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# ROOTS OF UNITY AS QUOTIENTS OF TWO ROOTS OF A POLYNOMIAL

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**Abstract**

Let $K$ be a number field. For $f \in K[x]$, we give an upper bound on the least positive integer $T = T(f)$ such that no quotient of two distinct $T$th powers of roots of $f$ is a root of unity. For each $\varepsilon > 0$ and each $f \in \mathbb{Q}[x]$ of degree $d \geq d(\varepsilon)$ we prove that $\log T(f) < (2 + \varepsilon)\sqrt{d \log d}$. In the opposite direction, we show that the constant 2 cannot be replaced by a number smaller than 1. These estimates are useful in the study of degenerate and nondegenerate linear recurrence sequences over a number field $K$.

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## 1. Introduction

Let $K$ be a number field, that is, a finite extension of the field of rational numbers $\mathbb{Q}$, and let

$$f(x) := x^d + a_{d-1}x^{d-1} + \cdots + a_0 = \prod_{i=1}^{d}(x - \alpha_i) \in K[x]$$

be a monic polynomial of degree $d \geq 2$. It may happen that the quotient of two distinct roots of $f$, say, $\alpha_i/\alpha_j$, $1 \leq i < j \leq d$, is a root of unity (for example, for $f(x) = x^d - 2 \in \mathbb{Q}[x]$). We say that the polynomial $f$ is *torsion-free* if no quotient of two distinct roots of $f$ is a root of unity. In the case where $f(x) = \prod_{i=1}^{d}(x - \alpha_i)$ is not torsion-free, the polynomial $f_T(x) := \prod_{i=1}^{d}(x - \alpha_i^T)$ is torsion-free for some $T \in \mathbb{N}$. The smallest positive integer $T = T(f)$ with this property is called the *torsion order* of $f$, so that torsion-free polynomials have torsion order one. For example, for the polynomial $x^d - 2$, we have $T(x^d - 2) = d$. The torsion order of a polynomial can be greater than its degree, for example, $T(\Phi_d) = d$, where $\Phi_d$ is the $d$th cyclotomic polynomial of degree $\varphi(d)$. Note that the torsion order $T(f)$ is independent of the field $K$. If $f \in K[x]$

is not monic and has a leading coefficient $a \in K \setminus \{0\}$ then the torsion order of $f$ is defined as $T(f/a)$.

Here is the main result of this paper.

Theorem 1. *Let $K$ be a number field with $k := [K : \mathbb{Q}]$. For every $f \in K[x]$ of degree $d \geq 2$,*

$$\log T(f) < (1.053\,14 + \sqrt{6k})\sqrt{d \log(kd)}. \tag{1}$$

*Furthermore, if $\varepsilon > 0$ then*

$$\log T(f) < (1 + \varepsilon)(1 + \sqrt{k})\sqrt{d \log d} \tag{2}$$

*provided that $d$ is large enough. On the other hand, for all $\varepsilon > 0$ there exists $d_0(\varepsilon)$ such that for each positive integer $d \geq d_0(\varepsilon)$ there is a monic polynomial $\Phi \in \mathbb{Z}[x] \subset K[x]$ of degree $d$ for which*

$$\log T(\Phi) > (1 - \varepsilon)\sqrt{d \log d}. \tag{3}$$

On replacing $\varepsilon$ by $\varepsilon/2$ in (2) and selecting $K := \mathbb{Q}$, we find that

$$\exp((1 - \varepsilon)\sqrt{d \log d}) < \max_{f \in \mathbb{Q}[x], \deg f = d} T(f) < \exp((2 + \varepsilon)\sqrt{d \log d}) \tag{4}$$

for each $\varepsilon > 0$ and each sufficiently large $d$. Note that the difference between the upper and lower bounds in (4) is only in the constants 1 and 2.

The torsion order $T$ of the polynomial $f(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0$ is related to the properties of the linear recurrence sequence

$$x_{n+d} = -a_{d-1}x_{n+d-1} - \cdots - a_0 x_n,$$

where $n = 1, 2, \ldots$. Such a sequence is called *degenerate* if its characteristic polynomial $f$ has a pair of distinct roots whose ratio is a root of unity and *nondegenerate* otherwise. On replacing $f$ by $f_T$, where $T$ is the torsion order of $f$, one reduces the study of arbitrary linear recurrence sequences to the study of nondegenerate sequences. Theorem 1.2 in [4] asserts that, for some $t \leq T(f)$, each subsequence $x_{tn+s}$, where $n = 1, 2, \ldots$, is either identically zero or nondegenerate. Other applications of the torsion order to the Skolem–Mahler–Lech theorem were mentioned by Berstel and Mignotte [2] (see also [4, Ch. 2]). Schinzel [11] used $T(f)$ to treat an old problem of Pólya [9] on the description of rational functions $\sum_{n=0}^{\infty} u_n x^n \in \mathbb{Q}(x)$ whose numerators are divisible by only finitely many primes. In this sense the bound in (4) gives the best possible (up to a constant) estimate in [11, Theorem 2]. See also [14], where the case $K = \mathbb{Q}$ was considered. Robba [10] investigated the case of a number field, but his upper bound $T(f) \leq 2^{kd+1}$ is weaker than that given in (1).

It seems likely that if $f \in \mathbb{Q}[x]$ is irreducible over the field $\mathbb{Q}$ then the upper bound for $T(f)$ should be given by the inequality

$$\varphi(T(f)) \leq d, \tag{5}$$

where $\varphi$ is Euler's function. Note that equality holds in (5) for cyclotomic polynomials $f = \Phi_d$. Since $\liminf_{n\to\infty}(\varphi(n) \log\log n)/n = e^{-\gamma}$, where $\gamma = 0.577\,21\ldots$ is Euler's constant (see, for example, [5, Theorem 324]), the upper bound (5) via [11, Lemma 3] would imply the inequality

$$T(f) < e^{\gamma} d(\log \log d + 4) \tag{6}$$

for every $d \geq 2$. Unfortunately, the proof of Corollary 2.1 (which is equivalent to (5)) in our paper [3] contains an error. The best known bound in this direction is due to Schinzel [11] who showed that

$$T(f) < e^{3\gamma/2} d^{3/2} (\log \log d + 4)^{3/2}$$

for every irreducible polynomial $f \in \mathbb{Q}[x]$ of degree $d \geq 2$ (which is short of the conjectured bound (6)). In [1, 3, 6, 14], one can find different proofs of the inequality

$$[K(\zeta) : K] \leq d = [K(\alpha) : K]$$

when $\alpha, \alpha'$ are two algebraic numbers conjugate over $K$ and $\zeta := \alpha/\alpha'$ is a root of unity. This shows that one can get rid of at least one root of unity among the ratios of roots of an irreducible polynomial $f \in K[x]$ of degree $d$ by taking some power $t$ for which $\varphi(t) \leq d$.

In the next section we shall prove (3) and give another example which shows that for some $f \in K[x]$ the torsion degree $T(f)$ may tend to infinity as $k \to \infty$ for cyclotomic extensions $K$ of $\mathbb{Q}$. In Section 3 we state two results on the least common multiple of positive integers $b_1, b_2, \ldots, b_m$, whose sum (or the sum of the values of Euler's function $\varphi(b_1), \varphi(b_2), \ldots, \varphi(b_m)$) does not exceed some fixed integer $n$. We give two more lemmas in Section 4, and then complete the proofs of (1) and (2) in Section 5.

## 2. Examples

Let $X(d)$ be the largest integer for which

$$\sum_{p \leq X(d)} (p-1) \leq d, \tag{7}$$

where the sum is taken over prime numbers $p$. Set

$$\Phi(x) := g(x) \prod_{p \leq X(d)} (x^{p-1} + \cdots + x + 1), \tag{8}$$

where $g(x) := \prod_{i=1}^{e} (x - i) \in \mathbb{Z}[x]$ is a polynomial of degree

$$e := d - \sum_{p \leq X(d)} (p-1).$$

Then $\Phi(x) \in \mathbb{Z}[x]$ is a monic polynomial of degree $d$. (Note that $\Phi(x) \in \mathbb{Z}[x] \subset K[x]$ for every number field $K$.) Put

$$Y(d) := \prod_{p \leq X(d)} p, \tag{9}$$

where the product is taken over primes $p$. Clearly, $\Phi_T$ is a torsion-free polynomial for $T = Y(d)$, since the $T$th powers of the roots of $\Phi$ are $i^T$, where $i = 1, \dots, e$, and 1, because $\zeta_p^{jY(d)} = 1$ whenever $p \le X(d)$ and $j = 1, \dots, p-1$. (Throughout, $\zeta_r := e^{2\pi i/r}$ is the primitive $r$th root of unity.) On the other hand, if some prime $p \le X(d)$ does not divide $T$ then $\Phi_T$ is not torsion-free, because the quotient of two distinct roots of $\Phi_T$, say $\zeta_p^{2T}$ and $\zeta_p^T$, is a root of unity. Consequently, for the polynomial defined in (8),

$$T(\Phi) = Y(d).$$

Now, using the prime number theorem $p_i \sim i \log i$ as $i \to \infty$, from (7) and (9) one can easily derive that

$$X(d) \sim \sqrt{d \log d} \quad \text{and} \quad \log Y(d) \sim X(d)$$

as $d \to \infty$. Hence the inequality $\log T(\Phi) = \log Y(d) > (1-\varepsilon)\sqrt{d \log d}$ holds for every $d \ge d_0(\varepsilon)$ and every $\varepsilon > 0$ . This proves (3).

Suppose next that $K = \mathbb{Q}(\zeta_m)$, where $\varphi(m) = k$, so that $k = [K : \mathbb{Q}]$. Let us take

$$G(x) := g_1(x) \prod_{m<p\le X_1(d)} (x^p - \zeta_m),$$

where the product is taken over primes $p$, the number $X_1(d)$ is the largest integer for which $\sum_{m<p\le X_1(d)} p \le d$, and $g_1(x) := \prod_{i=1}^{e_1}(x-i)$ with $e_1 := d - \sum_{m<p\le X_1(d)} p$. By the same argument as above, we see that $G \in K[x]$ is a monic polynomial of degree $d$ whose torsion order is

$$T(G) = m \prod_{m<p\le X_1(d)} p > m \exp((1-\varepsilon)\sqrt{d \log d})$$

for each $\varepsilon > 0$ and each sufficiently large $d$. Since $m > k$, it follows that

$$\log T(G) > \log k + (1-\varepsilon)\sqrt{d \log d}$$

for each $\varepsilon > 0$ and each sufficiently large $d$.

## 3. Two results for the least common multiple

LEMMA 2. *Let $b_1, \dots, b_m$ be some positive integers satisfying $b_1 + \cdots + b_m \le n$, where $n \ge 2$. Then*

$$\operatorname{lcm}(b_1, \dots, b_m) < \exp(1.053\,14\sqrt{n \log n}),$$

*where* lcm *stands for the least common multiple. Furthermore, for each $\varepsilon > 0$ there is an integer $n_0(\varepsilon)$ such that, for every $n \ge n_0(\varepsilon)$,*

$$\operatorname{lcm}(b_1, \dots, b_m) < \exp((1+\varepsilon)\sqrt{n \log n}).$$

The first part of Lemma 2 was proved by Massias [7]. In fact, $\operatorname{lcm}(b_1, \dots, b_m)$ is the order of an element consisting of $m$ disjoint cycles of lengths $b_1, \dots, b_m$ in the full symmetric group on $n$ elements $S_n$. Let $M(n)$ be the maximal order of an element in $S_n$. Landau proved that $\log M(n) \sim \sqrt{n \log n}$ as $n \to \infty$ and Shah [12] gave a more precise asymptotic formula

$$\log M(n) = \sqrt{n \log n}(1 + \log\log n/(2 \log n) + O(1/\log n)).$$

This implies the second part of Lemma 2. See also [8, 13] for further work on this problem. It is interesting to note that the upper bound 1.053 13 . . . is attained for the symmetric group $S_{1\,319\,766}$.

The following lemma was proved by Berstel and Mignotte [2].

Lemma 3. *Let $b_1, \ldots, b_m$ be positive integers satisfying $\varphi(b_1) + \cdots + \varphi(b_m) \le n$, where $n \ge 2$. Then*

$$\operatorname{lcm}(b_1, \ldots, b_m) < \exp(\sqrt{6n \log n}).$$

*Furthermore, the constant $\sqrt{6}$ can be replaced by any constant strictly greater than 1 if $n$ is sufficiently large.*

## 4. Some divisibility results for the torsion order of a polynomial

Let $f \in K[x]$ be a separable polynomial of degree $d \ge 2$. We say that two of its roots $\alpha$ and $\alpha'$ belong to the same equivalence class if their quotient is a root of unity. Suppose that there are $s := s(f)$ distinct equivalence classes. It is easy to see that each equivalence class contains the same number of elements, say, $\ell := \ell(f)$ roots of $f$, so

$$d = s\ell = s(f)\ell(f). \tag{10}$$

Now, let $f \in K[x]$ be an irreducible polynomial of degree $d \ge 2$, and let $\alpha$ be one of its roots. Suppose that $r$ is the largest positive integer for which

$$\mathbb{Q}(\zeta_r) \subseteq K(\alpha). \tag{11}$$

Then $K(\zeta_r) \subseteq K(\alpha)$. Since $K(\zeta_r)$ is a normal extension of $K$, the field $K(\zeta_r)$ is contained in the field $K(\alpha')$ for any conjugate $\alpha'$ of $\alpha$ over $K$. Hence $\mathbb{Q}(\zeta_r)$ is contained in every field $K(\alpha')$ too. Thus $r := r(f, K)$ is independent of the choice of $\alpha$ and depends only on the polynomial $f$ and the field $K$. Of course,

$$r(f, K) = w(K(\alpha)), \tag{12}$$

where $w(F)$ stands for the number of distinct roots of unity lying in the field $F$.

We claim that if $\alpha \ne \alpha'$ and $\alpha/\alpha'$ is a root of unity (so that $\ell \ge 2$) then

$$\alpha^{\ell r} = (\alpha')^{\ell r}. \tag{13}$$

Indeed, write $\gamma(C)$ for the product of conjugates of $\alpha$ belonging to the equivalence class $C$. From $\alpha, \alpha' \in C$ we obtain $\alpha^\ell = \zeta\gamma(C)$ and $(\alpha')^\ell = \zeta'\gamma(C)$ for some roots of unity $\zeta, \zeta'$. It is clear that $s = s(f)$ equivalence classes are blocks of imprimitivity of the Galois group $\operatorname{Gal}(K(\alpha)/K)$. Hence every automorphism $\sigma \in \operatorname{Gal}(K(\alpha)/K)$ satisfying $\sigma(\alpha) = \alpha$ is a permutation of the set $C$, so it maps $\gamma(C)$ to $\gamma(C)$. Thus $\gamma(C) \in K(\alpha)$.

This yields $\zeta = \alpha^{\ell}/\gamma(C) \in K(\alpha)$, so that $\zeta \in \mathbb{Q}(\zeta_r)$, by (11). Hence $\zeta^r = 1$. By the same argument, $(\zeta')^r = 1$. Therefore,

$$\left(\frac{\alpha}{\alpha'}\right)^{\ell r} = \left(\frac{\zeta\gamma(C)}{\zeta'\gamma(C)}\right)^r = \frac{\zeta^r}{(\zeta')^r} = 1.$$

This proves (13) and so implies the following result.

Lemma 4. *The torsion order of an irreducible polynomial $f \in K[x]$ satisfies*

$$T(f) \mid \ell(f) r,$$

*where $r := r(f, K)$ is defined in (11) and (12).*

Next we extend this result to the product of two irreducible polynomials.

Lemma 5. *Let $f_1, f_2 \in K[x]$ be two distinct irreducible polynomials having $\ell_1 := \ell(f_1)$ and $\ell_2 := \ell(f_2)$ elements in their equivalence classes and with $r_1 := r(f_1, K)$ and $r_2 := r(f_2, K)$ defined as in (11) and (12). Then*

$$T(f_1 f_2) \mid \mathrm{lcm}(\ell_1 \,\mathrm{lcm}(r_1, r_2), \ell_2 \,\mathrm{lcm}(r_1, r_2)).$$

Proof. In view of Lemma 4 it suffices to prove that if $\alpha$ is a root of $f_1$ and $\beta$ is a root of $f_2$ such that $\alpha/\beta$ is a root of unity then

$$\alpha^L = \beta^L,$$

where $L := \mathrm{lcm}(\ell_1 r, \ell_2 r)$ and $r := \mathrm{lcm}(r_1, r_2)$. Suppose that $\alpha$ and $\beta$ belong to the equivalence classes $C_1$ and $C_2$. As above, let $\gamma(C_1)$ and $\gamma(C_2)$ be the product of conjugates of $\alpha$ from $C_1$ and the product of conjugates of $\beta$ from $C_2$. Then

$$\alpha^{\ell_1} = \xi_1 \gamma(C_1) \quad \text{and} \quad \beta^{\ell_2} = \xi_2 \gamma(C_2) \tag{14}$$

with some roots of unity $\xi_1, \xi_2$.

Let $S$ be the set of $\deg f_1 + \deg f_2$ roots of the polynomial $f_1 f_2$. The equivalence class of $S$ containing $\alpha$ consists of $C_1 \cup C_2$. Consider the Galois group $\mathrm{Gal}(K(\alpha, \beta)/K)$ as permutations of the set $S$. Each element $\sigma \in \mathrm{Gal}(K(\alpha, \beta)/K)$ that fixes $\alpha$ (or $\beta$) permutes the class $C_1$ and the class $C_2$. Hence the products $\gamma(C_1)$ and $\gamma(C_2)$ both lie in the intersection $K(\alpha) \cap K(\beta)$. Thus $\xi_1 \in K(\alpha)$ and $\xi_2 \in K(\beta)$, by (14). From (14), the quotient $\zeta := \gamma(C_1)^{L/\ell_1 r}/\gamma(C_2)^{L/\ell_2 r}$ is a root of unity, because $\alpha/\beta$ is a root of unity. Note that $\zeta, \xi_1 \in K(\alpha)$ yields $\zeta, \xi_1 \in \mathbb{Q}(\zeta_{r_1})$, by the definition of $r_1$. Similarly, $\zeta, \xi_2 \in K(\beta)$ yields $\zeta, \xi_2 \in \mathbb{Q}(\zeta_{r_2})$. It follows that $\zeta, \xi_1, \xi_2$ lie in the compositum

$$\mathbb{Q}(\zeta_{r_1})\mathbb{Q}(\zeta_{r_2}) = \mathbb{Q}(\zeta_{r_1}, \zeta_{r_2}) = \mathbb{Q}(\zeta_{\mathrm{lcm}(r_1, r_2)}) = \mathbb{Q}(\zeta_r).$$

Now, using the fact that the root of unity $\zeta \xi_1^{L/\ell_1 r} \xi_2^{-L/\ell_2 r}$ belongs to the field $\mathbb{Q}(\zeta_r)$ (so that its $r$th power is 1), we obtain

$$\frac{\alpha^L}{\beta^L} = \frac{\xi_1^{L/\ell_1} \gamma(C_1)^{L/\ell_1}}{\xi_2^{L/\ell_2} \gamma(C_2)^{L/\ell_2}} = \left(\frac{\xi_1^{L/\ell_1 r} \zeta}{\xi_2^{L/\ell_2 r}}\right)^r = 1,$$

which completes the proof of the lemma. □

## 5. Proof of the upper bound for torsion order

Without loss of generality, we may assume that $f \in K[x]$ is monic. Let us write $f$ in the form

$$f(x) = f_1(x)^{k_1} f_2(x)^{k_2} \cdots f_m(x)^{k_m}$$

for distinct irreducible monic polynomials $f_1, \ldots, f_m \in K[x]$ and $k_1, \ldots, k_m \in \mathbb{N}$. Obviously, $T(f) = T(f_1 \cdots f_m)$ and the degree of the polynomial $f_1 \cdots f_m$ is smaller than $\deg f$ provided that at least one $k_i$ is greater than 1. So we may assume that $k_1 = \cdots = k_m = 1$. Then, by Lemmas 4 and 5, for $f = f_1 \cdots f_m$,

$$T(f) \mid \operatorname{lcm}(\ell_1, \ldots, \ell_m) \operatorname{lcm}(r_1, \ldots, r_m) \tag{15}$$

with $r_i := r(f_i, K)$ defined in (11) and (12), because $\operatorname{lcm}(\ell_i \operatorname{lcm}(r_i, r_j), \ell_j \operatorname{lcm}(r_i, r_j))$ divides $\operatorname{lcm}(\ell_i, \ell_j) \operatorname{lcm}(r_i, r_j)$ for all indices $i, j$ in the range $1 \le i, j \le m$. From (10) we see that $\ell_i \mid d_i$, where $d_i = \deg f_i$. Thus $\operatorname{lcm}(\ell_1, \ldots, \ell_m)$ divides $\operatorname{lcm}(d_1, \ldots, d_m)$. Using the equality $\sum_{i=1}^m d_i = d = \deg f$ from Lemma 2 we deduce that

$$\operatorname{lcm}(\ell_1, \ldots, \ell_m) \le \operatorname{lcm}(d_1, \ldots, d_m) < \exp(1.053\,14\sqrt{d \log d}). \tag{16}$$

Suppose that $\alpha_i$ is a root of $f_i$ for $i = 1, \ldots, m$. Then we see that $\mathbb{Q}(\zeta_{r_i}) \subseteq K(\alpha_i)$ from (11). Hence $[\mathbb{Q}(\zeta_{r_i}) : \mathbb{Q}] = \varphi(r_i)$ divides

$$[K(\alpha_i) : \mathbb{Q}] = [K : \mathbb{Q}] \cdot [K(\alpha_i) : K] = kd_i.$$

In particular, $\varphi(r_i) \le kd_i$ for each $i = 1, \ldots, m$ and thus

$$\sum_{i=1}^m \varphi(r_i) \le k \sum_{i=1}^m d_i = kd.$$

Therefore, Lemma 3 implies that

$$\operatorname{lcm}(r_1, \ldots, r_m) < \exp(\sqrt{6kd \log(kd)}). \tag{17}$$

Now, from (15), multiplying (16) and (17) we derive (1).

Next, suppose that $\varepsilon > 0$ and that $d$ is large enough. Then, by Lemmas 2 and 3, inequality (16) becomes

$$\operatorname{lcm}(\ell_1, \ldots, \ell_m) \le \operatorname{lcm}(d_1, \ldots, d_m) < \exp((1 + \varepsilon)\sqrt{d \log d}),$$

whereas (17) is replaced by

$$\operatorname{lcm}(r_1, \ldots, r_m) < \exp((1 + \varepsilon/2)\sqrt{kd \log(kd)}) < \exp((1 + \varepsilon)\sqrt{kd \log d}).$$

Multiplying these two inequalities, we derive (2) in view of (15).

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