HOCHSCHILD (CO)HOMOLOGY OF $\mathbb{Z}_2 \times \mathbb{Z}_2$-GALOIS COVERINGS OF QUANTUM EXTERIOR ALGEBRAS

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Abstract

Let $A_q = k\langle x, y \rangle/(x^2, xy + qyx, y^2)$ be the quantum exterior algebra over a field $k$ with $\text{char} \ k \neq 2$, and let $\Lambda_q$ be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of $A_q$. In this paper the minimal projective bimodule resolution of $\Lambda_q$ is constructed explicitly, and from it we can calculate the $k$-dimensions of all Hochschild homology and cohomology groups of $\Lambda_q$. Moreover, the cyclic homology of $\Lambda_q$ can be calculated in the case where the underlying field is of characteristic zero.


Keywords and phrases: Hochschild (co)homology, quantum exterior algebra, Galois covering.

1. Introduction

Let $\Lambda$ be a finite-dimensional algebra (associative with identity) over a field $k$. We consider the enveloping algebra $\Lambda^e = \Lambda^{\text{op}} \otimes_k \Lambda$. For a finitely-generated bimodule $\Lambda X$, the $i$th Hochschild homology and cohomology of $\Lambda$ with coefficients in $X$, denoted by $HH_i(\Lambda, X)$ and $HH^i(\Lambda, X)$, are the groups $\text{Tor}^i_{\Lambda^e}(\Lambda, X)$ and $\text{Ext}_i^{\Lambda^e}(\Lambda, X)$ respectively, for each $i \geq 0$ [7]. Of particular interest is the case $X = \Lambda$, and in this case we shall write $HH_i(\Lambda) = HH_i(\Lambda, \Lambda)$ and $HH^i(\Lambda) = HH^i(\Lambda, \Lambda)$. It is well known that the Hochschild homology and cohomology of an algebra are subtle invariants of associative algebras under Morita equivalence, tilting equivalence, derived equivalence, and so on, and have played a fundamental role in the representation theory of Artin algebras: Hochschild cohomology is closely related to simple connectedness, separability and deformation theory [1, 8, 14, 18, 25]; Hochschild homology is closely related to the oriented cycle and the global dimension of algebras [3, 16, 19].
In [18], Happel asked the following question. If the Hochschild cohomology groups $HH^n(\Lambda)$ of a finite-dimensional algebra $\Lambda$ over a field $k$ vanish for all sufficiently-large $n$, is the global dimension of $\Lambda$ finite? In 2005, Buchweitz, Green, Madsen and Solberg gave a negative answer to this question by exhibiting the Hochschild cohomology behaviour of a family of pathological algebras $A_q = k\langle x, y \rangle/(x^2, xy + qyx, y^2)$ (the so-called quantum exterior algebras) [5]. Moreover, these algebras have been studied to exhibit some other pathologies and thus give negative answers to some open problems, such as Tachikawa’s conjecture, Ringel’s problem, and so on [20, 23, 26]. Recently, this class of algebras has been extensively studied. Their Hochschild homology and cohomology have been calculated explicitly, and the Hochschild cohomology rings of $A_q$ have been determined via generators and relations [5, 27].

During the last few years, several results and tools from algebraic topology, such as covering theory, have been adapted to the representation theory of noncommutative finite-dimensional associative algebras over a field $k$ [13]. A comparison of the Hochschild cohomology of the algebras involved in a Galois covering of the Kronecker algebra with a cyclic group of order 2 was initiated in [22]. A Cartan–Leray spectral sequence related to the Hochschild–Mitchell (co)homology of a Galois covering of linear categories was obtained in [9]. The skew category, Galois covering and smash product of a category over a ring are studied in [10, 11]. It is well known that there are strong connections between skew group algebras, Galois coverings and smash products of graded algebras [12]. Moreover, it is proved in [17] that a finite-dimensional quiver algebra is Koszul if and only if its finite Galois covering algebra with Galois group $G$ satisfying $\text{char } k \nmid |G|$ is Koszul. As an example, the Galois covering algebra $\Lambda_q$ of quantum exterior algebra $A_q$ with Galois group $G$ satisfying $\text{char } k \nmid |G|$ is Koszul again. In this note we consider the case where $G$ is the noncyclic Abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$. We first provide a minimal projective resolution of $\Lambda_q$. Applying this minimal projective resolution, we calculate the Hochschild (co)homology of $\Lambda_q$, and the cyclic homology of $\Lambda_q$ can be calculated in the case where the underlying field is of characteristic zero. These examples will be helpful to understand deeply the Hochschild homology and cohomology behaviour of Galois covering algebras of Koszul algebras with finite Galois groups.

2. Minimal projective bimodule resolutions

Throughout this paper, we fix a base field $k$ with $\text{char } k \neq 2$ and let $\mathbb{Z}_2 = \{0, \bar{1}\}$ be the residue group modulo 2. We always suppose that

$$A_q = k\langle x, y \rangle/(x^2, xy + qyx, y^2)$$

is the quantum exterior algebra, with $q \in k \setminus \{0\}$, unless otherwise specified.

Let $Q$ be the quiver given by four points $\bar{0}, \bar{1}, \bar{0}', \bar{1}'$ and eight arrows $x_{\bar{0}}$, $y_{\bar{0}}$, $x_{\bar{1}}$, $y_{\bar{1}}$,
\( x_0', y_0', x_1', y_1' \) as follows:

\[
\begin{array}{c}
\bullet \quad x_0' \\
\downarrow y_0' \\
\bullet \quad y_1' \\
\downarrow y_1 \\
\bullet \quad x_1 \\
\downarrow x_0 \\
\bullet \quad x_0 \\
\end{array}
\]

We sometimes write arrows \( x_i, y_i, x'_i, y'_i \) instead of \( x_i, y_i, x'_i, y'_i \) respectively, \( i = 0, 1 \), and assume that, for any nonnegative integers \( k, j \), if \( k \equiv j \) (mod 2), then \( x_j = x_k, x'_j = x'_k, y_j = y_k \) and \( y'_j = y'_k \). For an arrow \( \rho \in Q \), we denote the length of \( \rho \) by \( l(\rho) \), and let \( o(\rho), t(\rho) \) be the origin and terminus of \( \rho \) respectively. Denote by \( I \) the ideal of the path algebra \( kQ \) generated by

\[
R := \{x_i x'_i, y_i y_{i+1}, x'_i x_i, y'_i y'_{i+1}, x_i y_i + q y_{i} x_{i+1}, x'_i y'_i + q y'_i x'_{i+1} \mid i = 0, 1 \}.
\]

For information on quivers we refer to [2]. Set \( \Lambda_q = kQ / I \). Then \( \Lambda_q \) is just the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-Galois covering of the quantum exterior algebra over \( k \), which is a Koszul algebra (see [17]). Denote by \( e_0, e'_0, e_1, e'_1 \) the primitive orthogonal idempotents corresponding to the points \( 0, 0', 1, 1' \) respectively. Then we can order the paths in \( Q \) in left length-lexicographic order by choosing

\[
e_0 < e'_0 < e_1 < e'_1 < x_0 < x'_0 < x_1 < x'_1 < y_0 < y'_0 < y_1 < y'_1,
\]

namely, \( u_1 \ldots u_s < v_1 \ldots v_t \) with \( u_i \) and \( v_i \) being arrows if \( s < t \), or \( s = t \) but \( u_i = v_i \) for \( 0 \leq i < r \) and \( u_r < v_r \) for some \( 1 \leq r \leq s \). Thus, \( \Lambda_q \) has an ordered \( k \)-basis

\[
\mathcal{B} = \{e_i, e'_i, x_i, x'_i, y_i, y'_i, y_{i+1}, y'_i x'_{i+1} \mid i = 0, 1 \},
\]

if we identify the elements of \( \mathcal{B} \) with their images in \( \Lambda_q \). We always write a composition of paths from left to right.

Now we construct a minimal projective bimodule resolution \((P_\bullet, \delta_\bullet)\) of \( \Lambda_q \). For each \( n \geq 0 \), we first construct certain elements

\[
\{f_j^{(n,i)}, f_j^{(n,i')} \mid 0 \leq j \leq n, i = 0, 1 \}.
\]

Let

\[
f_0^{(0,i)} = e_i, \quad f_0^{(0,i')} = e'_i,
\]

\[
f_0^{(1,i)} = x_i, \quad f_1^{(1,i)} = y_i, \quad f_0^{(1,i')} = x'_i, \quad f_1^{(1,i')} = y'_i.
\]

For any \( \rho \in \{x_0, y_0, x_1, y_1\} \), we define

\[
\rho^{(l)} = \begin{cases} 
\rho' & \text{if } l \text{ is odd; } \\
\rho & \text{if } l \text{ is even.}
\end{cases}
\]

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Then we can define
\[ \{ f_j^{(n,i)}, f_j^{(n,i')} \}_{j=0, \ 1}, \]
for all \( n \geq 2 \) inductively by setting
\[
\begin{align*}
  f_j^{(n,i)} &= q^j f_j^{(n-1,i)} x_{i+j}^{(n+j+1)} + f_{j-1}^{(n-1,i)} y_{i+j}^{(n+j+1)} , \\
  f_j^{(n,i')} &= q^j f_j^{(n-1,i')} x_{i+j}^{(n+j)} + f_{j-1}^{(n-1,i')} y_{i+j}^{(n+j+1)} ,
\end{align*}
\]
and
\[
\begin{align*}
  f_1^{(n,i)} &= f_1^{(n,i')} = f_{n+1}^{(n,i)} = f_{n+1}^{(n,i')} = 0, \quad i = 0, 1.
\end{align*}
\]
Let 
\[
\Gamma^{(n)} = \{ f_j^{(n,i)}, f_j^{(n,i')} \mid 0 \leq j \leq n, \ i = 0, 1 \}.
\]
Clearly, \( |\Gamma^{(n)}| = 4(n+1) \).
Recall that a nonzero element
\[
x = \sum_{i=1}^{s} \alpha_i p_i \in k Q
\]
is said to be uniform if there exist vertices \( u \) and \( v \) in \( Q_0 \) such that \( o(p_i) = u \) and \( t(p_i) = v \) for all \( p_i, \ i = 1, 2, \ldots, s \). Note that \( f_j^{(n,i)} \) and \( f_j^{(n,i')} \) are linear combinations of some paths in \( k Q \) for all \( 0 \leq j \leq n \), which are uniform. Thus for any \( f \in \Gamma^{(n)} \), we usually denote by \( o(f) \) and \( t(f) \) the common origins and termini of all the paths occurring in \( f \). Set
\[
\Gamma_{ij}^{(n)} = \{ f \in \Gamma^{(n)} \mid o(f) = i \text{ and } t(f) = j \}, \quad i, j = 0, 1, 0', 1'.
\]
Let \( \otimes := \otimes_k \). Let
\[
P_n := \bigsqcup_{f \in \Gamma^{(n)}} \Lambda_q o(f) \otimes t(f) \Lambda_q \quad (\forall \ n \geq 0).
\]
Note that
\[
\begin{align*}
  f_j^{(n,i)} &= x_i f_j^{(n-1,i')} + q^{n-j} y_i f_{j-1}^{(n-1,i+1)} , \\
  f_j^{(n,i')} &= x'_i f_j^{(n-1,i)} + q^{n-j} y'_i f_{j-1}^{(n-1,i+1)}'.
\end{align*}
\]
So we can define \( \delta_n : P_n \rightarrow P_{n-1} \) by setting
\[
\begin{align*}
  \delta_n(o(f_j^{(n,i)})) \otimes t(f_j^{(n,i)})) &= x_i \otimes t(f_j^{(n-1,i)}) + (-1)^n q^j o(f_j^{(n-1,i)}) \otimes x_{i+j}^{(n+j+1)} \\
&\quad + q^{n-j} y_i \otimes t(f_{j-1}^{(n-1,i+1)}) \\
&\quad + (-1)^n o(f_{j-1}^{(n-1,i)}) \otimes y_{i+j}^{(n+j)} ;
\end{align*}
\]
The complex

\[ \delta_n(o(f_j^{(n,i)}) \otimes t(f_j^{(n,i')})) = x_i' \otimes t(f_j^{(n-1,i)}) + (-1)^n q^j o(f_j^{(n-1,i')}) \otimes x_{i+j}^{(n+j)} \]

\[ + q^{n-j} y_i' \otimes t(f_{j-1}^{(n-1,i'+1)}) \]

\[ + (-1)^n o(f_{j-1}^{(n-1,i')}) \otimes y_{i+j+1}^{(n+j+1)}. \]

**Theorem 2.1.** The complex \((P_\bullet, \delta_\bullet)\)
\[
\cdots \rightarrow P_{n+1} \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_2} P_2 \xrightarrow{\delta_1} P_1 \xrightarrow{\delta_0} P_0 \rightarrow 0
\]
is a minimal projective bimodule resolution of the covering algebra \(\Lambda_q = k Q / I\).

**Proof.** We consider the minimal projective bimodule resolution of \(\Lambda_q\) constructed in [6, Section 9]. Let \(X = \{x_0, y_0, x_1, y_1, x_0', y_0', x_1', y_1'\}\). Since \(\Lambda_q\) is a Koszul algebra (see [17]), we only need to prove that \(\Gamma^{(n)}\) is a \(k\)-basis of the \(k\)-vector space

\[ K_n := \bigcap_{p+q=n-2} X^p Q X^q. \]

Note that \(X K_{n-1} \cap K_{n-1} X \subset K_n\). By Formulae (2.1) and (2.2),
\[ f_j^{(n,0)} , f_j^{(n,0')}, f_j^{(n,1)}, f_j^{(n,1')} \in K_n, \]
by induction on \(n\). Clearly, \(\Gamma^{(n)}\) is \(k\)-linearly independent.

Denote by \((R^\perp)\) the ideal of \(k Q\) generated by
\[ R^\perp := \{y_i x_{i+1} - q x_i y_i', y_i' x_{i+1} - q x_i y_i' | i = 0, 1\}. \]

Then the quadratic dual of the Koszul algebra \(\Lambda_q\) is just the algebra \(\Lambda_q^! = k Q / (R^\perp)\), which is isomorphic to the Yoneda algebra \(E(\Lambda_q)\) of \(\Lambda_q\) (see [4, Theorem 2.10.1]). Thus the Betti numbers of a minimal projective resolution of \(\Lambda_q\) over \(\Lambda_q^!\) are \(\dim_k K_n = 4(n+1)\). Hence \(\Gamma^{(n)}\) is a \(k\)-basis of \(K_n\).

Finally, the maps \(\delta_\bullet\) are determined by [6, p. 354]; see also [15]. \(\square\)

### 3. Hochschild homology and cyclic homology

In this section we calculate the \(k\)-dimensions of Hochschild homology groups and cyclic homology groups (in the case \(\text{char } k = 0\)) of the covering algebra \(\Lambda_q\). Let \(X\) and \(Y\) be the sets of uniform elements in \(k Q\); then one defines
\[ X \times Y = \{(p, q) \in X \times Y | t(p) = o(q) \text{ and } t(q) = o(p)\}. \]

We denote by \(k(X \times Y)\) the vector space that has as basis the set \(X \times Y\).

Applying the functor \(\Lambda_q \otimes \Lambda_q^! (\cdot)\) to the minimal projective bimodule resolution \((P_\bullet, \delta_\bullet)\), we have the following result.

**Lemma 3.1.** We have \(\Lambda_q \otimes \Lambda_q^! (P_\bullet, \delta_\bullet) = (N_\bullet, \tau_\bullet)\), where \(N_n \cong k(B \otimes \Gamma^{(n)})\) and \(\tau_n : N_n \rightarrow N_{n-1}\) is given by

\[
\cdots \rightarrow N_{n+1} \xrightarrow{\tau_{n+1}} N_n \xrightarrow{\tau_n} \cdots \xrightarrow{\tau_2} N_2 \xrightarrow{\tau_1} N_1 \xrightarrow{\tau_0} N_0 \rightarrow 0
\]
Clearly, we only need to determine the dimension of \( \tau_1 \Lambda_1 \). So the differentials \( \delta_n \) can be induced by \( \tau_n \) in the minimal projective resolution \( (P_\bullet, \delta_\bullet) \).

Note that \( HH_n(\Lambda_q) = \ker \tau_n / \operatorname{im} \tau_{n+1} \) by definition:

\[
\dim_k HH_n(\Lambda_q) = \dim_k \ker \tau_n - \dim_k \operatorname{im} \tau_{n+1} = \dim_k N_n - \dim_k \operatorname{im} \tau_n - \dim_k \operatorname{im} \tau_{n+1}.
\]

Consequently, to calculate the dimensions of Hochschild homology groups of \( \Lambda_q \), we only need to determine the dimensions \( N_n \) and \( \operatorname{im} \tau_n \).

For any \((b, f) \in B \circlearrowleft \Gamma^{(n)}\), \( l(b) \) and \( n \) must have the same parity. If \( n \) is odd, then

\[
B \circlearrowleft \Gamma^{(n)} = ([x_0] \circ \Gamma^{(n)}_{0'0}) \cup ([x_0'] \circ \Gamma^{(n)}_{0'0'}) \cup ([x_1] \circ \Gamma^{(n)}_{1'1}) \cup ([x_1'] \circ \Gamma^{(n)}_{1'1'}) \cup ([y_0] \circ \Gamma^{(n)}_{00}) \cup ([y_0'] \circ \Gamma^{(n)}_{00'}) \cup ([y_1] \circ \Gamma^{(n)}_{01}) \cup ([y_1'] \circ \Gamma^{(n)}_{01'});
\]

if \( n \) is even, then

\[
B \circlearrowleft \Gamma^{(n)} = ([e_0] \circ \Gamma^{(n)}_{00}) \cup ([e_0'] \circ \Gamma^{(n)}_{0'0'}) \cup ([e_1] \circ \Gamma^{(n)}_{1'1}) \cup ([e_1'] \circ \Gamma^{(n)}_{1'1'}) \cup ([y_0 x_1] \circ \Gamma^{(n)}_{1'0'}) \cup ([y_0' x_1'] \circ \Gamma^{(n)}_{1'0'}) \cup ([y_1 x_0] \circ \Gamma^{(n)}_{01}) \cup ([y_1' x_0'] \circ \Gamma^{(n)}_{01'}).
\]

Hence \( \dim_k N_n = 4(n + 1) \).
We now define an order on $\mathcal{B} \circ \Gamma^{(n)}$ as follows:

$$(b_1, f_{j_1}^{(n,i_1)}) < (b_2, f_{j_2}^{(n,i_2)}) \quad \text{if} \quad j_1 < j_2, \quad \text{or} \quad j_1 = j_2 \text{ but } b_1 < b_2,$$

for any $(b_1, f_{j_1}^{(n,i_1)}), (b_2, f_{j_2}^{(n,i_2)}) \in \mathcal{B} \circ \Gamma^{(n)}$ and $i_1, i_2 \in \{0, 0', 1, 1'\}$.

We still denote by $\tau_n$ the matrix of the differentials $\tau_n$ under the ordered bases above, and write

$$A_i = \begin{pmatrix} 1 & q^i & 0 & 0 \\ q^i & 1 & 0 & 0 \\ 0 & 0 & 1 & q^i \\ 0 & 0 & q^i & 1 \end{pmatrix}_{4 \times 4}, \quad B_i = \begin{pmatrix} q^{n-i} & 0 & 1 & 0 \\ 0 & q^{n-i} & 0 & 1 \\ 1 & 0 & q^{n-i} & 0 \\ 0 & 1 & 0 & q^{n-i} \end{pmatrix}_{4 \times 4}.$$

It follows from the descriptions of the differentials $\tau_n$ in Lemma 3.1 that if $n$ is odd, then

$$\tau_n = \begin{pmatrix} 0 & A_2 & -B_1 & 0 & 0 & A_4 \\ 0 & -B_2 & 0 & A_4 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & A_{n-1} & 0 & -B_{n-2} & 0 \end{pmatrix}_{4(n+1) \times 4n},$$

and if $n$ is even, then

$$\tau_n = \begin{pmatrix} A_0 & 0 & 0 & A_2 \\ 0 & B_2 & A_2 \\ \vdots & \vdots & \ddots & \ddots \\ 0 & B_{n-2} & A_{n-2} & 0 \\ B_n & 0 & 0 & B_n \end{pmatrix}_{4(n+1) \times 4n}.$$

Clearly,

$$\text{rank}(\tau_n) = \sum_{i \in \{1, 3, 5, \ldots, n-2\}} \text{rank} \begin{pmatrix} A_{i+1} \\ -B_i \end{pmatrix}_{4 \times 4}.$$
if \( n \) is odd; and
\[
\text{rank}(\tau_n) = \text{rank}(A_0) + \text{rank}(B_n) + \sum_{i \in \{2, 4, 6, \ldots, n-2\}} \text{rank} \left( \begin{array}{cc} B_i & A_i \end{array} \right),
\]
if \( n \) is even.

**Lemma 3.2.** Let \( \Lambda_q \) be the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-Galois covering of \( A_q \), with \( q \) an \( r \)th \( (r > 2) \) primitive root of unity. If \( n \) (>2) is odd,
\[
\text{rank}(\tau_n) = \begin{cases} 
2n - k - 1 & \text{if } r \text{ is odd and } n = 2kr - 1, \text{ for some } k \geq 1, \\
2n - 2 & \text{if } r \text{ is even and } n = kr - 1, \text{ for some } k \geq 1; \\
2n & \text{otherwise.}
\end{cases}
\]
and if \( n \) (>2) is even, then
\[
\text{rank}(\tau_n) = \begin{cases} 
2n - k + 1 & \text{if } r \text{ is odd and } n = 2kr, \text{ for some } k \geq 1, \\
2n & \text{if } r \text{ is even and } n = kr, \text{ for some } k \geq 1; \\
2n & \text{otherwise.}
\end{cases}
\]

**Proof.**

**Case I.** Suppose \( n \) is odd.

For \( i = 1, 3, 5, \ldots, n - 2 \), by elementary operations, each
\[
\begin{pmatrix} A_{i+1} \\ -B_i \end{pmatrix}
\]
can be changed into
\[
\begin{pmatrix} 0 & 0 & -q^{n-i} & -q^{n+1} \\ 0 & 0 & -q^{n+1} & -q^{n-i} \\ 0 & 0 & 1 & q^{i+1} \\ 0 & 0 & q^{i+1} & 1 \\ 0 & 0 & q^{2(n-i)} - 1 & 0 \\ 0 & 0 & 0 & q^{2(n-i)} - 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}_{8 \times 4}
\]
Note that \( \text{rank} \left( \begin{array}{cc} A_{i+1} \\ -B_i \end{array} \right) \) = 3 or 4, and \( \text{rank} \left( \begin{array}{cc} A_{i+1} \\ -B_i \end{array} \right) = 3 \) if and only if
\[
\begin{cases} 
q^{2(n-i)} = 1; \\
q^{2(i+1)} = 1 \end{cases} \iff \begin{cases} 
q^{2(n+1)} = 1; \\
q^{2(i+1)} = 1. 
\end{cases}
\]
Moreover, \( q^{2(n+1)} = 1 \) if and only if either of the following ((1) or (2)) is satisfied:
(1) \( r \) is odd and \( 2(n + 1) = 4kr \), for some \( k \geq 1 \);
(2) \( r \) is even and \( 2(n + 1) = 2kr \), for some \( k \geq 1 \).
Since \( i \) is odd, we have \( q^{2(i+1)} = 1 \) if and only if either of the following ((3) or (4)) is satisfied:

(3) \( r \) is odd and \( 2(i + 1) = 4k_1r \), for some \( k_1 \geq 1 \);
(4) \( r \) is even and \( 2(i + 1) = 2k_1r \), for some \( k_1 \geq 1 \).

If (1) and (3) are satisfied, then \( r \) is odd, \( n = 2kr - 1 \) and \( k_1 = 1, 2, \ldots, k - 1 \). So the number of \( i \) satisfying

\[
\text{rank } \left( \begin{array}{c} A_{i+1} \\ -B_i \end{array} \right) = 3
\]

is \( k - 1 \), and \( \text{rank}(\tau_n) = 2n - k - 1 \).

If (2) and (4) are satisfied, then \( r \) is even, \( n = kr - 1 \) and \( k_1 = 1, 2, \ldots, k - 1 \). So the number of \( i \) satisfying

\[
\text{rank } \left( \begin{array}{c} A_{i+1} \\ -B_i \end{array} \right) = 3
\]

is \( k - 1 \), and \( \text{rank}(\tau_n) = 2n - k - 1 \).

Otherwise, for each \( i \),

\[
\text{rank } \left( \begin{array}{c} A_{i+1} \\ -B_i \end{array} \right) = 4.
\]

So \( \text{rank}(\tau_n) = 4 \times ((n - 1)/2) = 2n - 2 \).

**Case II.** Suppose \( n \) is even.

For \( i = 2, 4, 6, \ldots, n - 2 \), by elementary operations, each \( (B_i \quad A_i) \) can be changed into

\[
\begin{pmatrix}
0 & 0 & 1 & q^{2(n-i)} & 0 & 1 & q^i & -q^{n-i} & -q^n \\
0 & 0 & 0 & 1 - q^{2(n-i)} & q^i & 1 & -q^n & -q^{n-i} \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}_{4 \times 8}.
\]

Note that \( \text{rank}(B_i \quad A_i) = 3 \) or 4, and \( \text{rank}(B_i \quad A_i) = 3 \) if and only if

\[
\begin{cases}
q^{2(n-i)} = 1; \\
q^{2i} = 1.
\end{cases} \iff \begin{cases}
q^{2n} = 1; \\
q^{2i} = 1.
\end{cases}
\]

Moreover, we have \( q^{2n} = 1 \) if and only if either of the following ((5) or (6)) is satisfied:

(5) \( r \) is odd and \( 2n = 4kr \), for some \( k \geq 1 \);
(6) \( r \) is even and \( 2n = 2kr \), for some \( k \geq 1 \).
Since $i$ is even, we have $q^{2i} = 1$ if and only if either of the following ((7) or (8)) is satisfied:

(7) $r$ is odd and $2i = 4k_1r$, for some $k_1 \geq 1$;
(8) $r$ is even and $2i = 2k_1r$, for some $k_1 \geq 1$.

If (5) and (7) are satisfied, then $r$ is odd, $n = 2kr$ and $k_1 = 1, 2, \ldots, k - 1$. So the number of $i$ satisfying $\text{rank}(B_i A_i) = 3$ is $k - 1$, and $\text{rank}(\tau_n) = 2n - k + 1$.

If (6) and (8) are satisfied, then $r$ is even, $n = kr$ and $k_1 = 1, 2, \ldots, k - 1$. So the number of $i$ satisfying $\text{rank}(B_i A_i) = 3$ is $k - 1$, and $\text{rank}(\tau_n) = 2n - k + 1$.

Otherwise, for each $i$, $\text{rank}(B_i A_i) = 4$. So

$$\text{rank}(\tau_n) = 4 + 4 \times \frac{n - 2}{2} = 2n.$$ 

The proof is complete. \&

**Lemma 3.3.** Let $\Lambda_q$ be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of $A_q$. If $q (\neq 0)$ is not an $r$th ($r > 2$) primitive root of unity, then for $n > 2$

$$\text{rank}(\tau_n) = \begin{cases} 
2n - 2 & \text{if } n \text{ is odd and } q (\neq 0) \text{ is not a root of unity; } \\
2n & \text{if } n \text{ is even and } q (\neq 0) \text{ is not a root of unity; } \\
\frac{3}{2}(n - 1) & \text{if } n \text{ is odd and } q = \pm 1; \\
\frac{3}{2}n + 1 & \text{if } n \text{ is even and } q = \pm 1.
\end{cases}$$

**Proof.**

**Case I.** Suppose $n$ is odd. If $q (\neq 0)$ is not a root of unity, then $q^{2(n+1)} \neq 1$ for $n > 2$, and $\text{rank}(\tau_n) = 2n - 2$; if $q = \pm 1$, then $q^{2(n+1)} = q^{2(i+1)} = 1$, and $\text{rank}(\tau_n) = (3/2)(n - 1)$.

**Case II.** Suppose $n$ is even. If $q (\neq 0)$ is not a root of unity, then $q^{2n} \neq 1$ for $n > 2$, and $\text{rank}(\tau_n) = 2n$; if $q = \pm 1$, then $q^{2n} = q^{2i} = 1$, and $\text{rank}(\tau_n) = (3/2)n + 1$. This completes the proof. \&

For $n = 0, 1, 2$, direct computations show that

$$\dim_k HH_0(\Lambda_q) = 4;$$
$$\dim_k HH_1(\Lambda_q) = 4;$$
$$\dim_k HH_2(\Lambda_q) = \begin{cases} 
4 & \text{if } q \neq \pm 1, \pm \sqrt{-1}; \\
5 & \text{if } q = \pm 1, \pm \sqrt{-1}.
\end{cases}$$

**Theorem 3.4.** Let $\Lambda_q$ be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of the quantum exterior algebra $A_q$. If $q$ is an $r$th ($r > 2$) primitive root of unity, then for $n > 2$
Let \( \Lambda_q \) be the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-Galois covering of the quantum exterior algebra \( A_q \). If \( q \neq 0 \) is not an \( r \)th \((r > 2)\) primitive root of unity, then for \( n > 2 \),
\[
\dim_k HH_n(\Lambda_q) = \begin{cases} 
2k + 2 & \text{if } r \text{ is even and } n = kr - 1, \text{ for some } k \geq 1, \\
4 & \text{otherwise}.
\end{cases}
\]

**Proof.** By Lemma 3.2 and the formula
\[
\dim_k HH_n(\Lambda_q) = \dim_k N_n - \dim_k \text{Im } \tau_n - \dim_k \text{Im } \tau_{n+1},
\]
we can get the result directly. \( \square \)

**Theorem 3.5.** Let \( \Lambda_q \) be the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-Galois covering of the quantum exterior algebra \( A_q \). If \( q \neq 0 \) is not an \( r \)th \((r > 2)\) primitive root of unity, then for \( n > 2 \),
\[
\dim_k HH_n(\Lambda_q) = \begin{cases} 
4 & \text{if } q \neq 0 \text{ is not a root of unity}; \\
4k + 2 & \text{otherwise}.
\end{cases}
\]

**Proof.** By Lemma 3.3, we have \( \dim_k (\tau_n) + \dim_k (\tau_{n+1}) = 4n \) if \( q \neq 0 \) is not a root of unity; and \( \dim_k (\tau_n) + \dim_k (\tau_{n+1}) = 3n + 1 \) if \( q = \pm 1 \). The theorem follows from the formula (3.1) as desired. \( \square \)

Denote by \( HC_n(\Lambda_q) \) the \( n \)th cyclic homology group of \( \Lambda_q \) (see [21]).

**Corollary 3.6.** Let \( \Lambda_q \) be the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-Galois covering of the quantum exterior algebra \( A_q \). If \( q \) is an \( r \)th \((r > 2)\) primitive root of unity and \( \text{char } k = 0 \), then
\[
\dim_k HC_n(\Lambda_q) = \begin{cases} 
1 & \text{if } r \text{ is odd and } n = 2kr + 2, \text{ for some } k \geq 1, \\
4 & \text{otherwise}.
\end{cases}
\]

**Proof.** By [21, Theorem 4.1.13],
\[
\dim_k HC_n(\Lambda_q) - \dim_k HC_n(k^4) = -\left(\dim_k HC_{n-1}(\Lambda_q) - \dim_k HC_{n-1}(k^4)\right) \\
+ \left(\dim_k HH_n(\Lambda_q) - \dim_k HH_n(k^4)\right).
\]

Thus
\[
\dim_k HC_n(\Lambda_q) - \dim_k HC_n(k^4) = \sum_{i=0}^{n} (-1)^{n-i} \left(\dim_k HH_i(\Lambda_q) - \dim_k HH_i(k^4)\right).
\]

It is well known that
\[
\dim_k HH_i(k^4) = \begin{cases} 
4 & \text{if } i = 0; \\
0 & \text{if } i \geq 1
\end{cases}
\]
and
\[
\dim_k HC_i(k^4) = \begin{cases} 
4 & \text{if } i \text{ is even}; \\
0 & \text{if } i \text{ is odd}.
\end{cases}
\]

By Theorem 3.4, we can obtain the result. \( \square \)
COROLLARY 3.7. Let \( \Lambda_q \) be the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-Galois covering of the quantum exterior algebra \( A_q \). If \( q (\neq 0) \) is not an \( r \)th \((r > 2)\) primitive root of unity and \( \text{char} \, k = 0 \), then

\[
\dim_k HC_n(\Lambda_q) = \begin{cases} 
\frac{n-1}{2} + 4 & \text{if } q = \pm 1 \text{ and } n \text{ is odd; } \\
\frac{n}{2} + 4 & \text{if } q = \pm 1 \text{ and } n \text{ is even; } \\
4 & \text{if } q (\neq 0) \text{ is not a root of unity. }
\end{cases}
\]

PROOF. Similarly to the proof of Corollary 3.6, we can get this corollary by Theorem 3.5.

For completeness, we also consider the degenerate case when \( q = 0 \). Then

\[ A_0 = k \langle x, y \rangle / (x^2, xy, y^2), \]

and its \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-Galois covering algebra \( \Lambda_0 = k Q / (R_0) \), where the quiver \( Q \) is as in Section 2 and

\[ R_0 := \{ x_i x'_i, y_i y_{i+1}, x'_i x_i, y'_i y'_{i+1}, x_i y_i, x'_i y'_i \mid i = 0, 1 \}. \]

THEOREM 3.8. Let \( \Lambda_0 \) be the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-Galois covering of \( A_0 \) as the above; then

\[ \dim_k HH_n(\Lambda_0) = 4. \]

PROOF. Clearly, \( \Lambda_0 \) is a quadratic monomial algebra. Denote by \( \Lambda_0^! \) the quadratic dual of \( \Lambda_0 \); then

\[ \Lambda_0^! = k Q / (y_0 x'_0, y_1 x'_1, y_0 x_0, y_1 x_1), \]

and

\[ \Gamma^n = \{ f^{(n,i)}_j = x_i x'_i x_i \ldots x^{(n-j+1)}_i y^{(n-j)}_i y^{(n-j)+1} \ldots y^{(n-j+1)}_{i+j-1}, f^{(n,i')} = x_i x'_i x_i \ldots x^{(n-j)}_i y^{(n-j+1)}_i y^{(n-j+1)+1} \ldots y^{(n-j+1)}_{i+j-1} \mid 0 \leq j \leq n, i = 0, 1 \} \]

is a \( k \)-basis of \( \Lambda_0^! \). Thus, by [24], we can get a minimal projection resolution of \( \Lambda_0 \),

\[ (P^0_\bullet, \delta^0_\bullet) : \cdots \rightarrow P^0_n \xrightarrow{\delta^0_n} P^0_{n-1} \rightarrow \cdots \rightarrow P^0_2 \xrightarrow{\delta^0_2} P^0_1 \xrightarrow{\delta^0_1} P^0_0 \rightarrow 0, \]

where

\[ P^0_n = \bigoplus_{f \in \Gamma^n} \Lambda_0 o(f) \otimes t(f) \Lambda_0, \]

and the differentials are given by

\[ \delta^0_n (o(f) \otimes t(f)) = L^{n-1} \otimes t(f) + (-1)^n o(f) \otimes R^{n-1}, \]
where $L^{n-1}$ and $R^{n-1}$ are the subpaths of $f$ satisfying $f = L^{n-1}g = hR^{n-1}$ for some $g, h \in \Gamma^{(n-1)}$.

Applying the functor $\Lambda_0 \otimes_{\Lambda_0^*} (\cdot)$ to $(P_0^0, \delta_0^0)$, we get the homology complex \((N_0^0, \tau_0^0)\) of $\Lambda_0$, and $\dim_k N_n^0 = 4(n + 1)$,

$$\dim_k \text{Im} \tau_n^0 = \begin{cases} 2n - 2 & \text{if } n \text{ is odd;} \\ 2n & \text{if } n \text{ is even.} \end{cases}$$

So $\dim_k H \text{H}_n(\Lambda_0) = 4$.

Similar to Corollary 3.6, we have the following result.

**Corollary 3.9.** Let $\Lambda_0$ be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of $A_0$ and char $k = 0$; then $\dim_k H \text{C}_n(\Lambda_0) = 4$.

### 4. Hochschild cohomology

In this section we calculate the $k$-dimensions of Hochschild cohomology groups of the covering algebra $\Lambda_q$. Let $X$ and $Y$ be the sets of uniform elements in $kQ$; then one defines

$$X//Y = \{(p, q) \in X \times Y \mid o(p) = o(q) \text{ and } t(p) = t(q)\}.$$

We denote by $k(X//Y)$ the vector space that has as basis the set $X//Y$.

Applying the functor $\text{Hom}_{\Lambda_q^*}(\cdot, \Lambda_q)$ to the minimal projective bimodule resolution $(P_*, \delta_*)$, we have the following result.

**Lemma 4.1.** We have $\text{Hom}_{\Lambda_q^*}(P_*, \Lambda_q) = (M^*, \varphi^*)$, where $M^* \cong k(\mathcal{B}//\Gamma^{(n)})$ and $\varphi^{n+1} : M^n \rightarrow M^{n+1}$ is given by

$$\varphi^{n+1}(b, f_j^{(n,i)}) = (x'_ib, f_j^{(n+1,i)'}) + (-1)^{n+1}q^i(bx_{i+j}^{(n+1)}, f_j^{(n+1,i)});$$

$$+ q^{n-j}(y_{i+1}b, f_{j+1}^{(n+1,i+1)}) + (-1)^{n+1}(by_{i+j}^{(n+j)}, f_{j+1}^{(n+1,i)});$$

$$\varphi^{n+1}(b, f_j^{(n,i)'}). = (x_i'b, f_j^{(n+1,i)}) + (-1)^{n+1}q^i(bx_{i+j}^{(n+j)}, f_j^{(n+1,i)'})$$

$$+ q^{n-j}(y_{i+1}b, f_{j+1}^{(n+1,i+1)}) + (-1)^{n+1}(by_{i+j}^{(n+j)}, f_{j+1}^{(n+1,i)})$$

**Proof.** Clearly,

$$M^n = \text{Hom}_{\Lambda_q^*}(P_n, \Lambda_q) = \text{Hom}_{\Lambda_q^*} \left( \bigsqcup_{f \in \Gamma^{(n)}} \Lambda_q o(f) \otimes t(f) \Lambda_q, \Lambda_q \right)$$

$$\cong \bigsqcup_{f \in \Gamma^{(n)}} (o(f) \otimes t(f) \Lambda_q) \cong \bigsqcup_{f \in \Gamma^{(n)}} (o(f) \Lambda_q t(f)).$$

Thus $\mathcal{B}//\Gamma^{(n)}$ forms a $k$-basis of $M^n$ by definition.
Applying the commutative diagram

\[
\begin{array}{ccc}
\cdots \rightarrow \text{Hom}_{\Lambda_q^e}(P_n, \Lambda_q) & \xrightarrow{\delta_{n+1}} & \text{Hom}_{\Lambda_q^e}(P_{n+1}, \Lambda_q) \rightarrow \cdots \\
\downarrow & & \downarrow \\
\cdots \rightarrow k(B/\Gamma^{(n)}) & \xrightarrow{\varphi^{n+1}} & k(B/\Gamma^{(n+1)}) \rightarrow \cdots
\end{array}
\]

we can induce the differentials \(\varphi^n\) by \(\delta_n\) in the minimal projective resolution \((P_\bullet, \delta_\bullet)\).

Clearly, for any \((b, f) \in B/\Gamma^{(n)}\), \(l(b)\) and \(n\) must have the same parity. Thus \(\dim_k M^n = 4(n+1)\).

Now, we can define an order on \(B/\Gamma^{(n)}\) as follows:

\[
(b_1, f^{(n,i_1)}_{j_1}) < (b_2, f^{(n,i_2)}_{j_2}) \text{ if } j_1 < j_2 \text{ or } j_1 = j_2 \text{ but } b_1 < b_2,
\]

for any

\[
(b_1, f^{(n,i_1)}_{j_1}), (b_2, f^{(n,i_2)}_{j_2}) \in B/\Gamma^{(n)} \quad \text{and} \quad i_1, i_2 \in \{0, 0', 1, 1'\}.
\]

We still denote by \(\varphi^n\) the matrix of the differentials \(\varphi^n\) under the ordered bases above, and write

\[
C_i = \begin{pmatrix}
-q^i & 1 & 0 & 0 \\
1 & -q^i & 0 & 0 \\
0 & 0 & -q^i & 1 \\
0 & 0 & 1 & -q^i
\end{pmatrix}_{4 \times 4}, \quad D_i = \begin{pmatrix}
-1 & 0 & q^{n-i} & 0 \\
0 & -1 & 0 & q^{n-i} \\
q^{n-i} & 0 & -1 & 0 \\
0 & q^{n-i} & 0 & -1
\end{pmatrix}_{4 \times 4}.
\]

It follows from the description of the differentials \(\varphi^n\) in Lemma 4.1 that if \(n\) is odd, then

\[
\varphi^n = \begin{pmatrix}
C_0 & D_1 & 0 & 0 \\
C_2 & D_3 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots \\
C_{n-3} & D_{n-2} & 0 & 0 \\
C_{n-1} & D_n & & \\
0 & 0 & & \end{pmatrix}_{4n \times 4(n+1)};
\]

and if \(n\) is even, then
\[
\varphi^n = \begin{pmatrix}
0 & qD_2 & 0 \\
0 & -qC_0 & 0 \\
0 & qD_4 & -qC_2 \\
& & . & . \\
& & 0 & qD_n \\
& & 0 & -qC_{n-2}
\end{pmatrix}
\]

Clearly,

\[
\text{rank}(\varphi^n) = \begin{cases} 
\sum_{i \in \{0, 2, 4, \ldots, n-1\}} \text{rank}(C_iD_{i+1}) & \text{if } n \text{ is odd;} \\
\sum_{i \in \{0, 2, 4, \ldots, n-2\}} \text{rank}\left(D_{i+2} - C_i\right) & \text{if } n \text{ is even.}
\end{cases}
\]

**Lemma 4.2.** Let \(\Lambda_q\) be the \(\mathbb{Z}_2 \times \mathbb{Z}_2\)-Galois covering of \(A_q\), with \(q\) an \(r\)th \((r > 2)\) primitive root of unity. If \(n > 2\) is odd, then

\[
\text{rank}(\varphi^n) = \begin{cases} 
2n - k + 1 & \text{if } r \text{ is odd and } n = 2kr + 1, \text{ for some } k \geq 1, \\
2n + 2 & \text{otherwise;}
\end{cases}
\]

and if \(n > 2\) is even, then

\[
\text{rank}(\varphi^n) = \begin{cases} 
2n - k + 1 & \text{if } r \text{ is odd and } n = 2kr + 2, \text{ for some } k \geq 1, \\
2n & \text{otherwise.}
\end{cases}
\]

**Proof.**

**Case I.** Suppose \(n\) is odd.

For \(i = 0, 2, 4, \ldots, n - 1\), by elementary operations, each \((C_i \ D_{i+1})\) can be changed into

\[
\begin{pmatrix}
-q^i & 1 & -q^{n-1} & q^{n-i-1} & q^{2(n-i-1)} - 1 & 0 & 0 & 0 \\
1 & -q^i & q^{n-i-1} & -q^{n-1} & 0 & q^{2(n-i-1)} - 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
\]

\(4 \times 8\).

Note that \(\text{rank}(C_i \ D_{i+1}) = 3\) or 4, and \(\text{rank}(C_i \ D_{i+1}) = 3\) if and only if

\[
\begin{cases} 
q^{2(n-i-1)} = 1; \\
q^{2i} = 1
\end{cases} \iff \begin{cases} 
q^{2(n-1)} = 1; \\
q^{2i} = 1.
\end{cases}
\]

Moreover, we have \(q^{2(n-1)} = 1\) if and only if either of the following ((1′) or (2′)) is satisfied:
(1') $r$ is odd and $2(n - 1) = 4kr$, for some $k \geq 1$;
(2') $r$ is even and $2(n - 1) = 2kr$, for some $k \geq 1$.
Since $i$ is even, we have $q^{2i} = 1$ if and only if either of the following ((3') or (4')) is satisfied:
(3') $r$ is odd and $2i = 4k_1r$, for some $k_1 \geq 0$;
(4') $r$ is even and $2i = 2k_1r$, for some $k_1 \geq 0$.
If (1') and (3') are satisfied, then $r$ is odd, $n = 2kr + 1$ and $k_1 = 0, 1, \ldots, k$. So the number of $i$ satisfying $\text{rank}(C_i \ D_{i+1}) = 3$ is $k + 1$, and $\text{rank}(q^n) = 2n - k + 1$.
If (2') and (4') are satisfied, then $r$ is even, $n = kr + 1$ and $k_1 = 0, 1, \ldots, k$. So the number of $i$ satisfying $\text{rank}(C_i \ D_{i+1}) = 3$ is $k + 1$, and $\text{rank}(q^n) = 2n - k + 1$.
Otherwise, $\text{rank}(q^n) = 4 \times ((n + 1)/2) = 2n + 2$.

CASE II. Suppose $n$ is even.
For $i = 0, 2, 4, \ldots, n - 2$, by elementary operations, each
\[
\begin{pmatrix}
D_{i+2} \\
-C_i
\end{pmatrix}
\]
can be changed into
\[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & q^{2(n-i-2)} - 1 & 0 \\
0 & 0 & q^{n-2} & q^{2(n-i-2)} - 1 \\
0 & 0 & -q^{n-i-2} & -q^{n-i-2} \\
0 & 0 & q^i & -1 \\
0 & 0 & -1 & q^i
\end{pmatrix}
\]
8×4
Note that $\text{rank}\begin{pmatrix}D_{i+2} \\
-C_i\end{pmatrix} = 3$ or 4, and $\text{rank}\begin{pmatrix}D_{i+2} \\
-C_i\end{pmatrix} = 3$ if and only if
\[
\begin{cases}
q^{2(n-i-2)} = 1; \\
q^{2i} = 1; \\
q^{2(n-2)} = 1;
\end{cases}
\]
Moreover, we have $q^{2(n-2)} = 1$ if and only if either of the following ((5') or (6')) is satisfied:
(5') $r$ is odd and $2(n - 2) = 4kr$, for some $k \geq 1$;
(6') $r$ is even and $2(n - 2) = 2kr$, for some $k \geq 1$.
Since $i$ is even, we have $q^{2i} = 1$ if and only if either of the following ((7') or (8')) is satisfied:
(7') $r$ is odd and $2i = 4k_1r$, for some $k_1 \geq 1$;
(8') $r$ is even and $2i = 2k_1r$, for some $k_1 \geq 1$. 

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If (5′) and (7′) are satisfied, then \( r \) is odd, \( n = 2kr + 2 \) and \( k_1 = 0, 1, \ldots, k \). So the number of \( i \) such that

\[
\text{rank} \left( \frac{D_{i+2}}{-C_i} \right) = 3
\]

is \( k + 1 \), and \( \text{rank}(\varphi^n) = 2n - k - 1 \).

If (6′) and (8′) are satisfied, then \( r \) is even, \( n = kr + 2 \) and \( k_1 = 0, 1, \ldots, k \). So the number of \( i \) such that

\[
\text{rank} \left( \frac{D_{i+2}}{-C_i} \right) = 3
\]

is \( k + 1 \), and \( \text{rank}(\varphi^n) = 2n - k - 1 \).

Otherwise, \( \text{rank}(\varphi^n) = 4 \times (n/2) = 2n \). The proof is complete. \( \square \)

**Lemma 4.3.** Let \( \Lambda_q \) be the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-Galois covering of \( A_q \). If \( q (\neq 0) \) is not an \( r \)-th (\( r > 2 \)) primitive root of unity, then for \( n > 2 \),

\[
\text{rank}(\varphi^n) = \begin{cases} 
2n + 2 & \text{if } n \text{ is odd and } q (\neq 0) \text{ is not a root of unity;} \\
2n & \text{if } n \text{ is even and } q (\neq 0) \text{ is not a root of unity;} \\
\frac{3}{2} (n + 1) & \text{if } n \text{ is odd and } q = \pm 1; \\
\frac{3}{2} n & \text{if } n \text{ is even and } q = \pm 1. 
\end{cases}
\]

**Proof.**

**Case I.** Suppose \( n \) is odd. If \( q (\neq 0) \) is not a root of unity, then \( q^{2(n-1)} \neq 1 \) for \( n > 2 \), and \( \text{rank}(\varphi^n) = 2n + 2 \). If \( q = \pm 1 \), then \( q^{2(n-1)} = q^{2i} = 1 \), and \( \text{rank}(\varphi^n) = (3/2) (n + 1) \).

**Case II.** Suppose \( n \) is even. If \( q (\neq 0) \) is not a root of unity, then \( q^{2(n-2)} \neq 1 \) for \( n > 2 \), and \( \text{rank}(\varphi^n) = 2n \). If \( q = \pm 1 \), then \( q^{2(n-2)} = q^{2i} = 1 \), and \( \text{rank}(\varphi^n) = (3/2)n \).

This completes the proof. \( \square \)

For \( n = 0, 1, 2 \), direct computations show that

\[
\begin{align*}
\dim_k HH^0(\Lambda_q) &= 1; \\
\dim_k HH^1(\Lambda_q) &= 2; \\
\dim_k HH^2(\Lambda_q) &= \begin{cases} 
3 & \text{if } q = \pm 1, \pm \sqrt{-1}; \\
1 & \text{if } q \neq \pm 1, \pm \sqrt{-1}. 
\end{cases}
\end{align*}
\]

**Theorem 4.4.** Let \( \Lambda_q \) be the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-Galois covering of the quantum exterior algebra \( A_q \). If \( q \) is an \( r \)-th (\( r > 2 \)) primitive root of unity, then for \( n > 2 \),
\[
\dim_k H H^n(\Lambda_q) = \begin{cases} 
  k + 1 & \text{if } r \text{ is odd and } n = 2kr \text{ or } n = 2kr + 2, \text{ for some } k \geq 1; \\
  2k + 2 & \text{if } r \text{ is even and } n = kr \text{ or } n = kr + 2, \text{ for some } k \geq 1; \\
  0 & \text{otherwise}.
\end{cases}
\]

**Proof.** Note that \(H H^n(\Lambda_q) = \ker \varphi^{n+1}/\text{im} \varphi^n\) by definition, and

\[
\dim_k H H^n(\Lambda_q) = \dim_k \ker \varphi^{n+1} - \dim_k \text{im} \varphi^n \\
= \dim_k M^n - \dim_k \text{im} \varphi^{n+1} - \dim_k \text{im} \varphi^n \\
= \dim_k M^n - \text{rank} \varphi^{n+1} - \text{rank} \varphi^n. \quad (4.1)
\]

The theorem follows from Lemma 4.2. \(\square\)

**Theorem 4.5.** Let \(\Lambda_q\) be the \(\mathbb{Z}_2 \times \mathbb{Z}_2\)-Galois covering of the quantum exterior algebra \(A_q\). If \(q \neq 0\) is not an \(r\)th \((r > 2)\) primitive root of unity, then for \(n > 2\),

\[
\dim_k H H^n(\Lambda_q) = \begin{cases} 
  0 & \text{if } q \neq 0 \text{ is not a root of unity}; \\
  n + 1 & \text{if } q = \pm 1.
\end{cases}
\]

**Proof.** By Lemma 4.3,

\[
\dim_k (\varphi^n) + \dim_k (\varphi^{n+1}) = 4n + 4
\]

if \(q \neq 0\) is not a root of unity; and

\[
\dim_k (\varphi^n) + \dim_k (\varphi^{n+1}) = 3n + 3
\]

if \(q = \pm 1\). The theorem follows from the formula (4.1) as desired. \(\square\)

**Corollary 4.6.** If \(q \neq 0\) is not an \(r\)th \((r > 2)\) primitive root of unity, then the Hilbert series of the \(\mathbb{Z}_2 \times \mathbb{Z}_2\)-Galois covering of the algebra \(\Lambda_q\) is

\[
\sum_{n=0}^{\infty} \dim_k H H^n(\Lambda_q) t^n = \begin{cases} 
  1 & \text{if } q = \pm 1; \\
  (1 - t)^{-2} & \text{if } q \neq 0 \text{ is not a root of unity}.
\end{cases}
\]

**Proof.** This follows from Theorem 4.5 and the fact that

\[
\sum_{n=0}^{\infty} (n + 1) t^n = \frac{1}{(1 - t)^2}.
\]

Theorem 4.5 shows that the \(\mathbb{Z}_2 \times \mathbb{Z}_2\)-Galois covering of the algebra \(\Lambda_q\) also give a family of counterexamples to Happel’s question in the case where \(q \neq 0\) is not a root of unity. For completeness, we also consider the degenerate case when \(q = 0\).
THEOREM 4.7. Let $\Lambda_0$ be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of $A_0$; then

$$\dim_k H H^n(\Lambda_0) = \begin{cases} 
1 & \text{if } n = 0; \\
5 & \text{if } n = 1; \\
2n & \text{if } n \geq 2 \text{ is even}; \\
2n + 2 & \text{if } n \geq 2 \text{ is odd}.
\end{cases}$$

PROOF. Denote the cohomology complex of $\Lambda_0$ by $(\mathcal{M}_0^\bullet, \varphi_0^\bullet)$. We can get

$$\dim_k M_0^n = 4(n + 1);$$

$$\dim_k \text{Im} \varphi_0^1 = 3;$$

$$\dim_k \text{Im} \varphi_0^n = \begin{cases} 
2n + 2 & \text{if } n \geq 2 \text{ is odd}; \\
0 & \text{if } n \geq 2 \text{ is even}.
\end{cases}$$

Thus, we can get this theorem by the formula (4.1). \qed

Similar to Corollary 4.6, we have the following result.

COROLLARY 4.8. The Hilbert series of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering algebra $\Lambda_0$ is

$$\sum_{n=0}^{\infty} \dim_k H H^n(\Lambda_0)t^n = 1 + t + \frac{4t(1 + t)}{(1 - t^2)^2}.$$ 

References


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