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# A NOTE ON THREE STOCHASTIC PROCESSES WITH IMMIGRATION

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**Abstract**

Three stochastic processes, the birth, death and birth-death processes, subject to immigration can be decomposed into the sum of each process in the absence of immigration and an independent process. We examine these independent processes through their probability generating functions (pgfs) and derive their expectations.

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## 1. Introduction

Bartlett [2] pointed out that the effect of immigration on a homogeneous stochastic process with pgf $[f(u,t)]^N$, $0 < t < \infty$, where $f(u,0) = u$, is such that the pgf of the process subject to Poisson immigration with parameter $\gamma$ is

$$\psi(u,t) = [f(u,t)]^N \exp\left\{\int_0^t \gamma\,[f(u,t-\tau) - 1]\,d\tau\right\}.$$

Here $\tau$ is the time of arrival of the immigrant, and $f(u, t-\tau)$ the pgf at time $t \geq \tau$ starting with one initial individual at time $\tau$, in the absence of immigration. Thus a process subject to immigration is the sum of the original process without immigration, and an independent process with pgf

$$\exp\left\{\int_0^t \gamma[f(u,t-\tau) - 1]\,d\tau\right\}.$$

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We list the relevant pgfs for a death, birth, and birth-death process with immigration, starting with initial populations of size $N$, for $0 < t < \infty$ (see Bartlett [2] and Cox and Miller [3]).

(1) Death:

(a) Pure death process, parameter $\mu$:

$$\phi_1(u,t) = \left[ue^{-\mu t} + 1 - e^{-\mu t}\right]^N .$$

(b) Death and immigration process, parameters $\mu$, $\gamma$:

$$\begin{aligned}\psi_1(u,t) &= \exp\left\{(\gamma/\mu)(1-e^{-\mu t})(u-1)\right\}\left[ue^{-\mu t} + 1 - e^{-\mu t}\right]^N \\ &= \chi_1(u,t)\phi_1(u,t).\end{aligned}$$

(2) Birth:

(a) Pure birth process, parameter $\lambda$:

$$\phi_2(u,t) = \left[\frac{ue^{-\lambda t}}{(1-u(1-e^{-\lambda t}))}\right]^N .$$

(b) Birth and immigration process, parameters $\lambda$, $\gamma$:

$$\begin{aligned}\psi_2(u,t) &= \left[\frac{e^{-\lambda t}}{1-u(1-e^{-\lambda t})}\right]^{\gamma/\lambda}\left[\frac{ue^{-\lambda t}}{(1-u(1-e^{-\lambda t}))}\right]^N \\ &= \chi_2(u,t)\phi_2(u,t).\end{aligned}$$

(3) Birth-Death, $\lambda \neq \mu$

(a) Birth-death process, parameters $\lambda \neq \mu$:

$$\phi_3(u,t) = \left[\frac{\mu(1-e^{-(\lambda-\mu)t}) - u(\mu - \lambda e^{-(\lambda-\mu)t})}{\lambda - \mu e^{-(\lambda-\mu)t} - \lambda u(1-e^{-(\lambda-\mu)t})}\right]^N .$$

(b) Birth-death and immigration process, parameters $\lambda \neq \mu$, $\gamma$:

$$\begin{aligned}\psi_3(u,t) &= \left[\frac{(\lambda-\mu)e^{-(\lambda-\mu)t}}{\lambda - \mu e^{-(\lambda-\mu)t} - \lambda u\,(1-e^{-(\lambda-\mu)t})}\right]^{\gamma/\lambda}\phi_3(u,t) \\ &= \chi_3(u,t)\phi_3(u,t).\end{aligned}$$

(4) Birth-Death, $\lambda = \mu$

(a) Birth-death process, parameters $\lambda = \mu$:

$$\phi_4(u,t) = \left[\frac{1-(\lambda t - 1)\,(u-1)}{1-\lambda t\,(u-1)}\right]^N .$$

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(b) Birth-death and immigration process, parameters $\lambda = \mu$, $\gamma$:

$$\psi_4(u,t) = \frac{1}{[1-\lambda t(u-1)]^{\gamma/\lambda}}\phi_4(u,t) = \chi_4(u,t)\phi_4(u,t).$$

Since the immigration process in each case is a Poisson process with pgf $e^{\gamma t(u-1)}$, it is of interest to compare this with the pgfs $\chi_1(u,t)$, $\chi_2(u,t)$, $\chi_3(u,t)$ and $\chi_4(u,t)$ of the respective independent processes above.

## 2. The death and immigration process

We first compare $e^{\gamma t(u-1)}$ with $\chi_1(u,t) = \exp\{(\gamma/\mu)(1-e^{-\mu t})(u-1)\}$, both Poisson processes, in the range $0 \le u \le 1$. For $x > -1$, we know that

$$\frac{x}{1+x} < 1-e^{-x} < x \quad \text{or} \quad \frac{1}{1+x} < \frac{1-e^{-x}}{x} < 1$$

(see Abramowitz and Stegun [1, Section 4.2.32]). Hence, since $\mu t > 0$ and $\gamma > 0$,

$$\frac{\gamma}{\mu}\left(\frac{\mu t}{1+\mu t}\right) < \frac{\gamma}{\mu}\left(1-e^{-\mu t}\right) < \gamma t, \tag{2.1}$$

and thus for $0 \le u \le 1$ (or $-1 \le u-1 \le 0$),

$$\exp\left\{\frac{\gamma}{\mu}\left(\frac{\mu t}{1+\mu t}\right)(u-1)\right\} \ge \exp\left\{\frac{\gamma}{\mu}\left(1-e^{-\mu t}\right)(u-1)\right\} \ge \exp\{\gamma t(u-1)\},$$
$$Q_1(u,t) \ge \chi_1(u,t) \ge P_1(u,t)$$

as illustrated in Figure 1. Figure 2 gives example plots for specific values of $\gamma$, $\mu$ and $t$.

From (2.1), we see that the slopes of the graphs in Figure 1 at $u = 1$ are such that

$$\frac{\partial P_1(u,t)}{\partial u} = \gamma t > \frac{\partial \chi_1(u,t)}{\partial u} = \frac{\gamma}{\mu}(1-e^{-\mu t}),$$

so that we may conclude that the expectation $\gamma t$ of the Poisson immigration process is larger than $\partial\chi_1(1,t)/\partial u = (\gamma/\mu)(1-e^{-\mu t})$, the expectation of the independent process with pgf $\chi_1(u,t)$. It follows that

$$\frac{\partial\psi_1(1,t)}{\partial u} < \frac{\partial\phi_1(1,t)}{\partial u} + \gamma t, \tag{2.2}$$

or the expectation of the death and immigration process is smaller than the sum of the expectations of a pure death processes and the Poisson immigration process. For in the death and immigration process, every individual of the initial population as well as every immigrant is subject to death, whereas from the right-hand side of (2.2) we can expect the initial population $N$ of the death process to become extinct, while the immigrants grow indefinitely without threat of death.

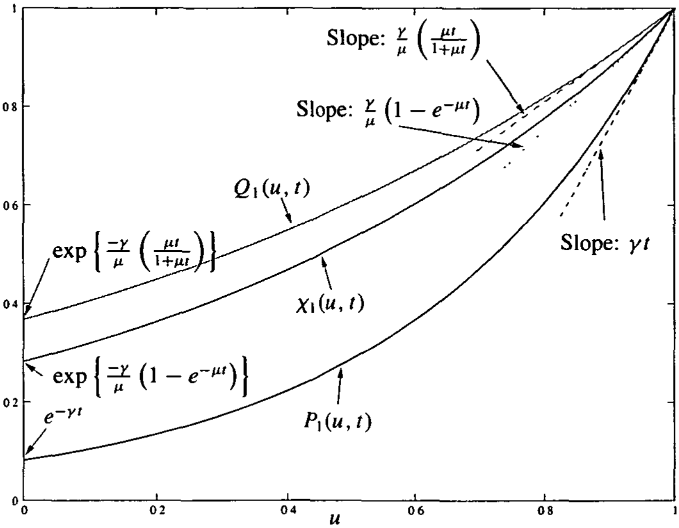


FIGURE 1. Generic graph for the death and immigration process.

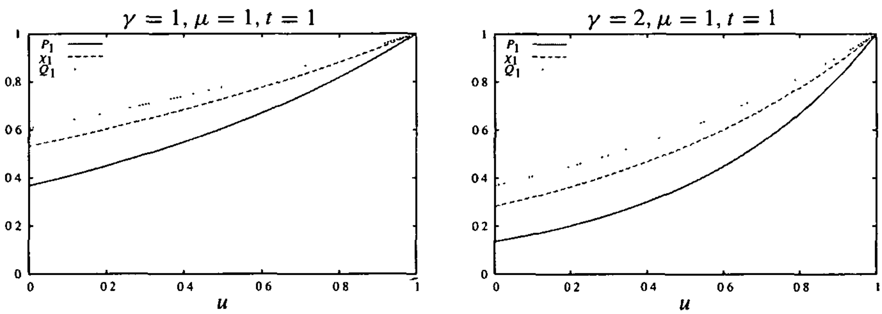


FIGURE 2. Specific examples of the death and immigration process.

## 3. The birth and immigration process

In this case, we compare the pgf $e^{\gamma t(u-1)}$ of the Poisson immigration process with the pgf of the negative binomial, $\chi_2(u,t) = e^{-\gamma t}/\big(1-u(1-e^{-\lambda t})\big)^{\gamma/\lambda}$, in the range $0 \le u \le 1$. We can show that in this range, for all $0 \le t < \infty$,

$$P_1(u,t) = e^{\gamma t(u-1)} \ge e^{-\gamma t}/\big(1-u(1-e^{-\lambda t})\big)^{\gamma/\lambda} = \chi_2(u,t).$$

Figure 3 indicates that both pgfs pass through the points $(0, e^{-\gamma t})$ when $u = 0$ and $(1,1)$ when $u = 1$.

Let us now consider the slopes of $P_1(u,t)$ and $\chi_2(u,t)$; for $P_1(u,t)$ the slopes in

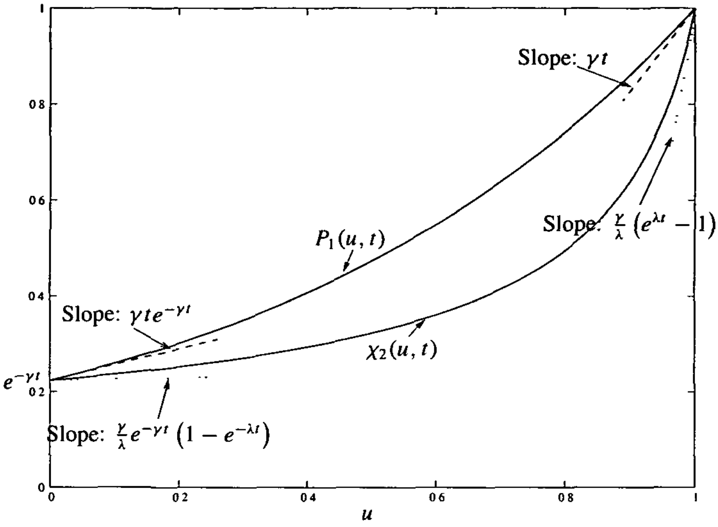


FIGURE 3. Generic graph for the birth and immigration process.

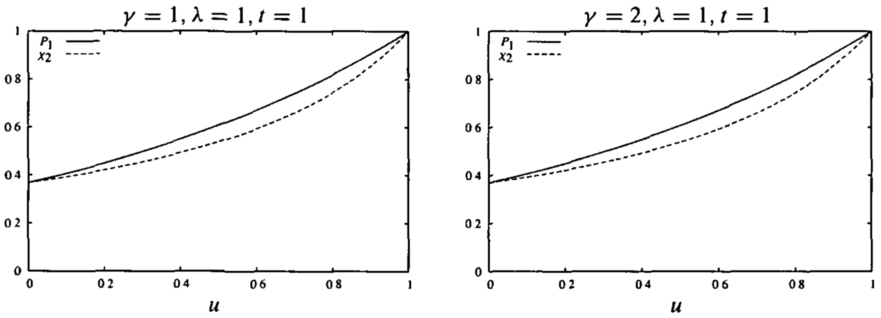


FIGURE 4. Specific examples of the birth and immigration process.

the range $0 \le u \le 1$ are $\gamma t e^{-\gamma t} \le \gamma t e^{\gamma t(u-1)} \le \gamma t$ with the bounds at $u = 0$ and $u = 1$ respectively. For $\chi_2(u, t)$, the slopes are

$$\frac{\gamma}{\lambda}e^{-\gamma t}\left(1 - e^{-\lambda t}\right) \le \frac{\gamma e^{-\gamma t}(1 - e^{-\lambda t})}{\lambda(1 - u(1 - e^{-\lambda t}))^{(\gamma+\lambda)/\lambda}} \le \frac{\gamma}{\lambda}\left(e^{\lambda t} - 1\right)$$

with the bounds once again at $u = 0$ and $u = 1$. We note that at $u = 0$

$$\frac{\partial P_1(0,t)}{\partial u} = \gamma t e^{-\gamma t} > \frac{\gamma}{\lambda}e^{-\gamma t}\left(1 - e^{-\lambda t}\right) = \frac{\partial \chi_2(0,t)}{\partial u},$$

while at $u = 1$

$$\frac{\partial P_1(1,t)}{\partial u} = \gamma t < \frac{\gamma}{\lambda}\left(e^{\lambda t} - 1\right) = \frac{\partial \chi_2(1,t)}{\partial u}. \tag{3.1}$$

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It follows from the continuity in $u$ of $P_1(u,t)$ and $\chi_2(u,t)$, that for all $0 < t < \infty$,

$$P_1(u,t) \geq \chi_2(u,t), \quad 0 \leq u \leq 1.$$

Also, from (3.1) the expectation of the birth and immigration process is larger than the sum of the expectations of a pure birth and an immigration process, since

$$\frac{\partial \psi_2(1,t)}{\partial u} = Ne^{\lambda t} + \frac{\gamma}{\lambda}\left(e^{\lambda t} - 1\right) > Ne^{\lambda t} + \gamma t = \frac{\partial \phi_2(1,t)}{\partial u} + \gamma t.$$

This is to be expected, since each additional immigrant into a population of size $n$ in the birth and immigration process raises the birth parameter from $\lambda n$ to $\lambda(n+1)$, an event which does not occur when one sums a pure birth processes and a Poisson process.

## 4. The birth-death and immigration process for $\lambda > \mu$, or $\lambda < \mu$

We now need to compare $P_1(u,t) = e^{\gamma t(u-1)}$ with

$$\chi_3 = \left[\frac{(\lambda-\mu)e^{-(\lambda-\mu)t}}{\lambda - \mu e^{-(\lambda-\mu)t} - \lambda u\left(1 - e^{-(\lambda-\mu)t}\right)}\right]^{\gamma/\lambda},$$

the pgfs of a Poisson and a negative binomial, in the range $0 \leq u \leq 1$. We assume that $\lambda \neq \mu$, and later consider the case $\lambda = \mu$ for which $\phi_4(u,t)$ and $\chi_4(u,t)$ differ from the pgfs in the present case. For simplicity, we shall take $\lambda > \mu$ in what follows.

We show that for some $u_0 < 1$, for all $0 \leq t < \infty$, in the range $0 \leq u \leq u_0$

$$P_1(u,t) = e^{\gamma t(u-1)} \leq \left[\frac{(\lambda-\mu)e^{-(\lambda-\mu)t}}{\lambda - \mu e^{-(\lambda-\mu)t} - \lambda u(1 - e^{-(\lambda-\mu)t})}\right]^{\gamma/\lambda} = \chi_3(u,t),$$

while for $u_0 < u \leq 1$, $P_1(u,t) > \chi_3(u,t)$. We can see from Figure 5 that the pgf $P_1(u,t)$ passes through the points $(0, e^{-\gamma t})$ and $(1,1)$, while $\chi_3(u,t)$ passes through $\left(0, [(\lambda-\mu)e^{-(\lambda-\mu)t}/(\lambda - \mu e^{-(\lambda-\mu)t}]^{\gamma/\lambda}\right)$ and $(1,1)$. We begin by comparing

$$e^{-\gamma t} \quad \text{with} \quad \left[\frac{(\lambda-\mu)e^{-(\lambda-\mu)t}}{\lambda - \mu e^{-(\lambda-\mu)t}}\right]^{\gamma/\lambda}$$

or taking the $\lambda/\gamma$-th power of both,

$$e^{-\lambda t} \quad \text{with} \quad \frac{(\lambda-\mu)e^{-(\lambda-\mu)t}}{\lambda - \mu e^{-(\lambda-\mu)t}} = \frac{\lambda-\mu}{\lambda e^{(\lambda-\mu)t} - \mu}.$$

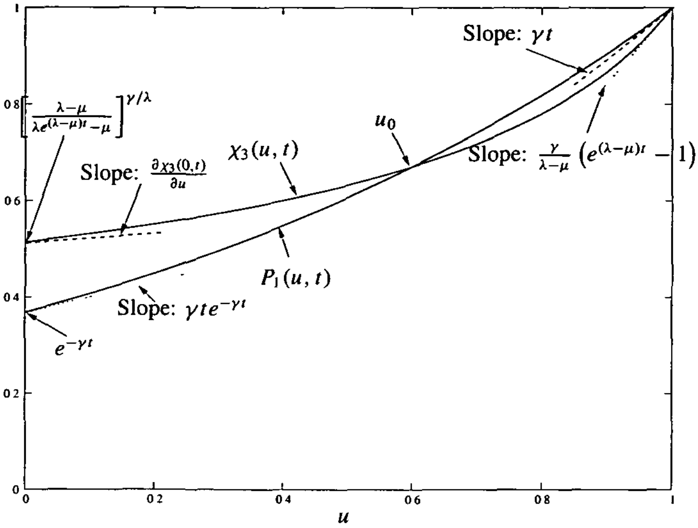


FIGURE 5. Generic graph for the birth-death and immigration process with $\lambda > \mu$.

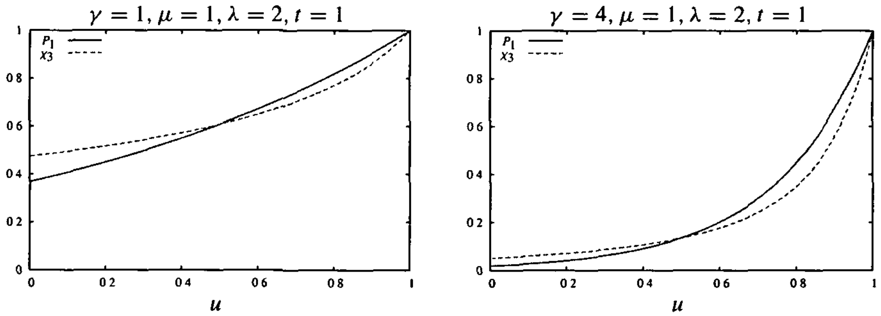


FIGURE 6. Specific examples of the birth-death and immigration process with $\lambda > \mu$.

Since for $0 \le t < \infty$,

$$\frac{\lambda e^{(\lambda-\mu)t} - \mu}{\lambda - \mu} = 1 + \lambda t + \lambda(\lambda - \mu)\frac{t^2}{2!} + \lambda(\lambda - \mu)^2\frac{t^3}{3!} + \cdots$$
$$\le 1 + \lambda t + \lambda^2\frac{t^2}{2!} + \lambda^3\frac{t^3}{3!} + \cdots \le e^{\lambda t}$$

we see that

$$e^{-\lambda t} \le \frac{\lambda - \mu}{\lambda e^{(\lambda-\mu)t} - \mu} \quad \text{or} \quad e^{-\gamma t} \le \left[\frac{\lambda - \mu}{\lambda e^{(\lambda-\mu)t} - \mu}\right]^{\gamma/\lambda},$$

as can be seen in Figure 5.

The slopes of $P_1(u, t)$ and $\chi_3(u, t)$ at $u = 0$ are given by

$$\gamma t e^{-\gamma t} \quad \text{and} \quad \frac{(\lambda - \mu)^{\gamma/\lambda}\gamma(e^{(\lambda-\mu)t} - 1)}{(\lambda e^{(\lambda-\mu)t} - \mu)^{(\gamma+\lambda)/\lambda}} = \frac{\partial \chi_3(0, t)}{\partial u},$$

while at $u = 1$, they are

$$\gamma t \quad \text{and} \quad \frac{\gamma(e^{(\lambda-\mu)t} - 1)}{\lambda - \mu}.$$

Since for $t > 0$ and $\lambda > \mu$,

$$\frac{\gamma}{\lambda - \mu}\left(e^{(\lambda-\mu)t} - 1\right) = \frac{\gamma}{\lambda - \mu}\left((\lambda - \mu)t + (\lambda - \mu)^2\frac{t^2}{2!} + \cdots\right) > \gamma t$$

we conclude that there is a crossover point for $P_1(u, t)$ and $\chi_3(u, t)$ at some $0 < u_0 < 1$. In this case since $\lambda > \mu$ the expectation of the birth-death and immigration process

$$\frac{\partial \psi_3(1, t)}{\partial u} = N e^{(\lambda-\mu)t} + \frac{\gamma}{\lambda - \mu}\left(e^{(\lambda-\mu)t} - 1\right)$$

is larger than the sum of the expectation of a birth-death and a Poisson immigration process

$$\frac{\partial \phi_3(1, t)}{\partial u} + \frac{\partial P_1(1, t)}{\partial u} = N e^{(\lambda-\mu)t} + \gamma t.$$

If $\lambda < \mu$ however, we see that

$$\frac{\psi_3(1, t)}{\partial u} < \frac{\partial \phi_3(1, t)}{\partial u} + \frac{\partial \phi_1(1, t)}{\partial u}.$$

Thus the expectation of the birth-death and immigration process is greater or smaller than the sum of the expectations of a birth-death process without immigration and a Poisson immigration process, depending on whether $\lambda > \mu$ or $\lambda < \mu$. For a population of size $n$, the birth and death parameters $\lambda n$, $\mu n$ both increase with an additional immigrant to $\lambda(n + 1)$, $\mu(n + 1)$, but for $\lambda > \mu$ the expectation of the birth-death and immigration process is greater than the sum of the expectations of a birth-death and a Poisson immigration process, while for $\lambda < \mu$ the reverse is the case. We now consider the case where $\lambda = \mu$.

## 5. The birth-death and immigration process for $\lambda = \mu$

The results for the birth-death and immigration process when the birth and death rates $\lambda$ and $\mu$ are equal differ from those when $\lambda \neq \mu$. We now need to compare

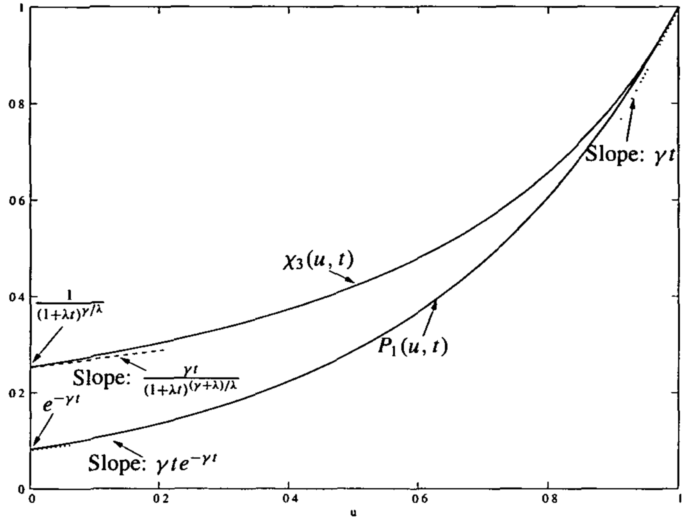


FIGURE 7. Generic graph for the birth-death and immigration process with $\lambda = \mu$.

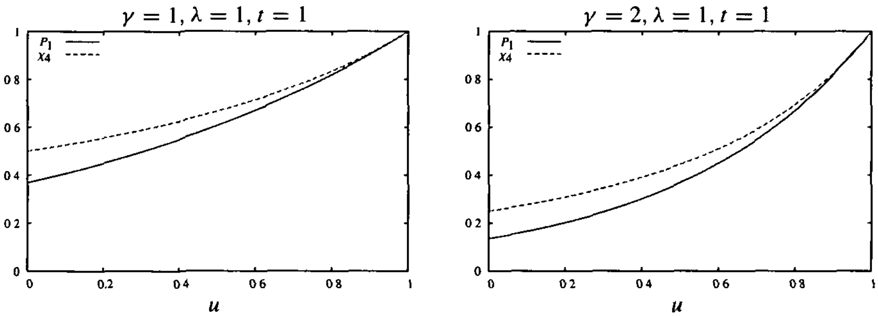


FIGURE 8. Specific examples of the birth-death and immigration process with $\lambda = \mu$.

$P_1(u, t) = e^{\gamma t(u-1)}$ with $\chi_4(u, t) = \left[1 - \lambda t(u - 1)\right]^{-\gamma/\lambda}$. We note that at $u = 0$, since $e^{\lambda t} > 1 + \lambda t$,

$$(1 + \lambda t)^{-\gamma/\lambda} > \left(e^{\lambda t}\right)^{-\gamma/\lambda} = e^{-\gamma t},$$

as can be seen from Figure 7, while at $u = 1$, $P_1(1, t) = \chi_4(1, t) = 1$. The slopes of $P_1(u, t)$ and $\chi_4(u, t)$ are given by

$$\frac{\partial P_1(u,t)}{\partial u} = \gamma t e^{\gamma t(u-1)} \quad \text{and} \quad \frac{\partial \chi_4(u,t)}{\partial u} = \frac{\gamma t}{[1 - \lambda t(u - 1)]^{(\gamma+\lambda)/\lambda}},$$

and are respectively $\gamma t e^{-\gamma t}$ and $(\gamma t)/\left((1 + \lambda t)^{(\gamma+\lambda)/\lambda}\right)$ at $u = 0$ and identically $\gamma t$ at $u = 1$. We can readily show that for any $0 \le u \le 1$

$$\chi_4(u, t) = [1 - \lambda t(u - 1)]^{-\gamma/\lambda} > e^{\gamma t(u-1)} = P_1(u, t). \tag{5.1}$$

For $0 \le u < 1$, writing $u - 1 = -x < 0, x > 0$, we have

$$1 - \lambda t(u-1) = 1 + \lambda t x < e^{\lambda t x}$$

so that

$$\frac{1}{1+\lambda t x} > e^{-\lambda t x} \quad \text{and} \quad \frac{1}{(1+\lambda t x)^{\gamma/\lambda}} > e^{-\gamma t x},$$

from which it follows that the inequality (5.1) is true.

The expectations of the Poisson and the negative binomial are both $\gamma t$; hence the expectation of the birth-death and immigration process and the sum of the expectations of the birth-death process in the absence of immigration and the Poisson process with mean $\gamma t$ are the same, that is,

$$\frac{\partial \psi_4(1,t)}{\partial u} = \frac{\partial \phi_4(1,t)}{\partial u} + \gamma t.$$

## 6. Concluding remarks

The birth, death and birth-death processes with immigration can each be decomposed into the original process in the absence of immigration, and an independent process. For the death process, this independent process is a Poisson process, though not the original Poisson immigration process with parameter $\gamma$. For each of the birth and birth-death processes, the independent process is a negative binomial process which differs for $\lambda \neq \mu$ and $\lambda = \mu$. The expectations of each of these processes with immigration are sometimes smaller and sometimes larger than the sums of the expectations of the original birth-death process in the absence of immigration and the Poisson immigration process with parameter $\gamma$.

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