On the Analysis of the Kronecker Product of Irreducible Representations of the Symmetric Group

By RAGY H. MAKAR

Introduction

The Kronecker product of two irreducible matrix representations $D(\lambda)$, $D(\mu)$ of the symmetric group on n letters, furnishes a representation of that group, which is, in general reducible. The question of what irreducible representations will appear in the analysis of such products has been dealt with by Prof. F. D. Murnaghan.[1] Indeed he has obtained the analysis of $D(n-p, \lambda_2, \ldots) \times D(n-q, \mu_2, \ldots)$, for the particular values, $p=1$, $q=1$, 2, 3, 4, 5; $p=2$, $q=2$, 3, 4; $p=3$, $q=3$, 4, applying a method which is a recurrence one, in the sense that to obtain such an analysis we have to look at some other analyses which come first in order.

In this short note, the same problem is dealt with, but in an entirely different way. Formulae are given which provide us with a direct method of obtaining the analysis of $D(n-p, \lambda_2 \ldots) \times D(\mu)$, where $p=1, 2, 3$, but (μ) is any partition of n. The formulae, in fact, turn the problem into that of multiplication of Schur-functions [2] (S-functions) of different weights, by the S-functions, $\{1\}$, $\{2\}$, $\{1^2\}$, $\{3\}$, $\{2, 1\}$, $\{1^3\}$. Multiplication by $\{1\}$, $\{2\}$, $\{1^2\}$, $\{3\}$, is governed by very simple rules stated in reference (3), pp. 480-481. A similar rule may be added for the multiplication by $\{1^3\}$. Multiplication by $\{2, 1\}$ can be performed by a very rapid method given in reference (1), pp. 308-309.

S-functions and notation.

The expression

$$\left\{\lambda\right\} = \left\{\lambda_1, \lambda_2, \ldots, \lambda_\kappa\right\} = \frac{1}{n!} \sum_{(a)} n_{(a)} \chi^{(\lambda)}_{(a)} s_1^{\alpha_1} s_2^{\alpha_2} \ldots s_n^{\alpha_n}, \qquad \ldots \ldots (1)$$

is called the S-function corresponding to the partition $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_\kappa)$ of n, and is said to be of weight n. It is shown that [3]

$$p \frac{\partial \{\lambda\}}{\partial S_p} = \sum_{i=1}^{\kappa} \left\{\lambda_1, \lambda_2, \ldots, \lambda_i - p, \ldots, \lambda_\kappa\right\},$$

---

[1] See reference (4) at the end.

[2] Reference (2), chapter VI.

[3] See reference (3), p. 461.

where the S-functions on the right-hand side are of weight $n - p$.

On a second differentiation this gives

$$pq \, \frac{\partial^2 \{\lambda\}}{\partial s_p \, \partial s_q} = \sum_{i=1}^{\kappa} \sum_{\delta=1}^{\kappa} \left\{ \lambda_1, \lambda_2, \ldots, \lambda_i - p, \ldots, \lambda_j - q, \ldots, \lambda_\kappa \right\},$$

and so on.

This justifies the use of the following notation:—

$$p \, \frac{\partial \{\lambda\}}{\partial s_p} = \left\{ \Lambda'_p \right\}, \; p = 1, 2, 3, \qquad \frac{\partial^2 \{\lambda\}}{\partial s_1^2} = \left\{ \Lambda'_{11} \right\}$$

$$\cdots \cdots (2)$$

$$2 \, \frac{\partial^2 \{\lambda\}}{\partial s_2 \, \partial s_1} = \left\{ \Lambda'_{21} \right\} \qquad\qquad \frac{\partial^3 \{\lambda\}}{\partial s_1^3} = \left\{ \Lambda'_{111} \right\}.$$

We also assume that:—

$\{\Lambda_p\}$, $p = 1, 2, 3$, is obtained from $\{\lambda_1, \lambda_2, \ldots, \lambda_\kappa\}$ by subtracting p in all possible ways,

$\{\Lambda_{pq}\}$, $p = 1, 2$, $q = 1$, is obtained by subtracting (p, q) in all possible ways,

$\{\Lambda_{111}\}$ is obtained by subtracting $(1, 1, 1)$ in all possible ways......(3)

The following identities are then written immediately:

$\{\Lambda'_p\} = \{\Lambda_p\}$, $p = 1, 2, 3$, $\qquad \{\Lambda'_{11}\} = \{2\Lambda_{11} + \Lambda_2\}$

$$\cdots \cdots (4)$$

$\{\Lambda'_{21}\} = \{\Lambda_{21} + \Lambda_3\}, \qquad\qquad \{\Lambda'_{111}\} = \{6\Lambda_{111} + 3\Lambda_{21} + \Lambda_3\}.$

Formulae for the analysis

(I) $D(n-1, 1) \times D(\lambda)$ $\qquad = \Sigma D(\mu) - D(\lambda),$
   $\{\Lambda_1\} \{1\}$ $\qquad\qquad\qquad = \Sigma \{\mu\}.$

(II) $D(n-2, 2) \times D(\lambda)$ $\qquad = \Sigma D(\mu) + \Sigma D(\nu) - \Sigma D(\gamma),$
   $\{\Lambda_{11} + \Lambda_2\} \{2\}$ $\qquad\qquad = \Sigma \{\mu\},$
   $\{\Lambda_{11}\} \{1^2\}$ $\qquad\qquad\quad = \Sigma \{\nu\},$
   $\{\Lambda_1\} \{1\}$ $\qquad\qquad\qquad = \Sigma \{\gamma\}.$

(III) $D(n-2, 1^2) \times D(\lambda)$ $\quad = \Sigma D(\mu) + \Sigma D(\nu) - \Sigma D(\gamma) + D(\lambda),$
   $\{\Lambda_{11}\} \{2\}$ $\qquad\qquad\quad = \Sigma \{\mu\},$
   $\{\Lambda_{11} + \Lambda_2\} \{1^2\}$ $\qquad = \Sigma \{\nu\},$
   $\{\Lambda_1\} \{1\}$ $\qquad\qquad\qquad = \Sigma \{\gamma\}.$

(IV) $D(n-3, 3) \times D(\lambda)$ $= \Sigma D(\mu) + \Sigma D(\nu) + \Sigma D(\epsilon) - \Sigma D(\gamma)$
$$- \Sigma D(\delta),$$

$\{\Lambda_{111} + \Lambda_{21} + \Lambda_3\}\{3\}$ $= \Sigma\{\mu\},$ $\{\Lambda_{11} + \Lambda_2\}\{2\} = \Sigma\{\gamma\},$
$\{2\Lambda_{111} + \Lambda_{21}\}\{2, 1\}$ $= \Sigma\{\nu\},$ $\{\Lambda_{11}\}\{1^2\}$ $= \Sigma\{\delta\},$
$\{\Lambda_{111}\}\{1^3\}$ $= \Sigma\{\epsilon\}.$

(V) $D(n-3, 2, 1) \times D(\lambda)$ $= \Sigma D(\mu) + \Sigma D(\nu) + \Sigma D(\epsilon) - \Sigma D(\gamma)$
$$- \Sigma D(\delta) + \Sigma D(\xi),$$

$\{2\Lambda_{111} + \Lambda_{21}\}\{3\}$ $= \Sigma\{\mu\},$ $\{2\Lambda_{11} + \Lambda_2\}\{2\} = \Sigma\{\gamma\},$
$\{4\Lambda_{111} + 2\Lambda_{21} + \Lambda_3\}\{2, 1\} = \Sigma\{\nu\},$ $\{2\Lambda_{11} + \Lambda_2\}\{1^2\} = \Sigma\{\delta\},$
$\{2\Lambda_{111} + \Lambda_{21}\}\{1^3\}$ $= \Sigma\{\epsilon\},$ $\{\Lambda_1\}\{1\}$ $= \Sigma\{\xi\}.$

(VI) $D(n-3, 1^3) \times D(\lambda)$ $= \Sigma D(\mu) + \Sigma D(\nu) + \Sigma D(\epsilon) - \Sigma D(\gamma)$
$$- \Sigma D(\delta) + \Sigma D(\xi) - D(\lambda),$$

$\{\Lambda_{111}\}\{3\}$ $= \Sigma\{\mu\},$ $\{\Lambda_{11}\}\{2\}$ $= \Sigma\{\gamma\},$
$\{2\Lambda_{111} + \Lambda_{21}\}\{2, 1\}$ $= \Sigma\{\nu\},$ $\{\Lambda_{11} + \Lambda_2\}\{1^2\} = \Sigma\{\delta\},$
$\{\Lambda_{111} + \Lambda_{21} + \Lambda_3\}\{1^3\}$ $= \Sigma\{\epsilon\},$ $\{\Lambda_1\}\{1\}$ $= \Sigma\{\xi\}.$

It is sufficient to establish formulae (VI); the others are established in a similar manner. We therefore prove that:

$$\chi^{(n-3, 1^3)}_{(a)} \times \chi^{(\lambda)}_{(a)} = \sum_{(\mu)} \chi^{(\mu)}_{(a)} + \sum_{(\nu)} \chi^{(\nu)}_{(a)} + \sum_{(\epsilon)} \chi^{(\epsilon)}_{(a)} - \sum_{(\gamma)} \chi^{(\gamma)}_{(a)} - \sum_{(\delta)} \chi^{(\delta)}_{(a)}$$
$$+ \sum_{(\xi)} \chi^{(\xi)}_{(a)} - \chi^{(\lambda)}_{(a)}, \ldots \ldots (5)$$

for all classes (a) of the symmetric group.

We note that [1]

$$\chi^{(n-3, 1^3)}_{(a)} = \tfrac{1}{6}(a_1 - 1)(a_1 - 2)(a_1 - 3) - (a_1 - 1)a_2 + a_3$$
$$= \tfrac{1}{6}a_1(a_1 - 1)(a_1 - 2) - a_1 a_2 + a_3 - \tfrac{1}{2}a_1(a_1 - 1) + a_2 + a_1 - 1.$$

(i) The coefficient of $s_1^{a_1} s_2^{a_2} \ldots s_n^{a_n}$ in $\{\Lambda'_{111}\}$. s_1^3 is by (2) the coefficient of $s_1^{a_1-3} s_2^{a_2} \ldots s_n^{a_n}$ in $\dfrac{\partial^3 \{\lambda\}}{\partial s_1^3}$, and by (1) this is

$$a_1(a_1 - 1)(a_1 - 2) n_{(a)} \chi^{(\lambda)}_{(a)} / n!$$

Since [2] $s_1^3 = \{3\} + 2\{2, 1\} + \{1^3\},$

we write $\{\Lambda'_{111}\} [\{3\} + 2\{2, 1\} + \{1^3\}] = \Sigma\{\mu'\},$

and the coefficient of $s_1^{a_1} s_2^{a_2} \ldots s_n^{a_n}$ is $n_{(a)} \sum_{(\mu')} \chi^{(\mu')}_{(a)} / n!$.

[1] Reference (2), p. 139.
[2] Reference (2), p. 86.

Thus:

$$\tfrac{1}{6} a_1 (a_1 - 1)(a_1 - 2) \chi^{(\lambda)}_{(a)} = \tfrac{1}{6} \sum_{(\mu')} \chi^{(\mu')}_{(a)}$$

(ii) The coefficient of $s_1^{a_1} s_2^{a_2} \ldots s_n^{a_n}$ in $\{\Lambda'_{21}\}. s_1 s_2$ is that of

$s_1^{a_1 - 1} s_2^{a_2 - 1} \ldots s_n^{a_n}$ in $2 \dfrac{\partial^2 \{\lambda\}}{\partial s_2 \partial s_1}$, which is $2 a_1 a_2 n_{(a)} \chi^{(\lambda)}_{(a)} / n!$

Since $\qquad\qquad s_1 s_2 = \{3\} - \{1^3\},$

we find

$$a_2 a_2 \chi^{(\lambda)}_{(a)} = \tfrac{1}{2} \sum_{(\nu')} \chi^{(\nu')}_{(a)}, \qquad\qquad \{\Lambda'_{21}\} [\{3\} - \{1^3\}] = \sum \{\nu'\}.$$

(iii) The coefficient of $s_1^{a_1} s_2^{a_2} \ldots s_n^{a_n}$ in $\{\Lambda'_3\}. s_3$ is that of

$$s_1^{a_1} s_2^{a_2} s_3^{a_3 - 1} \ldots s_n^{a_n} \text{ in } 3 \dfrac{\partial \{\lambda\}}{\partial s_3}, \text{ which is } a_3 \chi^{(\lambda)}_{(a)} / n!$$

$$s_3 = \{3\} - \{2, 1\} + \{1^3\},$$

so that

$$a_3 \chi^{(\lambda)}_{(a)} = \tfrac{1}{3} \sum_{(\epsilon')} \chi^{(\epsilon')}_{(a)}, \qquad\qquad \{\Lambda'_3\} [\{3\} - \{2, 1\} + \{1^3\}] = \sum \{\epsilon'\}.$$

(iv) $\qquad\qquad\qquad s_1^{2} = \{2\} + \{1^2\},$

so that

$$\tfrac{1}{2} a_1 (a_1 - 1) \chi^{(\lambda)}_{(a)} = \tfrac{1}{2} \sum_{(\nu')} \chi^{(\gamma')}_{(a)}, \qquad \{\Lambda'_{11}\} [\{2\} + \{1^2\}] = \sum \{\gamma'\}.$$

(v) $\qquad\qquad\qquad s_2 = \{2\} - \{1^2\},$

so that

$$a_2 \chi^{(\lambda)}_{(a)} = \tfrac{1}{2} \sum_{(\delta')} \chi^{(\delta')}_{(a)}, \qquad\qquad \{\Lambda'_2\} [\{2\} - \{1^2\}] = \sum \{\delta'\}.$$

(vi) $\qquad a_1 \chi^{(\lambda)}_{(a)} = \sum_{(\xi')} \chi^{(\xi')}_{(a)}, \qquad\qquad \{\Lambda'_1\} \{1\} = \sum \{\xi'\}.$

Adding up, we have

$$\chi^{(n - 3, 1^3)}_{a)} \times \chi^{(\lambda)}_{(a)} = \tfrac{1}{6} \sum_{(\mu')} \chi^{(\mu')}_{(a)} - \tfrac{1}{2} \sum_{(\nu')} \chi^{(\nu')}_{(a)} + \tfrac{1}{3} \sum_{(\epsilon')} \chi^{(\epsilon')}_{(a)} - \tfrac{1}{2} \sum_{(\gamma')} \chi^{(\gamma')}_{(a)} + \tfrac{1}{2} \sum_{(\delta')} \chi^{(\delta')}_{(a)}$$
$$+ \sum_{(\xi')} \chi^{(\xi')}_{(a)} - \chi^{(\lambda)}_{(a)}.$$

Using the identities (4), we see that the expression on the right-hand side may be simply written as the expression on the right-hand side of (5).

To establish the other formulae we only note that the following characters can be written as:

$$\chi_{(a)}^{(n-1,\,1)} = a_1 - 1,$$

$$\chi_{(a)}^{(n-2,\,2)} = \tfrac{1}{2}a_1(a_1 - 1) + a_2 - 1,$$

$$\chi_{(a)}^{(n-2,1^2)} = \tfrac{1}{2}a_1(a_1 - 1) - a_2 - a_1 + 1,$$

$$\chi_{(a)}^{(n-3,\,3)} = \tfrac{1}{6}a_1(a_1 - 1)(a_1 - 2) + a_1 a_2 + a_2 - \tfrac{1}{2}a_1(a_1 - 1) - a_2,$$

$$\chi_{(a)}^{(n-3,\,2,\,1)} = \tfrac{1}{3}a_1(a_1 - 1)(a_1 - 2) - a_3 - a_1(a_1 - 1) + a_1.$$

Application of the formulae

The simplicity of the formulae is seen through working a few examples. They give analyses of $D(n - p, \lambda_2, \ldots) \times D(n - q, \mu_2 \ldots)$, $p = 1, 2, 3$, q unrestricted, which are valid for all symmetric groups. They also give the analysis of the Kronecker product of any irreducible representation of a given symmetric group, by one or other of certain irreducible representations of that group.

It is sufficient to record here two examples which have been worked out applying formulae (II) and (VI).   If we write (λ) for $D(\lambda)$, these analyses are:

$(n - 2, 2) \times (n - 8, 4^2) = (n - 6, 4, 2) + (n - 7, 5, 2) + 2(n - 7, 4, 3)$
$+ (n - 7, 4, 2, 1) + (n - 7, 3^2, 1) + (n - 8, 6, 2) + 2(n - 8, 5, 3)$
$+ (n - 8, 5, 2, 1) + 3(n - 8, 4^2) + 3(n - 8, 4, 3, 1) + (n - 8, 4, 2^2)$
$+ (n - 8, 3^2, 1^2) + (n - 9, 6, 3) + 2(n - 9, 5, 4) + 2(n - 9, 5, 3, 1)$
$+ 2(n - 9, 4^2, 1) + (n - 9, 4, 3, 2) + (n - 9, 4, 3, 1^2) + (n - 10, 6, 4)$
$+ (n - 10, 5, 4, 1) + (n - 10, 4^2, 2).$

$(12, 1^3) \times (9, 6) = (10, 4, 1) + (10, 3, 1^2) + (9, 6) + 2(9, 5, 1)$
$+ (9, 4, 2) + 2(9, 4, 1^2) + (9, 3, 1^3) + 2(8, 6, 1) + (8, 5, 2)$
$+ 3(8, 5, 1^2) + (8, 4, 2, 1) + (8, 4, 1^3) + (7^2, 1) + (7, 6, 2)$
$+ 2(7, 6, 1^2) + (7, 5, 2, 1) + (7, 5, 1^3) + (6^2, 1^3).$

In deriving $\{\Lambda_1\}$, $\{\Lambda_{11}\}$, $\{\Lambda_2\}, \ldots$, as well as in performing the multiplication of S-functions, we apply the rules given in reference (2), pp. 98-99, to retain the descending order of parts of a partition.

In the second example (λ) is a partition of two parts, so that $\{\Lambda_{111}\}$ does not exist.

REFERENCES.

(1) A. C. Aitken : "The monomial expansion of determinantal symmetric functions," *Proc. Roy. Soc. Edinburgh*, (A), 61 (1943).

(2) D. E. Littlewood : "The theory of group characters and matrix representations of groups," (Oxford, 1940).

(3) F. D. Murnaghan : "On the representations of the symmetric group," *American Journ. of Math.*, 59 (1937).

(4) ———— : "The analysis of the Kronecker product of irreducible representations of the symmetric group," *ibid.*, 60 (1938).

THE UNIVERSITY,
     ABBASSIA, CAIRO,
          EGYPT.