalities (3.25) and (3.26) for $j = 1$ now follow from (6.23) and (6.29) as before.

This completes the proof of theorem 1.

7. Asymptotic Expansions of the Solutions

We begin this section by examining the usefulness of the form of the result (3.25). We recall that the known properties of the function $\Phi(z) \equiv \Phi_3(u, \theta, z)$ defined in § 5 are of an asymptotic nature, typified by (3.21) to (3.24). In (3.21) the quantity κ appearing in the error term $(1 + |z|^\kappa)O(u^{-m-1})$ lies in the range $0 \leqq \kappa < 1$; see (3.16). When κ is not zero this error term is unbounded for large $|z|$. Nevertheless, we can regard (3.21) as a satisfactory formula for $\Phi(z)$, because the *relative error*, which is obviously $z^{-1}(1 + |z|^\kappa)O(u^{-m-1})$, is bounded for large $|z|$. In other words, for a given (large) value of u, the number of correct *significant figures* in values of $\Phi(z)$ computed from (3.21), will have a positive lower bound, independent of z.

Examining now the effect of substituting in (3.25) approximate values of $\Phi(z)$ and $\Phi'(z)$ obtained from (3.21) and (3.22), we reach a different kind of conclusion concerning the resulting approximation for $w_j(u, \theta, z)$. The error term $(1 + |z|^\kappa)O(u^{-m-1})$ in (3.21) gives rise to a relative error $\exp\{(1 + |z|^\kappa)O(u^{p-m-1})\}$ in $w_j(u, \theta, z)$, which is unbounded if $\kappa > 0$. Thus

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for any given value of u the number of correct significant figures in values of $w_j(u, \theta, z)$ computed from (3.25) may decrease as $|z|$ increases, and for all sufficiently large $|z|$ no accuracy may be obtainable.

The equation (3.25), and similarly also (3.26), is accordingly of restricted value when $|z|$ is large. It would, however, be untrue to say that these equations have no value in these circumstances; in applications, for example, in which the *logarithms* of $w_j(u, \theta, z)$ and $w_j'(u, \theta, z)$ were the quantities of paramount importance the equations would be quite satisfactory.

To sharpen the results for other applications, we now impose the additional condition

$$\alpha > 1, \tag{7.1}$$

where α is the quantity introduced in (3.2). It then follows from (3.16) that $\kappa = 0$. The error term in (3.21) becomes $O(u^{-m-1})$ and gives rise to a relative error $O(u^{p-m-1})$ in $w_j(u, \theta, z)$ on substituting (3.21) in (3.25), which is perfectly satisfactory when $m \geqq p$.

We may note in passing that the condition (7.1) is fulfilled if $\mathbf{D}(\theta)$ is bounded for all θ, for then the defining inequality (3.2), if it holds at all, will hold for any positive value of α.

With the additional condition (7.1), it is possible to expand (3.25) and (3.26) in asymptotic series. From (4.1a) with $\kappa = 0$, we see that each $\phi_s(\theta, z)$ $(s \geqq 1)$ is bounded. Hence when $m \geqq p$ the expression

$$\exp\left[\pm u^p\left\{\sum_{s=p+1}^{m} \frac{\phi_s(\theta, z)}{u^s} + O\left(\frac{1}{u^{m+1}}\right)\right\}\right] \tag{7.2}$$

may be expanded as a polynomial in u^{-1} of degree $m - p$, together with an error term $O(u^{p-m-1})$. Combining this expansion with a similar one for $\{\Phi'(z)\}^{-\frac{1}{2}}$, we obtain, in the case $j = 1$ for example,

$$w_1(u, \theta, z) = \exp\left\{u^p z + u^p \sum_{s=1}^{p} \frac{\phi_s(\theta, z)}{u^s}\right\}\left\{1 + \sum_{s=1}^{m} \frac{A_s(\theta, z)}{u^s} + O\left(\frac{1}{u^{m+1}}\right)\right\}, \tag{7.3}$$

where the coefficients $A_s(\theta, z)$ are given by the asymptotic identity

$$1 + \sum_{s=1}^{\infty} \frac{A_s(\theta, z)}{u^s} \sim \left\{1 + \sum_{s=1}^{\infty} \frac{\phi_s'(\theta, z)}{u^s}\right\}^{-\frac{1}{2}} \exp\left\{\sum_{s=1}^{\infty} \frac{\phi_{s+p}(\theta, z)}{u^s}\right\}. \tag{7.4}$$

Clearly each $A_s(\theta, z)$ is a regular function of z in $\mathbf{D}(\theta)$, and bounded when $\theta \in \mathbf{\Theta}$, $z \in \mathbf{G}(\theta)$.

An alternative way of deriving the $A_s(\theta, z)$ is afforded by substituting (7.3) directly in the differential equation (3.1) and equating coefficients (see [2], § 4.2). The derivatives of the term $O(u^{-m-1})$ in (7.3) also have the form $O(u^{-m-1})$ at points whose distance from the boundary of $\mathbf{H}_1(\theta)$ has a positive

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lower bound; this is easily proved by application of Cauchy's integral formulae for derivatives. The validity of the resulting expressions for the $A_s(\theta, z)$ at the other points of $\mathbf{D}(\theta)$ follows by analytical continuation.

In order to simplify the procedure, it is convenient to use the notation

$$\text{(7.5)} \qquad \left.\begin{aligned} \beta_s(\theta, z) &= \phi_s(\theta, z) \quad &(s = 0, 1, \cdots, p) \\ &= 0 &(s \geqq p+1). \end{aligned}\right\}$$

Carrying out the substitution described, we arrive at the asymptotic identity

(7.6)

$$\left(\sum_{s=1}^{\infty} \frac{\gamma_s - f_s + u^{-p}\beta_s''}{u^s}\right)\left(1 + \sum_{s=1}^{\infty} \frac{A_s}{u^s}\right) + 2\left(1 + \sum_{s=1}^{\infty} \frac{\beta_s'}{u^s}\right) \sum_{s=1}^{\infty} \frac{A_s'}{u^{s+p}} + \sum_{s=1}^{\infty} \frac{A_s''}{u^{s+2p}} = 0,$$

where

$$\text{(7.7)} \qquad 1 + \sum_{s=1}^{\infty} \frac{\gamma_s}{u^s} = \left(1 + \sum_{s=1}^{\infty} \frac{\beta_s'}{u^s}\right)^2;$$

thus

$$\text{(7.8)} \qquad \gamma_s = 2\beta_s' + \sum_{r=1}^{s-1} \beta_r' \beta_{s-r}' \quad (s \geqq 1).$$

Equating to zero the coefficients of $u^{-1}, u^{-2}, \cdots, u^{-p}$ in (7.6), we obtain

$$\text{(7.9)} \qquad \gamma_s = f_s \quad (s = 1, 2, \cdots, p),$$

and equating to zero the coefficients of the higher powers of u^{-1} and using (7.9), we obtain

$$\text{(7.10)} \quad \sum_{r=p+1}^{s} (\gamma_r - f_r + \beta_{r-p}'') A_{s-r} + 2A_{s-p}' + 2\sum_{r=p+1}^{s} \beta_{r-p}' A_{s-r}' + A_{s-2p}'' = 0 \qquad (s \geqq p+1),$$

which becomes, on replacing s by $s+p$ and r by $r+p$,

$$\text{(7.11)} \quad 2A_s' = -A_{s-p}'' + \sum_{r=1}^{s} \{(f_{r+p} - \gamma_{r+p} - \beta_r'') A_{s-r} - 2\beta_r' A_{s-r}'\} \quad (s \geqq 1),$$

the first term on the right being absent when $s < p$.

Equation (7.9) is simply the defining relation for $\phi_1, \phi_2, \cdots, \phi_p$; compare (3.10) and (3.11). From (7.11) we obtain on integration

$$\text{(7.12)} \quad A_s = -\tfrac{1}{2}A_{s-p}' + \tfrac{1}{2}\sum_{r=1}^{s} \int \{(f_{r+p} - \gamma_{r+p} - \beta_r'') A_{s-r} - 2\beta_r' A_{s-r}'\} dz \quad (s \geqq 1).$$

The functions γ and β on the right of this equation are known in terms of $\phi_1, \phi_2, \cdots, \phi_p$, accordingly (7.12) is a recurrence relation for the A_s. It shows immediately that the A_s are regular in $\mathbf{D}(\theta)$.

On making obvious extensions to the above analysis, we may summarize the results of this section in the following theorem.

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THEOREM 2. *With the conditions of § 3 and the restriction $\alpha > 1$, there exist solutions $w_1(u, \theta, z)$ and $w_2(u, \theta, z)$ of the differential equation* (3.1) *such that*

$$\left.\begin{aligned} w_1(u, \theta, z) &= \exp\left\{u^p z + u^p \sum_{s=1}^{p} \frac{\phi_s(\theta, z)}{u^s}\right\}\left\{1 + \sum_{s=1}^{m} \frac{A_s(\theta, z)}{u^s} + O\left(\frac{1}{u^{m+1}}\right)\right\} \\ w_1'(u, \theta, z) &= u^p \exp\left\{u^p z + u^p \sum_{s=1}^{p} \frac{\phi_s(\theta, z)}{u^s}\right\}\left\{1 + \sum_{s=1}^{m} \frac{B_s(\theta, z)}{u^s} + O\left(\frac{1}{u^{m+1}}\right)\right\} \end{aligned}\right\} \tag{7.13}$$

$$(z \in \mathbf{H}_1(\theta)),$$

$$\left.\begin{aligned} w_2(u, \theta, z) &= \exp\left\{-u^p z - u^p \sum_{s=1}^{p} \frac{\phi_s(\theta, z)}{u^s}\right\}\left\{1 + \sum_{s=1}^{m} (-)^s \frac{A_s^*(\theta, z)}{u^s} + O\left(\frac{1}{u^{m+1}}\right)\right\} \\ w_2'(u, \theta, z) &= -u^p \exp\left\{-u^p z - u^p \sum_{s=1}^{p} \frac{\phi_s(\theta, z)}{u^s}\right\}\left\{1 + \sum_{s=1}^{m} (-)^s \frac{B_s^*(\theta, z)}{u^s} + O\left(\frac{1}{u^{m+1}}\right)\right\} \end{aligned}\right\} \tag{7.14}$$

$$(z \in \mathbf{H}_2(\theta)),$$

as $u \to +\infty$, where the O's are uniform with respect to z and $\theta \in \mathbf{\Theta}$. Here m is an arbitrary integer or zero, and $w_1(u, \theta, z)$, $w_2(u, \theta, z)$ are independent of m.

The functions $\phi_s(\theta, z)$ are given by (3.10) and (3.11), and $A_s(\theta, z)$ by (7.12) or, equivalently, (7.4). Also,

$$B_s = A_s + \beta_1' A_{s-1} + \beta_2' A_{s-2} + \cdots + \beta_s' + A_{s-p}'. \tag{7.15}$$

Corresponding formulae for the starred coefficients are

$$A_s^* = (-)^p \tfrac{1}{2} A_{s-p}^{*\prime} + \tfrac{1}{2} \sum_{r=1}^{s} (-)^r \int \{(\gamma_{r+p} - f_{r+p} - \beta_r'') A_{s-r}^* - 2\beta_r' A_{s-r}^{*\prime}\} dz \tag{7.16}$$

$$(s \geqq 1),$$

$$B_s^* = A_s^* - \beta_1' A_{s-1}^* + \beta_2' A_{s-2}^* - \cdots + (-)^s \beta_s' - (-)^p A_{s-p}^{*\prime}. \tag{7.17}$$

REMARK. Theorem A of Olver ([10], § 2) is the special case of theorem 2 obtained by taking $p = 1$, $f_1(\theta, z) = 0$. The condition $\alpha > 1$ is contained implicitly in Olver's inequality (2.2).

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