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THE POINTWISE ERGODIC THEOREM FOR TRANSFORMATIONS WHOSE ORBITS CONTAIN OR ARE CONTAINED IN THE ORBITS OF A MEASURE-PRESERVING TRANSFORMATION

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1. Introduction. Let $(\Omega, \mathscr{F}, P)$ be a probability space with $(\Omega, \mathscr{F})$ standard. Let T be a bimeasurable one-to-one map of Ω onto itself. Let $U: \Omega \rightarrow \Omega$ be another measurable transformation whose orbits are contained in the T-orbits; that is,

$$U(\omega) \in \{T^i\omega : i \in \mathbf{Z}\} \text{ for all } \omega \in \Omega,$$

where $\mathbf{Z}$ denotes the set of integers. (This is equivalent to saying that there is a measurable mapping $L: \Omega \rightarrow \mathbf{Z}$ such that $U(\omega) = T^{L(\omega)}(\omega)$, $\omega \in \Omega$.) Such pairs (T, U) arise quite naturally in ergodic theory and information theory. (For example, in ergodic theory, one can see such pairs in the study of the full group of a transformation [**1**]; in information theory, such a pair can be associated with the input and output of a variable-length source encoder [**2**] [**3**].) Neveu [**4**] obtained necessary and sufficient conditions for U to be measure-preserving if T is measure-preserving. However, if U fails to be measure-preserving, U might still possess many of the features of measure-preserving transformations. For example, the pointwise ergodic theorem might hold for U; that is, given a bounded measurable function $f: \Omega \rightarrow (-\infty, \infty)$, the sequence

$$\left\{n^{-1}\sum_{i=0}^{n-1} f(U^i(\omega))\right\}_{n=1}^{\infty}$$

may converge as $n \rightarrow \infty$ for almost all $\omega \in \Omega$. Since in many applications the property that U obeys the ergodic theorem is the property of U of interest (rather than the measure-preserving property), a useful problem to consider would be that of giving sufficient conditions on the pair (T, U) so that the pointwise ergodic theorem will hold for U if T is measure-preserving. This is one of the problems we address in this paper.

Before proceeding further, we need to identify the class of transformations for which the ergodic theorem holds. This was done by Gray

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and Kieffer [**2**]. They defined a probability measure Q on a measurable space $(\Lambda, \mathscr{M})$ to be *asymptotically mean stationary* (a.m.s.) with respect to a given transformation V on Λ if

$$\lim_{n\to\infty} n^{-1} \sum_{i=0}^{n-1} Q(V^{-i}E)$$

exists, for every measurable $E \subset \Lambda$. It was shown in [**2**] that the transformation V on the probability space $(\Lambda, \mathscr{M}, Q)$ obeys the pointwise ergodic theorem for every bounded measurable real-valued function on Λ if and only if Q is a.m.s. with respect to V.

Thus, the problem of finding a sufficient condition on the pair (T, U) so that U will obey the pointwise ergodic theorem when T is measure-preserving reduces to the problem of determining when P will be a.m.s. with respect to U given that P is stationary with respect to T. This problem is addressed in the second section of this paper. The special case of this problem where each $U(\omega)$ is an element of the unilateral orbit $\{\omega, T(\omega), \ldots .\}$ was considered in [**3**]. That case was simpler to handle than the general case since in that case the U-orbit $\{\omega, U(\omega), \ldots\}$ and T-orbit $\{\omega, T(\omega), \ldots\}$ are averaged in the "same direction". More precisely, there are non-negative integers of positive density $n_0 < n_1 < \ldots$ (depending on $\omega \in \Omega$) such that

$$U^i(\omega) = T^{n_i}(\omega), \quad i \geqq 0;$$

using this, the convergence of the averages

$$\left\{n^{-1} \sum_{i=0}^{n-1} f(T^i\omega)\right\}_{n=1}^{\infty}$$

was used to obtain convergence of the averages

$$\left\{n^{-1} \sum_{i=0}^{n-1} f(U^i\omega)\right\}_{n=1}^{\infty}$$

by considering functions f which vanish off the U-orbit, enabling one to obtain averages of the second type from averages of the first type. However in the general case this method breaks down because the bilateral character of the T-orbit $\{T^i\omega: i \in \mathbf{Z}\}$ gives us two directions along which we can average to obtain almost sure convergence (the direction $\omega \to T\omega \to T^2\omega \to \ldots$ and the direction $\omega \to T^{-1}\omega \to T^{-2}\omega \to \ldots$) whereas we must average along the U-orbit in the direction $\omega \to U(\omega) \to \ldots$, which doesn't conform to either of the two T-directions for all time. Roughly speaking, as we proceed in the U-orbit we frequently change direction vis-a-vis the T-orbit (i.e., we "hop" back and forth in the T-orbit). For these reasons, more care has to be taken with the proofs in this paper than was taken in [**3**].

In the third section of this paper, we consider the converse problem of determining when P will be a.m.s. with respect to T when P is stationary with respect to U.

2. Transformations whose orbits are contained in the orbits of a measure-preserving transformation. In this section, we assume that the invertible measurable transformation T on $(\Omega, \mathcal{F}, P)$ is measure-preserving. The measurable transformation U on Ω is then a transformation whose orbits are contained in the orbits of a measure-preserving transformation. Can we say that the ergodic theorem holds for U?

To help answer this question, we introduce another concept from [**2**]. If Q_1, Q_2 are probability measures on a measurable space Λ and V is a measurable transformation on Λ, we say that Q_2 *asymptotically dominates* Q_1 *with respect to* V if and only if for every measurable $E \subset \Lambda$ satisfying $Q_2(E) = 0$, we have

$$\lim_{n\to\infty} Q_1(V^{-n}E) = 0.$$

The following result from [**2**] gives the connection of this concept with the a.m.s. concept: Q_1 is a.m.s. with respect to V if and only if there exists a probability measure Q_2 on Λ, stationary with respect to V, such that Q_2 asymptotically dominates Q_1 with respect to V.

Here is the main result of this section. (In the following, $>$ denotes the transitive relation on Ω such that $\omega_1 > \omega_2$ if and only if $T^n\omega_2 = \omega_1$ for some $n > 0$.)

THEOREM 1. *Let T be an invertible measure-preserving transformation on the probability space $(\Omega, \mathcal{F}, P)$ and let $U: \Omega \to \Omega$ satisfy*

(a) $U(\omega) \in \{T^i\omega: i \in \mathbf{Z}\}$, $\omega \in \Omega$; *and*

(b) *For P-almost all $\omega \in \Omega$, the sets*

$$\{\omega' < \omega: U\omega' \geqq \omega\} \text{ and } \{\omega' > \omega: U\omega' \leqq \omega\}$$

are finite.

Then P is a.m.s. with respect to U. Furthermore, there exists $A \in \mathcal{F}$ with $P(A) > 0$ such that P_A is stationary with respect to U and asymptotically dominates P with respect to U, where P_A is the probability measure on $\mathcal{F}$ concentrated on A satisfying

$$P_A(F) = P(A \cap F)/P(A), \quad F \in \mathcal{F}.$$

Remark. If T is measure-preserving and U satisfies (a), let $L: \Omega \to \mathbf{Z}$ be a measurable function such that

$$U(\omega) = T^{L(\omega)}(\omega), \quad \omega \in \Omega.$$

Then assumption (b) of Theorem 1 holds if $\int_\Omega |L| dP < \infty$, or if, as shown in [**3**], the weaker assumption holds that

$$\limsup_{i\to\infty} i^{-1}|L(T^{-1}\omega)| < \infty \text{ for } P\text{-almost every } \omega \in \Omega.$$

The proof of Theorem 1 will occupy the rest of this section. As a by-product of our proof we will obtain some interesting results on the structure of the orbits of U. If V is a mapping from Ω into itself, a subset S of Ω is called a *unilateral V-orbit* if S takes the form $\{\omega, V(\omega), V^2(\omega), \ldots\}$ for some $\omega \in \Omega$, and is called a *bilateral V-orbit* if it takes the form $S = \{\omega_i : i \in \mathbf{Z}\}$, where $V(\omega_i) = \omega_{i+1}$, $i \in \mathbf{Z}$. The bilateral T-orbit containing $\omega \in \Omega$, namely $\{T^i\omega : i \in \mathbf{Z}\}$, we will denote by $O_T(\omega)$.

We consider three types of bilateral U-orbits. We define a bilateral U-orbit S to be of *type* 1 if S is finite; it is not hard to see that the bilateral U-orbit S is of type 1 if and only if S is a U-cyclic subset of Ω; i.e., a subset of Ω of form $\{\omega, U(\omega), \ldots, U^{n-1}(\omega)\}$ for some $\omega \in \Omega$ and some positive integer n, where $U^n(\omega) = \omega$. We consider two types of infinite bilateral U-orbits S: S is of *type* 2 if for some $\omega \in \Omega$ and $\{n_i : i \in \mathbf{Z}\} \subset Z$, we have

$$S = \{T^{n_i}\omega : i \in \mathbf{Z}\},$$

where

$$U(T^{n_i}\omega) = T^{n_{i+1}}(\omega), \quad i \in \mathbf{Z} \quad \text{and}$$
$$\lim_{i\to-\infty} n_i = -\infty, \ \lim_{i\to\infty} n_i = \infty;$$

S is of *type* 3 if

$$S = \{T^{n_i}\omega : i \in \mathbf{Z}\},$$

where

$$U(T^{n_i}\omega) = T^{n_{i+1}}(\omega) \quad (i \in \mathbf{Z}),$$
$$\lim_{i\to-\infty} n_i = \infty \quad \text{and} \quad \lim_{i\to\infty} n_i = -\infty.$$

We call a bilateral U-orbit *nice* if it is of one of these three types.

THEOREM 2. *Under the assumptions of Theorem* 1, *there exists a measurable subset Ω' of Ω of P-measure one such that:*

(a) $\{\omega \in \Omega'$: *there exists a bilateral U-orbit* $\supset \{\omega\}\}$ *is measurable;*

(b) *If $\omega \in \Omega'$, there are finitely many type* 2 *and* 3 *U-orbits* $\subset O_T(\omega)$*;*

(c) *If $\omega \in \Omega'$, any two nice U-orbits* $\subset O_T(\omega)$ *are either disjoint or identical;*

(d) *If $\omega \in \Omega'$, there exists $i \geqq 0$ such that $U^i(\omega)$ is an element of a nice bilateral U-orbit;*

(e) *If $\omega \in \Omega'$, then every bilateral U-orbit which is a subset of $O_T(\omega)$ is nice.*

If we can prove parts of (a)–(d) of Theorem 2, then Theorem 1 follows as a simple corollary. For, letting A be the measurable set in part (a) of Theorem 2, we have

(f) Given $\omega \in A$, there is a unique $\omega' \in A$ such that $U(\omega') = \omega$.
(g) $U^{-n}A \uparrow \Omega'$ as $n \to \infty$.

Statement (g) implies that $P(A) > 0$. Let

$$\Omega_0 = \{\omega \in \Omega: T^n\omega \neq \omega \text{ for all } n \geqq 1\}$$

and let $\Omega_1, \Omega_2, \ldots$ be finite T-cyclic subsets of Ω such that $\Omega_0, \Omega_1, \ldots$ are disjoint and $\sum_{i=0}^{\infty} P(\Omega_i) = 1$. Setting $A_i = A \cap \Omega_i$, $i \geqq 0$, and letting $L: A_0 \to \mathbf{Z}$ be the unique function such that $U = T^L$, we have from (f) that

$$\{T^j([L = j] \cap A_0) \cap A_0: j \in \mathbf{Z}\}$$

is a partition of A_0 and so if E is a measurable subset of A_0 we have

$$P(U^{-1}E \cap A_0) = \sum_{j \in \mathbf{Z}} P[\{L = j\} \cap T^{-j}E \cap A_0]$$
$$= \sum_{j \in \mathbf{Z}} P[T^j\{L = j\} \cap E \cap T^jA_0] = P(E).$$

If E is a measurable subset of $A_i (i > 0)$, then $U^{-1}E \cap A_i$ and E have the same cardinality and so

$$P(U^{-1}E \cap A_i) = P(E).$$

Thus, if E is a measurable subset of Ω, we have

$$P[U^{-1}E \cap A] = P[U^{-1}(E \cap A) \cap A] = \sum_{i=0}^{\infty} P[U^{-1}(E \cap A_i) \cap A_i]$$
$$= \sum_{i=0}^{\infty} P[E \cap A_i] = P(E \cap A),$$

and so the measure P_A is U-stationary. Also, $P_A(E) = 0$ implies $P(E \cap A) = 0$, which implies

$$P(U^{-n}E \cap U^{-n}A) = 0 \quad \text{for all } n > 0,$$

which implies $P(U^{-n}E) \to 0$ since $P(U^{-n}A) \to 1$ by (g). Hence P_A asymptotically dominates P with respect to U.

Let E_1, E_2, E_3 be the measurable subsets of Ω such that

$$E_1 = \{\omega \in \Omega: \{U^i(\omega)\}_{i=1}^{\infty} \text{ is finite}\}.$$
$$E_2 = \{\omega \in \Omega: \{U^i(\omega)\}_{i=1}^{\infty} \cap \{T^i(\omega)\}_{i=1}^{\infty} \text{ is infinite}\}.$$
$$E_3 = \{\omega \in \Omega: \{U^i(\omega)\}_{i=1}^{\infty} \cap \{T^{-i}(\omega)\}_{i=1}^{\infty} \text{ is infinite}\}.$$

To prove parts (a)–(d) of Theorem 2 it suffices to find sets $\Omega_i \subset E_i$ $(i = 1, 2, 3)$, such that $P(\Omega_i - E_i) = 0$ for each i and (a′) $\{\omega \in \Omega_i$: there exists a type i U-orbit $\supset \{\omega\}\}$ is measurable;

(b′) For $i = 2, 3$, if $\omega \in \Omega_i$ there are only finitely many type i U-orbits $\subset O_T(\omega) \cap E_i$;

(c′) For $i = 1, 2, 3$, if $\omega \in \Omega_i$ then any two type i U-orbits $\subset O_T(\omega) \cap E_i$ are either equal or disjoint;

(d′) For $i = 1, 2, 3$, if $\omega \in \Omega_i$ then for some $j \geqq 0$, $U^j(\omega)$ is an element of a type i U-orbit.

(To see this, one can show without much effort that parts (a)–(d) of Theorem 2 follow from (a′)–(d′) above with

$$\Omega' = (\Omega_1 \cup \Omega_2 \cup \Omega_3) \cap \left[\bigcap_{i \in Z} T^i \left[\bigcup_{j=1}^{3} (E_j - \Omega_j) \right] \right].$$

We can take $\Omega_1 = E_1$. (Note that $\{\omega \in E_1$: there exists a type 1 U-orbit $\supset \{\omega\}\}$ is the measurable set $\{\omega \in E_1: U^i(\omega) = \omega$ for some $i \geqq 1\}$.) It suffices to find Ω_2 and Ω_3. We show how to find Ω_2. (Ω_3 is found by a symmetric argument.) Define $\hat{U}: E_2 \rightarrow E_2$ so that $\hat{U}(\omega) = U^j(\omega)$, where j is the smallest integer $j > 0$ such that $U^j(\omega) > \omega$. From assumption (b) of Theorem 1 it follows that:

(1) For almost all $\omega \in \Omega$,

$$\{\omega' < \omega: \omega' \in E_2 \text{ and } \hat{U}\omega' \geqq \omega\}$$

is finite.

If $F \in \mathscr{F}$, let us call $\omega \in \Omega$ *F-regular* if for every $\omega' \in O_T(\omega) \cap F$ the sets

$$\{\omega'' \in F: \omega'' < \omega'\} \quad \text{and} \quad \{\omega'' \in F: \omega'' > \omega'\}$$

are nonempty. It is easy to see that almost every $\omega \in \Omega$ is F-regular. Let $\Omega^{(1)}$ be the set of all E_2-regular $\omega \in E_2$ such that if $\omega' \in O_T(\omega)$ then the sets

$$\{\omega'' \in E_2: \omega'' < \omega', \hat{U}(\omega'') \geqq \omega'\} \quad \text{and}$$
$$\{\omega'' \in E_2: \omega'' < \omega', U(\omega'') \geqq \omega'\}$$

are finite. From (1), we have that

$$P[E_2 - \Omega^{(1)}] = 0.$$

Also, letting

$$\hat{\Omega} = \{\omega \in \Omega^{(1)}: \text{there exists a bilateral } \hat{U}\text{-orbit} \supset \{\omega\}\},$$

$\hat{\Omega}$ is a measurable set because

$$\hat{\Omega} = \{\omega \in \Omega^{(1)}: \text{there exist infinitely many } \omega' \in O_T(\omega) \cap E_2 \text{ such that } \omega' < \omega,\ \omega \in \{\hat{U}^j(\omega')\}_{j=1}^{\infty} \text{ hold simultaneously}\}.$$

(To see this, if ω is in this latter set we can choose integers $0 = n_0 >$

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$n_1 > n_2 > \ldots$ and unilateral $\hat{U}$-orbits $S_1, S_2, \ldots$ containing ω such that for $i \geqq 1$,

$$S_k \cap \{T^j\omega: n_i \leqq j \leqq n_{i-1}\} \neq \emptyset \quad \text{for all } k \geqq i.$$

Choosing a subsequence $\{S_{i_k}\}$ of $\{S_i\}$ and a subset S of $O_T(\omega)$ so that $I_{S_{i_k}}$, the indicator function of S_{i_k}, converges pointwise on $O_T(\omega)$ as $k \to \infty$ to I_S, we have that S must be a bilateral $\hat{U}$-orbit $\supset \{\omega\}$.) For $k = 0, 1, \ldots$, set

$$G_k = \{\omega \in E_2: \text{there are exactly } k \text{ points } \omega' \text{ in } E_2 \text{ such that } \omega' < \omega \text{ and } U(\omega') \geqq \omega\}.$$

Let $\Omega^{(2)}$ be the set of all $\omega \in \Omega^{(1)}$ which are G_k-regular for every k. Then $P(E_2 - \Omega^{(2)}) = 0$ and, in addition, we have the property;

(2) If $\omega \in \Omega^{(2)}$, there are only finitely many $\hat{U}$-orbits $\subset O_T(\omega)$.

(To see this, let $\omega \in \Omega^{(2)} \cap G_k$. We show there can't be more than k $\hat{U}$-orbits $\subset O_T(\omega)$. Suppose there are $k + 1$ such orbits: $S_1, \ldots, S_{k+1}$. Find $\omega' < \omega$ so that the sets

$$S_i \cap \{\omega'': \omega'' < \omega'\} \quad (1 \leqq i \leqq k + 1),$$

are pairwise disjoint. Pick $\tilde{\omega} < \omega'$, $\tilde{\omega} \in G_k$. Then

$$\{\omega'': \omega'' < \tilde{\omega} \text{ and } \hat{U}(\omega'') \geqq \tilde{\omega}\}$$

has at least $k + 1$ elements, a contradiction.) Let $\Omega^{(3)}$ be the set of all $\omega \in \Omega^{(1)}$ for which there exist two bilateral $\hat{U}$-orbits S_1 and S_2 such that

$$\omega \in S_1 \cap S_2 \quad \text{and} \quad S_1 \cap S_2 \cap \{\omega': \omega' < \omega\} = \emptyset.$$

Then $\Omega^{(3)}$ is measurable because $\Omega^{(3)} = \{\omega \in \Omega^{(1)}$: for any $\omega' < \omega$ there exist $\omega_1, \omega_2 < \omega'$ such that for some $i_1, i_2 > 0$,

$$\hat{U}^{i_1}(\omega_1) = \hat{U}^{i_2}(\omega_2) = \omega$$

but

$$\hat{U}^{i_1-1}(\omega_1) \neq \hat{U}^{i_2-1}(\omega_2)\}.$$

(The proof is analogous to that given to show that Ω is measurable.) Let $\Omega^{(4)}$ be the set of all $\Omega^{(3)}$-regular elements ω of $\hat{\Omega}^{(2)}$ such that

$$\omega' \in O_T(\omega) \cap E_2$$

implies

$$\omega' \in O_T(\omega) \cap \Omega^{(1)}.$$

Then

$P(E_2 - \Omega^{(4)}) = 0$ and:

(3) If $\omega \in \Omega^{(4)}$, any two bilateral $\hat{U}$-orbits $\subset O_T(\omega)$ are equal or disjoint.

(To see this, if $\omega \in \Omega^{(4)}$, let $S_1, \ldots, S_k$ be the finite number of bilateral $\hat{U}$-orbits $\subset O_T(\omega)$. If there are two $\hat{U}$-orbits $\subset O_T(\omega)$ which are not disjoint, then there is a point $\omega' \in O_T(\omega) \cap \Omega^{(3)}$. Find $S_{r_1}, S_{r_2} \in \{S_1, \ldots, S_k\}$ with $r_1 \neq r_2$ and $\omega' > \omega_1 > \omega_2 > \ldots$ such that $S_{r_1}, S_{r_2} \supset \{\omega_i\}$ for each i. This forces $S_{r_1} = S_{r_2}$, a contradiction.) Let $\Omega^{(5)}$ be the set of all $\omega \in E_2$ such that $\hat{U}^i(\omega) \notin \Omega$ for all $i \geqq 0$, and let Ω_2 be the set of all $\Omega^{(5)}$-regular elements of $\Omega^{(4)}$. Then

$P[E_2 - \Omega_2] = 0$ and:

(4) If $\omega \in \Omega_2$, $\hat{U}^i(\omega) \in \hat{\Omega}$ for some $i \geqq 0$.

(To see this, if (4) is false, find $\omega \in \Omega_2 \cap \Omega^{(5)}$. Find $\omega > \omega_1 > \omega_2 \ldots$ such that each $\omega_j \in \Omega^{(5)}$. Let $S_1, S_2, \ldots$ be the unilateral $\hat{U}$-orbits starting at $\omega_1, \omega_2, \ldots$. Then no element of $\bigcup_{i=1}^{\infty} S_i$ is in $\hat{\Omega}$. Let S be a bilateral $\hat{U}$-orbit whose indicator function is a pointwise limit over $O_T(\omega)$ of the indicator functions of a subsequence of $S_1, S_2, \ldots$. Since $S \subset \hat{\Omega}$ and $S \subset \bigcup_{i=1}^{\infty} S_i$, we have a contradiction.)

We wish to show that (a′)–(d′) hold for $i = 2$. To this end, we will need the following lemma.

LEMMA 1. *For all $\omega \in \Omega_2$, the following is true for every $k \geqq 0$:*

(5) *For all $\omega' \in O_T(\omega) \cap E_2$, $U^k(\omega')$ and ω' can't lie in different bilateral $\hat{U}$-orbits.*

Proof. (By induction on k). Statement (5) is true for $k = 0$ by (3). Fix $k > 0$ and suppose (5) holds for smaller k. Let $\omega \in \Omega_2$, $\omega' \in O_T(\omega) \cap E_2$, and suppose S_1, S_2 are bilateral $\hat{U}$-orbits with $\omega' \in S_1$, $U^k(\omega') \in S_2$.

Case 1. $\hat{U}(\omega') = U^j(\omega')$ for some $j > k$. Then $U^k(\omega') < \omega'$ and if s is the smallest $s > 0$ such that $\hat{U}^s(U^k\omega') > \omega'$, then

$$\hat{U}^s(U^k\omega') = \hat{U}(\omega').$$

The left hand side of the preceding equation is in S_2 and the right hand side is in S_1, and so $S_1 = S_2$ by (3).

Case 2. $\hat{U}(\omega') = U^j(\omega')$ for some $1 \leqq j \leqq k$. Then

$$U^j\omega' \in S_1, \quad U^{k-j}(U^j\omega') \in S_2 \quad \text{and} \quad 0 \leqq k - j < k,$$

and so by the induction hypothesis $S_1 = S_2$.

We now continue with our proof that (a′)–(d′) hold for $i = 2$. Fix $\omega \in \Omega_2$. Let $\mathscr{S}$ be the set of all type 2 U-orbits contained in $O_T(\omega) \cap E_2$

and let $\hat{\mathscr{S}}$ be the set of all bilateral $\hat{U}$-orbits contained in $O_T(\omega) \cap E_2$. We show there is a one-to-one map ϕ of $\hat{\mathscr{S}}$ onto $\mathscr{S}$ which carries disjoint orbits into disjoint orbits and satisfies $\phi(S) \supset S(S \in \hat{\mathscr{S}})$. (This, coupled with (2)–(4), gives (b′)–(d′).) If $S \in \hat{\mathscr{S}}$, define $\phi(S)$ to be the bilateral U-orbit

$$\bigcup_{s \in S} \{s, U(s), \ldots, U^{j_s - 1}(s)\},$$

where j_s is the positive integer such that $U^{j_s}(s) = \hat{U}(s)$. Writing

$$\phi(S) = \{\omega_i : i \in \mathbf{Z}\} \quad \text{where} \quad U(\omega_i) = \omega_{i+1} (i \in \mathbf{Z}),$$

we have (since $\phi(S) \supset S$)

$$\liminf_{i \to -\infty} n_i = -\infty, \quad \limsup_{i \to \infty} n_i = \infty,$$

in which $\{n_i : i \in \mathbf{Z}\}$ are the integers such that

$$T^{n_i}(\omega) = \omega_i \quad (i \in \mathbf{Z}).$$

We must have

$$\lim_{i \to -\infty} n_i = -\infty.$$

(Otherwise there would exist $\omega' \in O_T(\omega)$ and $0 > i_1 > i_2 > \ldots$ such that $\omega_{i_k} < \omega'$ for odd k and $\omega_{i_k} > \omega$ for even k, which would imply that there are infinitely many $\omega'' < \omega'$ for which $U(\omega'') \geqq \omega'$, contradicting membership of ω in $\Omega^{(1)}$.) Similarly,

$$\lim_{i \to \infty} n_i = \infty,$$

and $\phi(S)$ is type 2. Thus, ϕ maps $\hat{\mathscr{S}}$ into $\mathscr{S}$. We now show ϕ is onto. Let $S \in \mathscr{S}$. Pick $\omega > \omega_1 > \omega_2 > \ldots$ from S. Let $\hat{S}_1, \hat{S}_2, \ldots$ be the unilateral $\hat{U}$-orbits starting at $\omega_1, \omega_2, \ldots$. Let $\hat{S}$ be a bilateral $\hat{U}$-orbit whose indicator function is a pointwise limit over $O_T(\omega)$ of the indicator functions of a subsequence of $\{\hat{S}_i\}$. Then

$$\hat{S} \subset \bigcup_{i=1}^{\infty} \hat{S}_i \subset S,$$

and so $\phi(\hat{S}) \subset S$, from which it can be deduced that $\phi(\hat{S}) = S$ (since $\phi(\hat{S})$ and S are both bilateral U-orbits which contain no U-cyclic subsets). We now show that ϕ maps disjoint orbits of $\hat{\mathscr{S}}$ into disjoint orbits of $\mathscr{S}$. (A fortiori, this will show that ϕ is one-to-one.) Let $S_1, S_2 \in \hat{\mathscr{S}}$ satisfy $S_1 \cap S_2 = \emptyset$. If

$$\phi(S_1) \cap \phi(S_2) \neq \emptyset,$$

there must exist $\omega' \in O_T(\omega) \cap E_2$ such that $\omega' \in S_1$ and $U^k(\omega') \in S_2$ for some $k \geqq 1$. By Lemma 1, this is impossible.

Thus (b′)–(d′) hold for $i = 2$. To see that (a′) holds, observe that $\{\omega \in \Omega_2$: there exists a type 2 U-orbit $\supset \{\omega\}\} = \{\omega \in \Omega_2$: there exists $\omega' \in O_T(\omega) \cap \hat{\Omega}$ and $j > 0$ such that $U(\omega') \leqq \omega', \ldots, U^{j-1}(\omega') \leqq \omega'$, $U^j(\omega') = \omega\}$, which is a measurable set. Therefore, we have (a′)–(d′), from which (a)–(d) of Theorem 2 follow, and then Theorem 1 follows. All that remains is to prove part (e) of Theorem 2.

Definition. If Λ is a measurable space, let Λ^∞ be the product measurable space of all bilateral sequences $(\omega_i: i \in \mathbf{Z})$ from Λ. If $E \subset \Omega$, a sequence $(\omega_i: i \in \mathbf{Z})$ in Ω^∞ is defined to be *E-regular* if $\omega_i \in E$ implies $\omega_j \in E$ for infinitely many $j > i$ and infinitely many $j < i$. If $V: \Omega \to \Omega$ is a measurable transformation, a bilateral V-orbit S is E-regular if there is an E-regular sequence $(\omega_i: i \in \mathbf{Z})$ from Ω^∞ such that

$$V(\omega_i) = \omega_{i+1} \quad (i \in \mathbf{Z}) \quad \text{and} \quad S = \{\omega_i: i \in \mathbf{Z}\}.$$

LEMMA 2. *Let the invertible $T: \Omega \to \Omega$ be measure-preserving and let $V: \Omega \to \Omega$ be any measurable transformation satisfying* (a), (b) *of Theorem* 1 *(with U replaced by V). If $E \in \mathscr{F}$, then for almost every $\omega \in \Omega$, if $S \subset O_T(\omega)$ is a nice bilateral V-orbit, then S is E-regular.*

Proof. Note that if $\{X_i\}_{i=-\infty}^{\infty}$ is a stationary process with state space Ω then the sequence $(X_i: i \in \mathbf{Z})$ is E-regular with probability one.

By Theorem 2, we may choose a measurable T-invariant subset Ω' of Ω, of measure one, so that if $\omega \in \Omega'$ and $\omega' \in O_T(\omega)$ there is at most one nice V-orbit containing ω' and

$$A = \{\omega \in \Omega': \text{there exists a nice } V\text{-orbit} \supset \{\omega\}\}$$

is a measurable set of positive P-measure. Then V is a one-to-one map of A onto A, and consequently the inverse function for V on A is a measurable function from A to A. Thus there is a measurable map $\psi: A \to A^\infty$ such that if $\psi(\omega) = (\omega_i: i \in \mathbf{Z})$ then

(i) $\omega_0 = \omega$;
(ii) $V(\omega_i) = \omega_{i+1} \quad (i \in \mathbf{Z})$; and

$(\omega_i: i \in \mathbf{Z})$ is the only sequence from A satisfying (i) and (ii). Since the probability measure P_A (see proof of Theorem 1) is V-invariant, we can find a stationary process $\{X_i: i \in \mathbf{Z}\}$ with state space Ω such that X_0 has distribution P_A and $X_{i+1} = V(X_i)$ a.s. $(i \in \mathbf{Z})$. Then $(X_i: i \in \mathbf{Z})$ is E-regular a.s. and $(X_i: i \in \mathbf{Z}) = \psi(X_0)$ a.s. Hence for P-almost all $\omega \in A$, $\psi(\omega)$ is E-regular. Choose a measurable $\Omega'' \subset \Omega'$, of P-measure one, so that if $\omega \in \Omega''$ and $\omega' \in O_T(\omega) \cap A$, then $\psi(\omega)$ is E-regular. Fix $\omega \in \Omega''$. Let $S \subset O_T(\omega)$ be a nice U-orbit. Fix $\omega' \in S$. Then $\omega' \in A$, $S = \{\psi(\omega')_i: i \in \mathbf{Z}\}$, and therefore S is E-regular since $\psi(\omega')$ is E-regular.

The following lemma is proved with the same argument used earlier to prove that the function ϕ maps $\hat{\mathscr{S}}$ into $\mathscr{S}$.

LEMMA 3. *Under the hypotheses of Theorem* 1, *for almost all* $\omega \in \Omega$, *if* $S \subset O_T(\omega)$ *is a bilateral* U*-orbit which is not nice, then* S *is of one of the following four types, where in the following,* $\{n_i: i \in \mathbf{Z}\}$ *is a set of integers such that*

$$S = \{T^{n_i}\omega: i \in \mathbf{Z}\} \quad \textit{and} \quad U(T^{n_i}\omega) = T^{n_{i+1}}\omega, i \in \mathbf{Z}:$$

(i) *S contains a U-cyclic subset and*

$$\lim_{i\to-\infty} n_i = -\infty;$$

(ii) *S contains a U-cyclic subset and*

$$\lim_{i\to-\infty} n_i = \infty;$$

(iii) $\lim_{i\to-\infty} n_i = \lim_{i\to\infty} n_i = -\infty$;
(iv) $\lim_{i\to-\infty} n_i = \lim_{i\to\infty} n_i = \infty$.

Proof of Theorem 2, *part* (e). We show that for almost all ω, there is no bilateral U-orbit $\subset O_T(\omega)$ of type (i) or (iii). (Types (ii) and (iv) are ruled out in the same manner.) Let

$$E = \{\omega \in \Omega: U^n(\omega) = \omega \text{ for some } n > 0\}.$$

Let Ω' be the set of all E-regular points in Ω. Then $P(\Omega') = 1$. It is not hard to see that there is a measurable transformation $V: \Omega \to \Omega$ satisfying the hypotheses of Theorem 1 (with U replaced by V) such that

(a) If $\omega \in \Omega' \cap E$, $V(\omega) = T^i\omega$, where i is the smallest $i > 0$ such that $T^i\omega \in E$;

(b) If $\omega \notin E$, $V(\omega) = U(\omega)$.

By Lemmas 2 and 3, we may choose $\Omega'' \subset \Omega$ of measure one such that

(c) If $\omega \in \Omega''$ and $S \subset O_T(\omega)$ is a nice bilateral V-orbit, then S is E-regular;

(d) If $\omega \in \Omega''$, all the non-nice bilateral U-orbits $\subset O_T(\omega)$ are of types (i)–(iv).

Fix $\omega \in \Omega''$ and $S \subset O_T(\omega)$ of type (i). Say $S = \{\omega_i: i \in \mathbf{Z}\}$, where $U(\omega_i) = \omega_{i+1}$ $(i \in \mathbf{Z})$. Pick i^* to be the smallest integer such that $\omega_{i^*} \in E$. Then

$$\{\omega_i: i < i^*\} \cup \{V^j(\omega_{i^*})\}_{j=0}^{\infty}$$

is a type 2 V-orbit $\subset O_T(\omega)$ which is not E-regular, a contradiction. Hence type (i) orbits are ruled out. Now set

$$E = \{\omega: \{U^i(\omega)\}_{i=1}^{\infty} \cap \{T^i(\omega)\}_{i=1}^{\infty} = \emptyset\}$$

and define $V: \Omega \to \Omega$ and Ω'' exactly as before. Fix $\omega \in \Omega''$ and $S \subset O_T(\omega)$ of type (iii). Say $S = \{\omega_i: i \in \mathbf{Z}\}$, where $U(\omega_i) = \omega_{i+1}$ $(i \in \mathbf{Z})$.

Pick i^* to be the smallest integer such that $\omega_{i^*} \in E$. Then

$$\{\omega_i : i < i^*\} \cup \{V^j(\omega_{i^*})\}_{j=0}^{\infty}$$

is a type 2 V-orbit $\subset O_T(\omega)$ which is not E-regular, a contradiction. Hence type (iii) orbits are ruled out.

3. Transformations whose orbits contain the orbits of a measure-preserving transformation. As we have done previously in this paper, we consider the pair (T, U) where T is invertible and the orbits of U are contained in the orbits of T, except now we require that U be measure-preserving (rather than T). The transformation T is thus a transformation whose orbits contain the orbits of a measure-preserving transformation. In this section of the paper, we address the problem of finding sufficient conditions on the pair (T, U) so that the pointwise ergodic theorem will hold for T. As discussed earlier in the paper, this reduces to the problem of determining conditions under which P is a.m.s. with respect to T. The following theorem, which can be considered as a type of converse to Theorem 1, gives such a condition.

THEOREM 3. *Given the probability space* $(\Omega, \mathscr{F}, P)$, *the invertible measurable transformation* $T: \Omega \to \Omega$, *and the measurable transformation* $U: \Omega \to \Omega$ *such that*

$$U(\omega) \in \{T^i\omega : i \in \mathbf{Z}\}, \omega \in \Omega.$$

Suppose that U *is measure-preserving, aperiodic, and that* $\int_\Omega |L| dP < \infty$, *where* $L: \Omega \to Z$ *is the map such that* $U = T^L$. *Then there exists a* T-*stationary probability measure* Q *on* $\mathscr{F}$ *such that* P *is absolutely continuous with respect to* Q.

Remarks. Since T is invertible, we have from [**2**] that P is a.m.s. with respect to T if and only if there is a T-stationary Q on $\mathscr{F}$ with respect to which P is absolutely continuous. The special case of Theorem 3 where $L > 0$ a.s. was proved in [**2**] by giving an explicit formula for a T-stationary Q with respect to which P is absolutely continuous. (That formula won't work here.)

Before proving Theorem 3, we need a couple of lemmas.

LEMMA 4. *Let the hypotheses of Theorem* 3 *hold and assume in addition that* U *is ergodic. Let* $\hat{\Omega}$ *be the product measurable space* $\Omega \times \mathbf{Z}$, *let* $\hat{L}$: $\hat{\Omega} \to \{-1, 0, 1\}$ *be the map*

$$\begin{aligned} \hat{L}(\omega, n) &= 1, L(\omega) > 0; \\ &= 0, L(\omega) = 0; \\ &= -1, L(\omega) < 0, \end{aligned}$$

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let $\Phi: \hat{\Omega} \to \Omega$ *be the map* $(\omega, n) \to T^n\omega$, *let* $\hat{P}$ *be the probability measure*

$$\hat{P}(E) = \frac{\sum_{j=1}^{\infty} [P(E_j \cap \{L > j\}) + P(E_{-j} \cap \{L < -j\})] + P(E_0)}{P[L = 0] + \int_{\Omega} |L| dP}$$

on $\hat{\Omega}$ (E_j *denotes the section of* $E \subset \hat{\Omega}$ *at* $j \in \mathbf{Z}$), *and let* $T: \hat{\Omega} \to \hat{\Omega}$ *be a measurable transformation such that*

$$\begin{aligned}
\hat{T}(x, 0) &= (x, 0), L(x) = 0; \\
\hat{T}(x, i) &= (x, i + 1), \quad L(x) > 0, 0 \leqq i < L(x) - 1; \\
&= (Ux, 0), \quad L(x) > 0, i = L(x) - 1; \\
&= (x, i - 1), \quad L(x) < 0, L(x) + 1 < i \leqq 0; \\
&= (Ux, 0), \quad L(x) < 0, i = L(x) + 1.
\end{aligned}$$

Then $\hat{T}$ *is an ergodic measure-preserving transformation on* $\hat{\Omega}$ (*regarded as probability space with probability measure* $\hat{P}$) *and*

(a) $\hat{P}(\Omega \times \{0\}) > 0$ *and* $P(E) = \hat{P}(E \times \{0\})/\hat{P}(\Omega \times \{0\})$, $E \in \mathscr{F}$;

(b) $T^{\hat{L}(\omega)}(\Phi(\omega)) = \Phi(\hat{T}(\omega)), \omega \in \hat{\Omega}$;

(c) *If* $(\omega, 0) \in \Omega \times \{0\}$, *then the first element from the list* $\hat{T}(\omega, 0)$, $\hat{T}^2(\omega, 0), \ldots$ *which lies in* $\Omega \times \{0\}$ *is* $(U\omega, 0)$.

Proof. This is straightforward.

Remark. The construction of $\hat{T}$ on $\hat{\Omega}$ in Lemma 4 is a generalization of the "tower construction" [**5**] of ergodic theory, which given that L has range the positive integers, allows us to construct $\hat{T}$ on $\hat{\Omega}$ so that $\hat{T}$ induces U on $\Omega \times \{0\}$ with return time given by L.

LEMMA 5. *Let* $L_1, L_2, \ldots$ *be a stationary sequence of integer-valued random variables with* $E|L_1| < \infty$ *and the integers* $\{\sum_{i=1}^n L_i\}_{n=1}^{\infty}$ *distinct. Then*

$$\mathrm{Prob}\left[\sum_{i=1}^{\infty} L_i \textit{ diverges to } \infty \textit{ or } -\infty\right] = 1.$$

Proof. Since the integers $\{\sum_{i=1}^n L_i\}_{n=1}^{\infty}$ are distinct, it is easy to show that

$$\text{(6)} \qquad \liminf_{n\to\infty} n^{-1}\left|\sum_{i=1}^{n} L_i\right| \geqq \frac{1}{2}.$$

Then,

$$\text{(7)} \qquad \lim_{N\to\infty} \mathrm{Prob}\left[\text{For infinitely many } n \geqq 1, \quad \sum_{i=1}^{n} L_i < 0 \text{ and } \sum_{i=1}^{n+1} L_i > N\right] = 0.$$

To see this, note that from (6) the event in (7) is contained in the event

$$\left[\text{For infinitely many } n, L_{n+1} > N + \frac{n}{4}\right],$$

which has probability no bigger than

$$\sum_{n=1}^{\infty} P\left[L_1 \geqq N + \frac{n}{4}\right],$$

which approaches 0 as $N \to \infty$ since $EL_1^+ < \infty$. Similarly using $EL_1^- < \infty$, one can show

$$\text{(8)} \quad \lim_{N\to\infty} \text{Prob}\left[\text{For infinitely many } n \geqq 1, \sum_{i=1}^{n+1} L_i < 0 \text{ and } \sum_{i=1}^{n} L_i > N\right] = 0.$$

Suppose

$$\limsup_{n\to\infty} \sum_{i=1}^{n} L_i = \infty \quad \text{and} \quad \liminf_{n\to\infty} \sum_{i=1}^{n} L_i = -\infty.$$

Then, given N, there exists M such that:

$$\sum_{i=1}^{n} L_i \notin [0, N], n \geqq M;$$

$$\sum_{i=1}^{n} L_i < 0, \text{ for infinitely many } n \geqq M;$$

$$\sum_{i=1}^{n} L_i < N, \text{ for infinitely many } n \geqq M.$$

Hence the event

$$\left[\limsup_{n\to\infty} \sum_{i=1}^{n} L_i = \infty, \liminf_{n\to\infty} \sum_{i=1}^{n} L_i = -\infty\right]$$

is contained in the union of the events in (7) and (8), and therefore has zero probability, completing the proof.

We are now ready to prove Theorem 3. We introduce the following notation: if Q is a probability measure on a measurable space Λ a.m.s. with respect to a measurable transformation V on Λ, we let $\bar{Q}$ denote the stationary mean for Q, i.e., $\bar{Q}$ is the V-stationary probability measure on Λ such that

$$\bar{Q}(E) = \lim_{n\to\infty} n^{-1} \sum_{i=0}^{n-1} Q(V^{-i}E).$$

If $\{X_i\}_{i=1}^{\infty}$ is a process with state space S, we say that the process is

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a.m.s. if the distribution of the process (which is a probability measure on S_1^∞, the set of all unilateral sequences $(s_1, s_2, \ldots)$ from S) is a.m.s. with respect to the shift transformation on S_1^∞.

Proof of Theorem 3. By the remarks after Theorem 3, we need only show that P is a.m.s. with respect to T. In showing this, because of the ergodic decomposition theorem, we can assume that U is ergodic. Let $\hat{\Omega}$, $\hat{L}$, Φ, $\hat{P}$, $\hat{T}$ be the objects given in Lemma 4. Let $\{S_i\}_{i=-\infty}^{\infty}$ be a stationary ergodic process with state space $\hat{\Omega}$ such that the distribution of S_0 is $\hat{P}$ and $S_{i+1} = \hat{T}(S_i)$, $i \in \mathbf{Z}$. Our first task will be to show that:

$$(9) \quad \text{Either } \sum_{i=1}^{\infty} \hat{L}(S_{-i}) = \infty \text{ a.s.} \quad \text{or} \quad \sum_{i=1}^{\infty} \hat{L}(S_{-i}) = -\infty \text{ a.s.}$$

Let $A = \Omega \times \{0\}$. Let $\{X_i\}_{i=1}^{\infty}$ be the process with state space Ω defined almost surely as a function of the process $\{S_i\}_{i=-\infty}^{\infty}$ as follows: Determine the times $0 > T_1 > T_2 > \ldots$ such that $S_{T_i} \in A$; X_i is then the Ω-coordinate of S_{T_i}, $i \geqq 1$. (Since $\text{Prob}[S_0 \in A] > 0$ and $\{S_i\}$ is ergodic, $\{X_i\}$ is almost surely defined.) Statement (9) will hold if:

$$(10) \quad \text{Either } \sum_{i=1}^{\infty} L(X_i) = \infty \text{ a.s.} \quad \text{or} \quad \sum_{i=1}^{\infty} L(X_i) = -\infty \text{ a.s.}$$

Let $\{\bar{X}_i\}_{i=1}^{\infty}$ be a stationary aperiodic ergodic process with state space Ω such that the distribution of $\bar{X}_0$ is P and $U(\bar{X}_i) = \bar{X}_{i-1}$, $i \geqq 2$. From Lemma 4(c) it follows that $U(X_i) = X_{i-1}$, $i \geqq 2$; using this fact, it is not hard to show that the process $\{X_i\}_{i=1}^{\infty}$ is a.m.s. and that if R is the distribution of $(X_1, X_2, \ldots)$, then $\bar{R}$ is the distribution of $(\bar{X}_1, \bar{X}_2, \ldots)$.

If $i < j$, then

$$\bar{X}_i = T^{L(\bar{X}_{i+1})+\cdots+L(\bar{X}_j)}(\bar{X}_j);$$

therefore the integers $\{L(\bar{X}_1) + \ldots + L(\bar{X}_j)\}_{j=1}^{\infty}$ must be distinct by aperiodicity of the process $\{\bar{X}_i\}$. Applying Lemma 5:

$$(11) \quad \text{Either } \sum_{i=1}^{\infty} L(\bar{X}_i) = \infty \text{ a.s.} \quad \text{or} \quad \sum_{i=1}^{\infty} L(\bar{X}_i) = -\infty \text{ a.s.}$$

Since the events $\{\sum_{i=1}^{\infty} L(\bar{X}_i) = \infty\}$, $\{\sum_{i=1}^{\infty} L(\bar{X}_i) = -\infty\}$ are tail events of the process $\{\bar{X}_i\}_{i=1}^{\infty}$, (10) follows from (11) by Theorem 3(c) of [**2**]. Suppose

$$(12) \quad \sum_{i=1}^{\infty} \hat{L}(S_{-i}) = \infty \text{ a.s.}$$

Under this assumption, we will show that P is a.m.s. with respect to T. (A symmetric argument can be given in case $\sum_{i=1}^{\infty} \hat{L}(S_{-i}) = -\infty$ a.s.) Let $\tilde{L}: \hat{\Omega} \to \{1, 2, \ldots, \infty\}$ be the measurable map:

$$\tilde{L}(\omega) = \inf\{i > 0; \Phi(\hat{T}^i\omega) = T(\Phi(\omega))\}.$$

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If we can show that

(13) $\operatorname{Prob}[\tilde{L}(S_{-i}) > i$ for infinitely many $i > 0] = 0$,

then it will follow from Theorem 1 that $\hat{P}$ is a.m.s. with respect to the transformation $\tilde{T}$ defined on almost all of $\hat{\Omega}$ by:

$$\tilde{T}(\omega) = \hat{T}^{\tilde{L}(\omega)}(\omega).$$

Since, if $(\omega, 0) \in A$, $\{\Phi(\tilde{T}^i(\omega, 0))\}_{i=0}^{\infty}$ is just $\{T^i\omega\}_{i=0}^{\infty}$, we could conclude from Lemma 4(a) that P is a.m.s. with respect to T. We conclude the proof of Theorem 3 by proving (13). From Lemma 4(b), we have for $i > 0$ that

$$T^{\hat{L}(S_{-i})+\cdots+\hat{L}(S_{j-1})}(\Phi(S_{-i})) = \Phi(S_j), \quad -i + 1 \leqq j \leqq 0,$$

and so $\tilde{L}(S_{-i}) > i$ would imply that $\Phi(S_j) \neq T(\Phi(S_{-i}))$ for these same j, which would imply that

$$\hat{L}(S_{-i}) + \ldots + \hat{L}(S_{j-1}) \neq 1$$

for these j, which would imply

$$\hat{L}(S_{-i}) + \ldots + \hat{L}(S_{-1}) \leqq 0,$$

since $|\hat{L}| \leqq 1$. Hence (13) must hold, since we are assuming that (12) holds.

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