STRUCTURE OF PSEUDO-SEMISIMPLE RINGS

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(Received 2 May 1989; revised 10 October 1989)

Communicated by B. J. Gardner

Abstract

A ring $R$ is called right pseudo-semisimple if every right ideal not isomorphic to $R$ is semisimple. Rings of this type in which the right socle $S$ splits off additively were characterized; such a ring has $S^2 = 0$. The existence of right pseudo-semisimple rings with zero right singular ideal $Z$ remained open, except for the trivial examples of semisimple rings and principal right ideal domains. In this work we give a complete characterization of right pseudo-semisimple rings with $S^2 = 0$. We also give examples of non-trivial right pseudo-semisimple rings with $Z = 0$; in fact it is shown that such rings exist as subrings in every infinite-dimensional full linear ring. A structure theorem for non-singular right pseudo-semisimple rings, with homogeneous maximal socle, is given. The general case is still open.


Throughout this paper, $S$, $Z$ and $J$ will stand for the right socle, the right singular ideal and the Jacobson radical of a ring $R$. A local ring $R$ will mean one in which $J \neq 0$ and $R/J$ is a division ring. For a subset $X$ of $R$, $X^0$ and $^0X$ will denote the right and left annihilators in $R$. It is true in general that $S \leq ^0J$, and if $R/J$ is semisimple (in particular if $R$ is local), then $S = ^0J$. We also note that $Z$ and $J$ contain no non-zero idempotents of $R$; hence a regular ring $R$ has $Z = J = 0$.

The split extension $R \rtimes M$ of a ring $R$ by an $(R - R)$-bimodule $M$, is the ring of all ordered pairs $(r, m)$, $r \in R$ and $m \in M$; with addition defined componentwise and multiplication defined by $(r, m) \cdot (r', m') = (rr', rm' + mr')$.

This research was supported in part by the NSERC of Canada, grants A4033 and A8778.

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The set of positive integers will be denoted by $\mathbb{N}$.

1. General results

**Lemma 1.1.** Let $R$ be a right pseudo-semisimple ring. If $R = A \oplus B$ for right ideals $A$ and $B$, then $A$ or $B$ is semisimple.

**Proof.** Assume that $A$ is not semisimple; then $A \cong R$. Hence $A = A_1 \oplus B_1$ with $A_1 \cong A \cong R$ and $B_1 \cong B$. Iterating this process, we obtain $R = A_n \oplus B_n \oplus \cdots \oplus B_1$ for every $n \in \mathbb{N}$. Hence $R$ contains the right ideal $\bigoplus_{i \in \mathbb{N}} B_i$ with $B_i \cong B$. This right ideal is not finitely generated, and therefore not isomorphic to $R$. Thus it is semisimple, and hence $B$ is semisimple.

**Corollary 1.2.** Let $R$ be a right pseudo-semisimple ring which is not semisimple.

1. If $R = A \oplus B$ for non-zero right ideals $A$ and $B$, then exactly one of them is semisimple and the other one is isomorphic to $R$. In particular neither $A$ nor $B$ is an ideal.

2. If $e$ is a non-trivial idempotent of $R$, then $eR(1 - e) \neq 0$.

**Lemma 1.3.** If $R$ is right pseudo-semisimple, then $R/S$ is a principal right ideal domain.

**Proof.** It is obvious that $R/S$ is a principal right ideal ring. Consider $a, b \in R$ with $a \notin S$ and $ab \in S$. Then $aR \cong R$, and hence $R = a^0 \oplus C$ where $C \cong aR \cong R$. Thus $a^0 \leq S$ by Lemma 1.1. Write $b = x + y$ with $x \in a^0$ and $y \in C$. Since $S \geq abR = ayR \cong yR$, we have $y \in S$. Therefore $b = x + y \in S$.

It follows by Lemma 1.3 that a ring $R$ with zero right socle is right pseudo-semisimple if and only if it is a principal right ideal domain. We call a right pseudo-semisimple ring $R$ non-trivial if $0 \neq S \neq R$.

**Lemma 1.4.** Let $R$ be a non-trivial right pseudo-semisimple ring. The following hold in $R$:

1. $S$ is the smallest essential right ideal;
2. $0^0 S = Z \leq S \cap J$;
3. $S = Z \oplus I$ where $Z$ and $I$ consist of homogeneous components (hence $I$ is also an ideal);
4. $S = 0^0 x$ for every $0 \neq x \in J$, in particular if $J \neq 0$, then $S = 0^0 J$.
(5) if \( a \notin S \), then \( aR \cap S = aS \) and \( S = aS \oplus K \) where \( K \) is isomorphic to a direct summand of \( R \) (hence finitely generated);

(6) \( SZ = 0 \) and \( Z \) is torsion-free divisible as a left \( R/S \)-module.

**PROOF.** (1) Consider a non-zero right ideal \( A \) of \( R \). Then either \( A \leq S \) or \( A \cong R \), and hence \( A \) contains a copy of \( S \). In either case \( A \cap S \neq 0 \).

(2) \( Z = 0 \) follows by (1). This also proves that \( Z \neq R \) and hence \( Z \leq S \). Then \( Z^2 \leq ZS = 0 \), and consequently \( Z \leq J \).

(3) If \( X \) and \( Y \) are minimal right ideals with \( X \leq Z \) and \( Y \notin Z \), then \( XS = 0 \) and \( YS \neq 0 \). Thus \( X \neq Y \).

(4) That \( S \leq J \leq x \) is obvious. Consider an element \( a \in R \) such that \( a \notin S \). Then \( a^0 \leq S \) and is a direct summand of \( R \) (since \( aR \cong R \)). Therefore \( a^0 \cap J = 0 \) and hence \( ax \neq 0 \). This proves that \( x \leq S \), and consequently \( S = xJ = x \).

(5) That \( aS = aR \cap S \) follows by Lemma 1.3. Let \( S = aS \oplus K \). Then \( R \cong aR + S = aR \oplus K \).

(6) The result is trivial in the case \( Z = 0 \). Assume that \( Z \neq 0 \). Since \( Z \leq J \), \( S = 0 \) by (4). Hence \( Z \) is a torsion-free \( R/S \)-module.

Next we prove that \( Z \leq A \) for every right ideal \( A \) not contained in \( S \). Write \( Z = (A \cap Z) \oplus B \). Then \( A \oplus B \cong R \). Since \( B \leq Z \), \( B^2 = 0 \) by (2), and hence \( B = 0 \). Thus \( Z = A \cap Z \) and \( Z \leq A \).

Now consider an element \( a \in R \) such that \( a \notin S \). Since \( aR \cong R \), \( aR = bR \) with \( b^0 = 0 \). If \( br \in Z \), then \( brS = 0 \), and hence \( rS = 0 \) and \( r \in Z \) by (2). Therefore \( bR \cap Z = bZ \).

Now

\[
\begin{align*}
aZ &= aRZ = bRZ = bZ = bR \cap Z = Z.
\end{align*}
\]

Hence \( Z \) is divisible as a left \( R/S \)-module.

**LEMMA 1.5.** Let \( R \) be a non-trivial right pseudo-semisimple ring.

(1) \( Z = S \) if and only if \( S^2 = 0 \) and \( S \leq J \leq S^0 \).

(2) \( Z \neq S \) if and only if \( S \) contains a countable set of non-zero orthogonal idempotents and \( S^0 = J < S \).

**PROOF.** (1) is obvious.

(2) Assume that \( Z \neq S \). Then \( S^2 \neq 0 \), and hence \( S \) contains a non-zero idempotent. By an argument similar to that given in Lemma 1.1, we conclude that \( S \) contains a countable set of non-zero orthogonal idempotents \( \{e_n\} \).

Write \( S = (J \cap S) \oplus X \). It is clear that the projections of the \( e_n \) into \( X \) are still non-zero orthogonal idempotents, and hence \( X \) is not finitely generated. It then follows by Lemma 1.4(5) that \( J \neq R \). Thus \( J \leq S \), and since \( S^2 \neq 0 \), \( J < S \).
Let $B = S^0$. Then $J \leq B$ and so $J \leq B \cap S$. Moreover $(B \cap S)^2 = 0$, and hence $B \cap S \leq J$. Thus $B \cap S = J$ and $S = (B \cap S) \oplus X$. Again by Lemma 1.4(5) we get $B \neq R$ and hence $B \leq S$. Therefore $B = B \cap S = J$.

The converse is obvious.

**Corollary 1.6.** If $R$ is a right pseudo-semisimple ring, then the right socle of $R$ contains the left socle.

**Proof.** The result is obvious in the trivial cases. So assume that $R$ is non-trivial and consider the two cases of Lemma 1.5.

If $Z = S$, study a minimal left ideal $A$. Assume that $A^2 \neq 0$. Then $A = Re$ for some idempotent $e$ such that $0 \neq e \neq 1$. However, by Lemma 1.1, either $e \in S$ or $1 - e \in S$, in contradiction to $S^2 = 0$. Hence $A^2 = 0$, and $A \leq S$ by Lemma 1.3.

If $Z \neq S$, then $S^0 < S$ by Lemma 1.5. It follows readily that $S$ is essential as a left ideal, and therefore contains the left socle.

The following generalization of our theorem in [3] characterizes right pseudo-semisimple rings with socle square zero.

**Theorem 1.7.** Let $R$ be a ring with $S^2 = 0$. Then $R$ is right pseudo-semisimple if and only if $R/S$ is a principal right ideal domain and $S$ is torsion-free divisible as a left $R/S$-module.

**Proof.** The 'only if' part follows from Lemma 1.3 and Lemma 1.4 ((2) and (6)).

Conversely, assume that $R$ satisfies the conditions. Consider a right ideal $A$ of $R$ which is not contained in $S$, and select $x \in A - S$. Since $R/S$ is divisible, $S = xS \leq A$. Now $R/S$ is a principal right ideal domain implies $A = aR$ with $a^0 \leq S$. But then $a^0 = 0$ as $R/S$ is torsion-free. Hence $A = aR \cong R$.

According to Lemma 1.5, a non-trivial right pseudo-semisimple ring with $S^2 = 0$ satisfies $0 \neq S \leq J \leq S^0$. We list examples of the four possible cases.

**Examples 1.8.** (1) $S = J = S^0$: any local ring $R$ with $J^2 = 0$.

(2) $S = J < S^0$: $R = F[X] \rtimes F(X)$, the split extension of the polynomial ring over a field $F$ by the rational function field, made into an $F[X]$-bimodule via the natural multiplication on the left and multiplication by the constant coefficient on the right; compare [3].

(3) $S < J = S^0$: the localization of (2) at $(X)$.

(4) $S < J < S^0$: the localization of (2) at $(X) \cap (X + 1)$.
Theorem 1.9. Let \( R \) be a ring with \( S \neq 0 \) and \( S^2 = 0 \). Then \( R \) is right and left pseudo-semisimple if and only if \( R \) is a local ring with radical square zero.

Proof. The 'if' part is obvious. Conversely, assume that \( R \) is right and left pseudo-semisimple. Then the left-right analogue of Corollary 1.6 implies that \( S \) is the left socle of \( R \). Consider a minimal left ideal \( A \). Since \( A^2 = 0 \), \( A = Rx \) with \( x \in J \), and \( A \cong R/0x \). Since \( S = 0x \) by Lemma 1.4(4), \( S \) is a maximal left ideal. Then \( S = J \) as \( S \leq J \) and the result follows.

Proposition 1.10. Let \( R \) be a non-trivial right pseudo-semisimple ring. Then \( R/Z \) is right pseudo-semisimple with \( Z(R/Z) = 0 \). Moreover \( R/Z \) is semisimple if and only if \( R \) is a local ring with radical square zero.

Proof. Let \( T/Z \) be the right socle of \( R/Z \). Then it is obvious that \( S \leq T \). Moreover \( T/S \) is contained in the right socle of \( R/S \). Since \( R/S \) is a domain by Lemma 1.3, \( T = S \) or \( T = R \).

(i) Consider the case \( T = S \). Let \( A/Z \) be a right ideal of \( R/Z \) such that \( A/Z \notin S/Z \). Then \( A \notin S \) and hence \( A = aR \) for some \( a \in R \) with \( a^2 = 0 \). Since \( Z = aZ \) by Lemma 1.4(6), we obtain \( R/Z \cong aR/aZ = A/Z \).

Thus \( R/Z \) is right pseudo-semisimple.

Next we prove that \( Z(R/Z) = 0 \). If \( Z = S \), then \( R/Z \) is a domain by Lemma 1.3, and the result holds trivially. So, assume that \( Z \neq S \). Then by Lemma 1.4(3), \( S = Z \oplus I \) for a non-zero ideal \( I \) of \( R \). Consider \( x + Z \in Z(R/Z) \). Then \( xS \leq Z \), and hence \( xI \leq Z \cap I = 0 \). Consequently \( xR \not\cong R \), and therefore \( x \in S \). Then \( xZ = 0 \) by Lemma 1.4(6), and \( xS = x(Z \oplus I) = 0 \). Thus \( x \in Z \).

(ii) Now, assume that \( T = R \). Then \( R/Z \) is semisimple, and clearly \( Z(R/Z) = 0 \). We claim that \( Z \) is a maximal ideal. Let \( u \) be a central idempotent in \( R/Z \). Since \( Z \) is a nil ideal by Lemma 1.4(2), we may assume that \( u = e + Z \) for some idempotent \( e \in R \). According to Lemma 1.1, \( e \in S \) or \( (1 - e) \in S \); we may assume that \( e \in S \). Then \( Z \cap eR = 0 \). Since \( eR(1 - e) \leq Z \), \( eR(1 - e) = 0 \). It then follows by Corollary 1.2(2) that \( e = 0 \) or \( e = 1 \). Thus \( R/Z \) has no non-trivial central idempotents, and is therefore simple artinian. This proves our claim. Since \( Z \leq S \cap J \) by Lemma 1.4(2), we obtain \( S = J = Z \). It then follows by Lemma 1.3 that
\( R/J \) is a division ring. Hence \( R \) is a local ring with \( J^2 = 0 \). The rest is obvious.

2. Maximal socle

We turn to the second type in Lemma 1.5. Here we do not know of an effective criterion for pseudo-simplicity. However, right pseudo-semisimple rings \( R \) of this type are characterized in the special case where \( S \) is a maximal right ideal. In view of Lemma 1.3, this additional assumption is automatically satisfied if \( R \) is regular.

We start by listing some properties of rings \( R \) with maximal socle. Note that such rings are local if and only if \( S^2 = 0 \). The proofs are straightforward, and hence are omitted.

**Lemma 2.1.** Let \( R \) be a non-local ring with maximal socle (that is, \( R/S \) is a division ring and \( S^2 \neq 0 \)). Then \( R \) has the following properties:

1. \( S \) is the only proper essential right ideal;
2. every right ideal is semisimple or a direct summand;
3. if \( R = A \oplus B \) for right ideals \( A \) and \( B \), then precisely one of them is semisimple;
4. \( 0^S = Z \leq J < S \), and \( J^2 = 0 \);
5. \( J \leq A \) for every right ideal \( A \) not contained in \( S \);
6. \( R \) is regular if and only if \( J = 0 \) if and only if \( R \) is semiprime.

Consider an idempotent \( g \) in the socle of an arbitrary ring \( R \). It is well known that \((1 - g)R \cong R\) holds if and only if \( R \cong R \oplus gR \) if and only if there exist \( t, t^* \in R \) such that \( t^* t = 1 \) and \( t t^* = 1 - g \) (hence \( R(1 - g) \cong R \) also holds). We call \( t \) a shift for \( g \).

Now assume that for every isomorphism type of indecomposable idempotents \( f \) in \( S \), there is a representative \( f' \) for which there exists a shift. Then \( R \oplus fR \cong R \oplus f'R \cong R \). It follows that \( R \) has a shift for every idempotent \( e \in S \). Indeed, \( eR = \bigoplus_{i=1}^n e_i R \) with \( e_i \) indecomposable, and hence

\[
R \oplus eR = R \oplus e_1 R \oplus \cdots \oplus e_n R \cong R.
\]

Such a ring \( R \) is said to have enough shifts.

**Theorem 2.2.** Let \( R \) be a non-local ring with maximal socle. Then \( R \) is right pseudo-semisimple if and only if \( R \) has enough shifts.

**Proof.** From (1) and (2) of Lemma 2.1, the proof is obvious.
COROLLARY 2.3. Let $R$ be a ring with maximal socle. The following are equivalent:

1. $R$ is right and left pseudo-semisimple and regular;
2. $R$ is right pseudo-semisimple and $J = 0$;
3. $R$ is semiprime and has enough shifts.

PROOF. That (1) implies (2) and that (2) implies (3) are obvious.
Assume (3). Since $R$ is semiprime, $S^2 \neq 0$ and therefore $R$ is non-local. Then (3) implies (1) follows from Lemma 2.1(6) and Theorem 2.2.

The next proposition effectively reduces the study of pseudo-simplicity for rings with maximal socle to the non-singular case.

PROPOSITION 2.4. Let $R$ be a non-local ring with maximal socle. Then $R/Z$ is right pseudo-semisimple if and only if $R$ is right pseudo-semisimple or $R = A \oplus B$, where $A$ is a local ring with radical square zero and $B$ is semisimple.

PROOF. It is clear that $R/Z$ is semisimple for any ring $R = A \oplus B$ as described above; the ‘if’ part then follows from Proposition 1.10. Conversely, assume that $\overline{R} = R/Z$ is right pseudo-semisimple. The right socle of $\overline{R}$ can either be $\overline{S}$ or $\overline{R}$.

In the first case, for any right ideal $C \notin S$ we have $\overline{C} = c\overline{R}$ with $c^0 \leq Z$. Since $Z \leq C$ by Lemma 2.1 ((4) and (5)), we obtain $C = cR$. Also $C$ is a direct summand of $R$, and hence is projective. Thus $c^0$ is a direct summand of $R$, and consequently $c^0 = 0$. Therefore $C \cong R$, and $R$ is right pseudo-semisimple.

In the second case, we have $Z = J$ and $J^2 = 0$. Since $\overline{R}$ is semisimple, $\overline{R} = \bigoplus_{i=1}^n T_i$, where each $T_i$ is a simple artinian ring. Then $1 = \sum_{i=1}^n e_i$ where the $e_i$ are orthogonal idempotents of $R$, $\overline{e_i}$ is central in $\overline{R}$ and $\overline{e_i} \overline{R} = T_i$. Since for $i \neq j$, $e_i e_j = 0$ and $R/S$ is a domain, all the $e_i$, except possibly one, are in $S$. We denote the exceptional one by $e$. By Lemma 2.1(3), $(1 - e)R \leq S$ and therefore $(1 - e)Re \leq Z \cap (1 - e)R = 0$. Also $eR(1 - e) \leq Z = 0S \leq (1 - e)R = e$, and hence $eR(1 - e) = 0$. Thus $e$ is a central idempotent in $R$. Let $A = eRe$ and $B = (1 - e)R(1 - e)$. Then $B$ is semisimple. As $A/J(A) = e\overline{R}$ is simple artinian, $J(A)$ is a maximal ideal in $A$. However $J(A) = A \cap J \leq A \cap S$; and so $J(A) = A \cap S$. Now $A/J(A) = A/(A \cap S) \cong A + S/S = R/S$, a division ring. It is obvious that $J(A) \neq 0$ and $J(A)^2 = 0$. Hence $A$ is a local ring with radical-square zero.
Corollary 2.5. Let $R$ be a non-local ring with homogeneous maximal socle. Then $R$ is right pseudo-semisimple if and only if $R/Z$ is.

We end this section by showing that any non-trivial right pseudo-semisimple ring $R$ with $Z = 0$ can be embedded in one with maximal socle.

Proposition 2.6. Let $R$ be a non-trivial right pseudo-semisimple ring with $Z = 0$. Then $R$ is isomorphic to a subring of a right pseudo-semisimple ring $R_*$ with $Z(R_*) = 0$ and $S(R_*)$ maximal.

Proof. Let $\Sigma = \{c \in R : c^0 = 0$ and $cS = S\}$. Clearly $\Sigma$ is multiplicatively closed and $1 \in \Sigma$. If $xc = 0$ for $x \in R$ and $c \in \Sigma$, then $xS = xcS = 0$; hence $x \in Z = 0$. Thus $\Sigma$ consists of regular elements. Now we prove that $\Sigma$ is a right Ore set. Let $c \in \Sigma$ and $r \in R$. If $r \in S$, then $r \in cS$; consequently $r1 = cr'$ with $r' \in R$. Assume that $r \notin S$, and let $B = \{b \in R : rb \in cR\}$. It is clear that $S \leq B$. If $S = B$, then $cR \cap rR = rB \leq S$. This implies $\overline{cR} \cap \overline{rR} = 0$ in $\overline{R}$, in contradiction of the fact that $\overline{R}$ is a principal right ideal domain. Thus $S < B$, and therefore $B = c' R$ with $c' \in \Sigma$ (see Lemma 1.4(5)). Then $rc' R \leq cR$, and hence $rc' = cr'$ for $r' \in R$.

Let $R_* = R_\Sigma$, the localization of $R$ with respect to $\Sigma$, and identify $R$ with its image in $R_*$. One can easily check that $S_*$ is an essential right ideal in $R_*$, and is semisimple as a right $R_*$-module. Thus $S_*$ is the right socle of $R_*$. We prove that $S_*$ is a maximal right ideal. Clearly $S_* \neq R_*$. If $S_* < M$ for some right ideal $M$ of $R_*$, then $M = DR_*$ for a right ideal $D$ of $R$ with $S < D$. Hence $D = dR$ with $d \in \Sigma$, and $M = dRR_* = dR_* = R_*$. Next we prove that $R_*$ is right pseudo-semisimple. Let $A$ be a right ideal of $R$. Then $A \leq S$ or $A = aR$ with $a^0 = 0$. Thus $AR_* \leq S_*$ or $AR_* = aRR_* = aR_*$.

Let $x \in Z(R_*)$. Then $xS_* = 0$, and hence $xS = 0$. Since $x = rc^{-1}$ for some $c \in \Sigma$, $rS = xcS = xS = 0$. Thus $r \in Z = 0$, and hence $x = 0$.

Remark. We note that in Proposition 2.6, $\Sigma$ is actually the largest right Ore set of $R$, and hence $R_*$ is the maximal right classical ring of fractions of $R$.

3. Subrings of full linear rings

If $R$ is a ring with $Z = 0$, then the maximal right quotient ring of $R$ is a regular right self-injective ring having $R$ as a subring. Moreover if $S$ is
essential in $R$, then $Q = \text{End } S_R$, and is therefore a product of full linear rings (compare [5, Chapter 12]); it is just one full linear ring if and only if $S$ is homogeneous.

In this section we discuss the existence of pseudo-semisimple rings which are subrings of full linear rings; in view of Lemma 1.5(2), non-trivial examples can only occur with linear rings of infinite dimensional vector spaces.

Throughout this section, $Q$ will stand for the endomorphism ring of a vector space $V$ of infinite dimension over a division ring $D$. We shall call an element $t \in Q$ a shift endomorphism if it is an isomorphism onto a subspace of codimension one. For such $t$ we choose a complement $U$ of $tV$, so that $V = tV \oplus U$ and $\dim U = 1$. We define $t^* = t^{-1}$ on $tV$ and $t^* = 0$ on $U$. Let $e = 1 - tt^*$. Then $e$ is the projection onto $U$ along $tV$, and hence is of rank one. A subring $R$ of $Q$ is said to contain a shift, if $t, t^* \in R$ for some shift $t$; it is clear that $e \in R$ and $(1 - e)R \cong R$ (also $R(1 - e) \cong R$).

**Lemma 3.1.** Let $L$ be a non-zero left ideal of $Q$ consisting of linear transformations of finite rank, and let $T$ be a subring of $Q$ having $L$ as a two sided ideal. If $T$ contains a shift and $T/L$ is a division ring, then $T$ is a right pseudo-semisimple ring with $S(T) = L$ and $Z(T) = 0$; moreover $T$ is regular if and only if $\bigcap \{\ker x : x \in L\} = 0$.

**Proof.** Let $f$ be an indecomposable idempotent in $L$. Since $L$ is a left ideal in $Q$, $f$ stays indecomposable in $Q$, and therefore $fQ$ is a minimal right ideal. Let $ft \neq 0$ for some $t \in T$. Then there exists $q \in Q$ with $ftq = f$. Consequently $ftqf = f$ and $qf \in L \leq T$. This proves that $fT$ is a minimal right ideal in $T$.

Given $x \in L$, there exists $p \in Q$ such that $xp = x$. Then $g = px$ is an idempotent in $L$, and $xT = xgT \cong gT$. One may write $g = g_1 + \cdots + g_n$ where the $g_i \in Q$ are orthogonal idempotents of rank one. However $g_i = g_i g \in L$, and it follows by the preceding argument that $g_i T$ is a minimal right ideal. Therefore $xT \cong gT$ is semisimple. Hence $L$ is contained in the right socle of $T$. Since $T$ contains a shift, $T$ is not semisimple. Then $T/L$ is a division ring implies that $L$ is the right socle of $T$.

Our argument also shows that any minimal right ideal in $L$ is generated by a rank one idempotent. Since rank one idempotents are isomorphic in $Q$, they are also isomorphic in $T$ (again since $L$ is a left ideal in $Q$). Thus $L$ is homogeneous. Let $t$ be the given shift in $T$. Then it is clear that the rank one idempotent $e = 1 - tt^*$ is in $L$, and hence all rank one idempotents in $L$ are isomorphic to $e$. Then $T$ is right pseudo-semisimple by Theorem 2.2.
Given \( 0 \neq y \in Q \), then \( 0 \neq yQe \leq L \leq T \) and \( Qe \leq L \leq T \). Hence \( T \) is essential in \( Q_T \), and therefore \( Z(T) = 0 \). (This also proves that \( Q \) is the maximal quotient ring of \( T \).)

Now we prove the last statement of the theorem. In view of Lemma 1.5(2) and Lemma 2.1(6), \( T \) is regular if and only if \( L^0 = 0 \). Let \( W = \bigcap \{ \ker x : x \in L \} \). If \( W \neq 0 \), then there exists \( q \in Q \) such that \( 0 \neq qeV \leq W \) (since \( QeV = V \)), and therefore \( 0 \neq qe \in L^0 \); thus \( L^0 \neq 0 \). Conversely, assume that \( W = 0 \) and let \( r \in L^0 \). Then \( rV \leq \ker x \) for every \( x \in L \), and therefore \( r = 0 \). This proves that \( L^0 = 0 \) holds if and only if \( W = 0 \).

At this point it is convenient to discuss some examples. We start with [2, Example 4.26], which is originally due to G. M. Bergman, and represents a regular, but not unit-regular ring, in which perspectivity is transitive. This example was suggested to us by K. R. Goodearl through a communication by K. M. Rangaswami. Similar examples can be obtained from the more general construction to be discussed in Proposition 3.5.

**Example 3.2.** (A regular right and left pseudo-semisimple ring which is not semisimple.) Let \( V = F[[t]] \), the power series ring over a field \( F \) considered as an \( F \)-space, \( Q = \text{End } V_F \), and \( F((t)) \) the Laurent series ring, that is, the quotient field of \( F[[t]] \). Let

\[
L = \{ x \in Q : \exists n \in \mathbb{N} \ (xt^nV = 0) \}, \\
T = \{ x \in Q : \exists n \in \mathbb{N}, \ a \in F((t)) \ (x - a) t^nV = 0) \}.
\]

It is obvious that \( t \) is a shift and \( t^* \in T \). One can verify that \( L \) is a left ideal of \( Q \) consisting of linear transformations of finite rank, \( T \) is a subring of \( Q \) having \( L \) as a two sided ideal and \( T/L \cong F((t)) \). Moreover \( \bigcap \{ \ker x : x \in L \} = \bigcap_{n \in \mathbb{N}} t^nV = 0 \). Thus \( T \) is right pseudo-semisimple and regular by Lemma 3.1. According to Corollary 2.3, \( T \) is also left pseudo-semisimple.

**Example 3.3.** (A non-singular right pseudo-semisimple ring which is not left pseudo-semisimple.) Modifying the above example by taking \( V = F[[t]] \oplus F((t)) \), one obtains a right pseudo-semisimple ring \( T \) with \( Z(T) = 0 \). However

\[
J(T) = \bigcap \{ \ker x : x \in L \} = F((t))
\]

Thus \( T \) is not left pseudo-semisimple in view of Corollary 2.3.

A right pseudo-semisimple ring \( R \) in which \( Z \neq S \) satisfies \( 0 \leq Z \leq J < S \). Examples 3.2 and 3.3 correspond to the cases \( 0 = Z = J \) and \( 0 = Z < J \), respectively. Examples of the other two cases can be obtained using split extensions.
Let $A$ be any right pseudo-semisimple ring with $Z(A) = 0$ and $A/S(A)$ a division ring. Let $R = A \times A/S(A)$. Then $R$ has right singular ideal $0 \times A/S(A)$, right socle $S(A) \times A/S(A)$ and Jacobson radical $J(A) \times A/S(A)$; and $R$ is right pseudo-semisimple by Proposition 2.4.

For the case $0 < Z = J$ (respectively $0 < Z < J$), take $R = T \times T/L$ where $T$ is the ring of Example 3.2 (respectively 3.3).

**Lemma 3.4.** Let $t \in Q = \text{End} V_D$ be a shift. If $q$ is a non-zero polynomial over the centre of $D$, then $q(t)$ has infinite rank.

**Proof.** Let $q = a_m X^m + \cdots + a_n X^n$ where $m \leq n$ and $a_m \neq 0$. Without loss of generality we may assume that $a_m = 1$. Let $K = \ker q(t)$. Then clearly $t^i K \leq K$ for every $i \in \mathbb{N}$. Now

$$0 = t^m q(t) K = (1 + a_{m+1} t + \cdots + a_n t^{n-m}) K,$$

and so $K \leq tK$. Thus $K = t^i K \leq t^i V$ for every $i \in \mathbb{N}$. Writing $V = tV \oplus U$, we get

$$V = t^i V \oplus t^{i-1} U \oplus \cdots \oplus tU \oplus U.$$

Therefore $K \cap \bigoplus_{i=0}^{\infty} t^i U = 0$, and hence $K$ has infinite codimension. Thus $q(t)$ is of infinite rank.

Let $A$ denote the prime subring of $D$, that is, the subring of $D$ generated by the identity element. We shall say that the pair $(t, L)$ is permissible if $t$ is a shift endomorphism and $L$ a non-zero left ideal of $Q$ consisting of linear transformations of finite rank such that:

$(P_1)$ $\forall 0 \neq q \in A[X] \forall x \in Q \ (x \in L \Leftrightarrow xq(t) \in L)$;

$(P_2)$ $\forall 0 \neq q \in A[X] \exists y \in L \ (\ker q(t) \cap \ker y = 0)$.

**Remark.** One particular choice of $L$ is the ideal consisting of all linear transformations of finite rank. For this choice, a shift $t \in Q$ is such that $(t, L)$ is permissible if and only if $\ker q(t)$ has finite dimension and $\text{Im} q(t)$ has finite codimension for all $0 \neq q \in A[X]$.

A shift $t$ satisfying the above requirements exists in every full linear ring $Q = \text{End} V_D$. Indeed $A[X]$ is countable, and therefore the central localization $D[X, X^{-1}]_\alpha$ at the non-zero elements of $A[X]$ is a countable dimensional $D$-space. Consequently $V \cong D[X] \oplus D[X, X^{-1}]^{(\dim V)}$. Define $t$ as componentwise multiplication by $X$. This yields a shift with $\ker q(t) = 0$ for every $0 \neq q \in A[X]$. Moreover, if $q$ is of degree $n$, then by the Euclidean Algorithm

$$D[X] = q(X) D[X] \oplus D \oplus DX \oplus \cdots \oplus DX^{n-1}.$$
Clearly $q[X]D[X, X^{-1}] = D[X, X^{-1}]$. Therefore $\text{Im} q(t)$ has finite codimension.

The following proposition ensures that subrings as described in Lemma 3.1 exist in every infinite-dimensional full linear ring.

**Proposition 3.5.** Let $(t, L)$ be a permissible pair. Then
\[ T = \{ x \in Q : \exists p, 0 \neq q \in A[X] \ (xq(t) - p(t) \in L) \} \]
is a non-singular right pseudo-semisimple ring with $L$ as its right socle and $T/L \cong A(X)$, the quotient field of $A[X]$.

**Proof.** From Lemma 3.4 and condition $(P_1)$, a routine verification establishes that $T$ is a subring of $Q$, and $\varphi: x \rightarrow p/q$ is a well defined ring homomorphism of $T$ into $A(X)$. To show that $\varphi$ is surjective consider any $p/q \in A(X)$. By $(P_2)$ we have $y \in L$ such that $\ker q(t) \cap \ker y = 0$. Hence $q(t)|_{\ker y}$ is one-to-one. Let $V = q(t)\ker y \oplus C$. Then the mapping $x$ given by
\[ x(q(t)v) = p(t)v, \quad v \in \ker y; \quad x|_{C} = 0 \]
is a well defined element in $Q$. Let $\alpha = xq(t) - p(t)$. Then $\alpha|_{\ker y} = 0$. Write $V = \ker y \oplus W$. Since $yW \cong W$, there exists $\beta \in Q$ such that $\beta y|_{W} = 1$. It then follows that $\alpha = \alpha \beta y \in L$. This shows that $x \in T$ as well as $\varphi(x) = p/q$.

It is clear that $\ker \varphi = \{ x \in Q : \exists 0 \neq q \in A[X] \ (xq(t) \in L) \}$. Then by $(P_1)$, $L = \ker \varphi$. Hence $L$ is an ideal in $T$ and $T/L \cong A(X)$. Now $t1 - t = 0$ and $t^*t - 1 = 0$ imply that $t, t^* \in T$. The result now follows from Lemma 3.1.

**Theorem 3.6.** A ring $R$ is a non-singular right pseudo-semisimple ring with homogeneous maximal socle if and only if

1. $R$ is a subring of a full linear ring $Q$,
2. there exists a permissible pair $(t, L)$ in $Q$ with $L$ an ideal in $R$ and $R/L \cong A(X)$, and
3. the ring $T$ corresponding to $(t, L)$, as in Proposition 3.5, is a subring of $R$.

**Proof.** The 'if' part follows from Lemma 3.1 as $t, t^* \in T \leq R$.

'Only if'. Since $R$ is right non-singular and $S$ is homogeneous, the maximal quotient ring $Q$ of $R$ is a full linear ring; $Q = \text{End} \ V_D$. Also $Q \cong \text{End} S_R$, and hence $S$ is a left ideal of $Q$ consisting of linear transformations of finite rank. By Theorem 2.2, $R$ has a shift for some indecomposable idempotent $e \in S$. As $e$ is a rank one projection, $t$ is a shift.
endomorphism in $Q$. We verify conditions $(P_1)$ and $(P_2)$ for the pair $(t, S)$.

Let $0 \neq q \in A[X]$. Then $q(t)$ is not of finite rank by Lemma 3.4 and hence not in $S$. It follows by Lemma 2.1 ((2) and (3)) that $R = fR \oplus (1 - f)R$ such that $q(t)R = fR$ and $1 - f \in S$. Let $f = q(t)r$, $r \in R$. Then $rq(t)$ is also an idempotent, and $rq(t) \notin S$; otherwise $q(t) = fq(t) = q(t)rq(t) \in S$, a contradiction.

For the non-trivial implication of $(P_1)$, assume that $xq(t) \in S$ for some $x \in Q$. Then $xf = xq(t)r \in S$. Also $x(1 - f) \in S$. Thus $x \in S$.

To prove $(P_2)$, note that $1 - rq(t) \in S$ by Lemma 2.1(3). Clearly $\ker q(t) \cap \ker (1 - rq(t)) = 0$.

Now $(P_1)$ and $(P_2)$ being established, we may form the subring $T$ of $Q$ according to Proposition 3.5. Let $x \in T$. Then $xq(t) - p(t) = s \in S$, for some $p$, $0 \neq q \in A[X]$. With $f$ and $r$ as before, we obtain

$$xf = xq(t)r = (p(t) + s)r \in R,$$

and $x(1 - f) \in S \subseteq R$. Hence $x \in R$.

**Added in Proof**

Using Lemma 3.1, the referee suggested the following example of a regular pseudo-semisimple ring which is not semisimple (a similar example was suggested by Mark L. Teply). Let $Q$ be the ring of $\mathbb{N}_0 \times \mathbb{N}_0$ column-finite matrices over a field $F$, let $L = \text{Socle } Q$ (set of matrices with a finite number of non-zero rows), and let $M$ be the subset of $Q$ consisting of all matrices of the form.

$$
\begin{bmatrix}
   a_0 & a_1 & a_2 \\
   b_1 & a_0 & a_1 & a_2 & \cdots \\
   b_2 & b_1 & a_0 & a_1 & a_2 \\
   \vdots & \ddots & \ddots & \ddots & \ddots \\
   b_2 & b_2 & b_1 & \cdots & a_0
\end{bmatrix},
$$

where only a finite number of the $b_i$ are non-zero. Let $T = L + M$. It is clear that $T$ is a ring which contains the standard shift

$$
\begin{bmatrix}
   0 \\
   1 \\
   \vdots
\end{bmatrix},
$$

$\theta = \begin{bmatrix}
   1 \\
   \vdots
\end{bmatrix},$
$L$ is a two sided ideal in $T$, and $T/L \cong F((t^*))$ where

$$t^* = \begin{bmatrix}
0 & 1 \\
1 & 1 \\
1 & 0 \\
& & \ddots
\end{bmatrix}.$$

The authors are thankful to the referee for other comments and suggestions.

References